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FEYNMAN-KAC PENALISATIONS OF SYMMETRIC STABLE PROCESSES

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Abstract

In K. Yano, Y. Yano and M. Yor (2009), limit theorems for the one-dimensional symmetric α-stable process normalized by negative (killing) Feynman-Kac functionals were studied. We consider the same problem and extend their results to positive Feynman-Kac functionals of multi-dimensional symmetric α-stable processes.

1 Introduction

In [9], [10], B. Roynette, P. Vallois and M. Yor have studied limit theorems for Wiener processes normalized by some weight processes. In [16], K. Yano, Y. Yano and M. Yor studied the limit theorems for the one-dimensional symmetric stable process normalized by non-negative functions of the local times or by negative (killing) Feynman-Kac functionals. They call the limit theorems for Markov processes normalized by Feynman-Kac functionals the *Feynman-Kac penalisations*. Our aim is to extend their results on Feynman-Kac penalisations to positive Feynman-Kac functionals of multi-dimensional symmetric α-stable processes.

Let $\mathbf{M}^\alpha = (\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P}_x, X_t)$ be the symmetric α-stable process on $\mathbb{R}^d$ with $0 < \alpha \le 2$, that is, the Markov process generated by $-(1/2)(-\Delta)^{\alpha/2}$, and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ the Dirichlet form of $\mathbf{M}^\alpha$ (see (2.1),(2.2)). Let μ be a positive Radon measure in the class $\mathscr{K}_\infty$ of Green-tight Kato measures (Definition 2.1). We denote by A_t^μ the positive continuous additive functional (PCAF in abbreviation) in the Revuz correspondence to μ: for a positive Borel function f and γ-excessive function g,

$$\langle g\mu, f\rangle = \lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^d}\mathbb{E}_x\left[\int_0^t f(X_s)dA_s^\mu\right]g(x)dx. \tag{1.1}$$

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We define the family $\{\mathbb{Q}^{\mu}_{x,t}\}$ of normalized probability measures by

$$\mathbb{Q}^{\mu}_{x,t}[B] = \frac{1}{Z^{\mu}_t(x)} \int_B \exp(A^{\mu}_t(\omega))\mathbb{P}_x(d\omega), \ \ B \in \mathscr{F}_t,$$

where $Z^{\mu}_t(x) = \mathbb{E}_x[\exp(A^{\mu}_t)]$. Our interest is the limit of $\mathbb{Q}^{\mu}_{x,t}$ as $t \to \infty$, mainly in transient cases, $d > \alpha$. They in [16] treated negative Feynman-Kac functionals in the case of the one-dimensional recurrent stable process, $\alpha > 1$. In this case, the decay rate of $Z^{\mu}_t(x)$ is important, while in our cases the growth order is.

We define

$$\lambda(\theta) = \inf\left\{ \mathscr{E}_\theta(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\}, \ \ 0 \le \theta < \infty, \tag{1.2}$$

where $\mathscr{E}_\theta(u,u) = \mathscr{E}(u,u) + \theta \int_{\mathbb{R}^d} u^2 dx$. We see from [5, Theorem 6.2.1] and [12, Lemma 3.1] that the time changed process by A^{μ}_t is symmetric with respect to μ and $\lambda(0)$ equals the bottom of the spectrum of the time changed process. We now classify the set $\mathscr{K}_\infty$ in terms of $\lambda(0)$:

(i) $\lambda(0) < 1$

In this case, there exist a positive constant $\theta_0 > 0$ and a positive continuous function h in the Dirichlet space $\mathscr{D}(\mathscr{E})$ such that

$$1 = \lambda(\theta_0) = \mathscr{E}_{\theta_0}(h,h)$$

(Lemma 3.1, Theorem 2.3). We define the multiplicative functional (MF in abbreviation) L^h_t by

$$L^h_t = e^{-\theta_0 t} \frac{h(X_t)}{h(X_0)} e^{A^{\mu}_t}. \tag{1.3}$$

(ii) $\lambda(0) = 1$

In this case, there exists a positive continuous function h in the extended Dirichlet space $\mathscr{D}_e(\mathscr{E})$ such that

$$1 = \lambda(0) = \mathscr{E}(h,h)$$

([14, Theorem 3.4]). Here $\mathscr{D}_e(\mathscr{E})$ is the set of measurable functions u on $\mathbb{R}^d$ such that $|u| < \infty$ a.e., and there exists an $\mathscr{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathscr{D}(\mathscr{E})$ such that $\lim_{n\to\infty} u_n = u$ a.e. We define

$$L^h_t = \frac{h(X_t)}{h(X_0)} e^{A^{\mu}_t}. \tag{1.4}$$

(iii) $\lambda(0) > 1$

In this case, the measure μ is *gaugeable*, that is,

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[e^{A^{\mu}_\infty} \right] < \infty$$

([15, Theorem 3.1]). We put $h(x) = \mathbb{E}_x[e^{A^{\mu}_\infty}]$ and define

$$L^h_t = \frac{h(X_t)}{h(X_0)} e^{A^{\mu}_t}. \tag{1.5}$$

The cases **(i), (ii)**, and **(iii)** are corresponding to the *supercriticality*, *criticality*, and *subcriticality* of the operator, $-(1/2)(-\Delta)^{\alpha/2}+\mu$, respectively ([15]). We will see that L_t^h is a martingale MF for each case, i.e., $\mathbb{E}_x[L_t^h]=1$. Let $\mathbf{M}^h=(\Omega,\mathbb{P}_x^h,X_t)$ be the transformed process of $\mathbf{M}^\alpha$ by L_t^h:

$$\mathbb{P}_x^h(B)=\int_B L_t^h(\omega)\mathbb{P}_x(d\omega),\quad B\in\mathscr{F}_t.$$

We then see from [3, Theorem 2.6] and Proposition 3.8 below that if $\lambda(0)\leq 1$, then $\mathbf{M}^h$ is an h^2dx-symmetric Harris recurrent Markov process.
To state the main result of this paper, we need to introduce a subclass $\mathscr{K}_\infty^S$ of $\mathscr{K}_\infty$; a measure $\mu\in\mathscr{K}_\infty$ is said to be in $\mathscr{K}_\infty^S$ if

$$\sup_{x\in\mathbb{R}^d}\left(|x|^{d-\alpha}\int_{\mathbb{R}^d}\frac{d\mu(y)}{|x-y|^{d-\alpha}}\right)<\infty. \tag{1.6}$$

This class is relevant to the notion of *special* PCAF's which was introduced by J. Neveu ([6]); we will show in Lemma 4.4 that if a measure μ belongs to $\mathscr{K}_\infty^S$, then $\int_0^t(1/h(X_s))dA_s^\mu$ is a *special* PCAF of $\mathbf{M}^h$. This fact is crucial for the proof of the main theorem below. In fact, a key to the proof lies in the application of the Chacon-Ornstein type ergodic theorem for special PCAF's of Harris recurrent Markov processes ([2, Theorem 3.18]).
We then have the next main theorem.

Theorem 1.1. (i) *If $\lambda(0)\neq 1$, then*

$$\mathbb{Q}_{x,t}^\mu\overset{t\to\infty}{\longrightarrow}\mathbb{P}_x^h\quad\text{along }(\mathscr{F}_t), \tag{1.7}$$

that is, for any $s\geq 0$ and any bounded $\mathscr{F}_s$-measurable function Z,

$$\lim_{t\to\infty}\frac{\mathbb{E}_x\left[Z\exp(A_t^\mu)\right]}{\mathbb{E}_x\left[\exp(A_t^\mu)\right]}=\mathbb{E}_x^h[Z].$$

(ii) *If $\lambda(0)=1$ and $\mu\in\mathscr{K}_\infty^S$, then (1.7) holds.*

Throughout this paper, $B(R)$ is an open ball with radius R centered at the origin. We use $c,C,...,etc$ as positive constants which may be different at different occurrences.

2 Preliminaries

Let $\mathbf{M}^\alpha=(\Omega,\mathscr{F},\mathscr{F}_t,\theta_t,\mathbb{P}_x,X_t)$ be the symmetric α-stable process on $\mathbb{R}^d$ with $0<\alpha\leq 2$. Here $\{\mathscr{F}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration and θ_t, $t\geq 0$, is the shift operators satisfying $X_s(\theta_t)=X_{s+t}$ identically for $s,t\geq 0$. When $\alpha=2$, $\mathbf{M}^\alpha$ is the Brownian motion. Let $p(t,x,y)$ be the transition density function of $\mathbf{M}^\alpha$ and $G_\beta(x,y)$, $\beta\geq 0$, be its β-Green function,

$$G_\beta(x,y)=\int_0^\infty e^{-\beta t}p(t,x,y)dt.$$

For a positive measure μ, the β-potential of μ is defined by

$$G_\beta\mu(x)=\int_{\mathbb{R}^d}G_\beta(x,y)\mu(dy).$$

Let P_t be the semigroup of $\mathbf{M}^\alpha$,

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy = \mathbb{E}_x[f(X_t)].$$

Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be the Dirichlet form generated by $\mathbf{M}^\alpha$: for $0 < \alpha < 2$

$$\begin{cases} \mathscr{E}(u,v) = \dfrac{1}{2}\mathscr{A}(d,\alpha) \displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} dx dy \\ \mathscr{D}(\mathscr{E}) = \left\{ u \in L^2(\mathbb{R}^d) : \displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} dx dy < \infty \right\}, \end{cases} \tag{2.1}$$

where $\Delta = \{(x,x) : x \in \mathbb{R}^d\}$ and

$$\mathscr{A}(d,\alpha) = \frac{\alpha 2^{d-1} \Gamma(\frac{\alpha+d}{2})}{\pi^{d/2} \Gamma(1-\frac{\alpha}{2})}$$

([5, Example 1.4.1]); for $\alpha = 2$

$$\mathscr{E}(u,v) = \frac{1}{2}\mathbf{D}(u,v), \quad \mathscr{D}(\mathscr{E}) = H^1(\mathbb{R}^d), \tag{2.2}$$

where $\mathbf{D}$ denotes the classical Dirichlet integral and $H^1(\mathbb{R}^d)$ is the Sobolev space of order 1 ([5, Example 4.4.1]). Let $\mathscr{D}_e(\mathscr{E})$ denote the extended Dirichlet space ([5, p.35]). If $\alpha < d$, that is, the process $\mathbf{M}^\alpha$ is transient, then $\mathscr{D}_e(\mathscr{E})$ is a Hilbert space with inner product $\mathscr{E}$ ([5, Theorem 1.5.3]). We now define classes of measures which play an important role in this paper.

Definition 2.1. (I) A positive Radon measure μ on $\mathbb{R}^d$ is said to be in the *Kato class* ($\mu \in \mathscr{K}$ in notation), if

$$\lim_{\beta \to \infty} \sup_{x \in \mathbb{R}^d} G_\beta \mu(x) = 0. \tag{2.3}$$

(II) A measure μ is said to be β-*Green-tight* ($\mu \in \mathscr{K}_\infty(\beta)$ in notation), if μ is in $\mathscr{K}$ and satisfies

$$\lim_{R \to \infty} \sup_{x \in \mathbb{R}^d} \int_{|y|>R} G_\beta(x,y) \mu(dy) = 0. \tag{2.4}$$

We see from the resolvent equation that for $\beta > 0$

$$\mathscr{K}_\infty(\beta) = \mathscr{K}_\infty(1).$$

When $d > \alpha$, that is, $\mathbf{M}^\alpha$ is transient, we write $\mathscr{K}_\infty$ for $\mathscr{K}_\infty(0)$. For $\mu \in \mathscr{K}$, define a symmetric bilinear form $\mathscr{E}^\mu$ by

$$\mathscr{E}^\mu(u,u) = \mathscr{E}(u,u) - \int_{\mathbb{R}^d} \widetilde{u}^2 d\mu, \quad u \in \mathscr{D}(\mathscr{E}), \tag{2.5}$$

where $\widetilde{u}$ is a quasi-continuous version of u ([5, Theorem 2.1.3]). In the sequel, we always assume that every function $u \in \mathscr{D}_e(\mathscr{E})$ is represented by its quasi continuous version. Since $\mu \in \mathscr{K}$ charges no set of zero capacity by [1, Theorem 3.3], the form $\mathscr{E}^\mu$ is well defined. We see from

[1, Theorem 4.1] that $(\mathscr{E}^\mu, \mathscr{D}(\mathscr{E}))$ becomes a lower semi-bounded closed symmetric form. Denote by $\mathscr{H}^\mu$ the self-adjoint operator generated by $(\mathscr{E}^\mu, \mathscr{D}(\mathscr{E}))$: $\mathscr{E}^\mu(u,v) = (\mathscr{H}^\mu u, v)$. Let P_t^μ be the L^2-semigroup generated by $\mathscr{H}^\mu$: $P_t^\mu = \exp(-t\mathscr{H}^\mu)$. We see from [1, Theorem 6.3(iv)] that P_t^μ admits a symmetric integral kernel $p^\mu(t,x,y)$ which is jointly continuous function on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. For $\mu \in \mathscr{K}$, let A_t^μ be a PCAF which is in the Revuz correspondence to μ (Cf. [5, p.188]). By the Feynman-Kac formula, the semigroup P_t^μ is written as

$$P_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu) f(X_t)]. \tag{2.6}$$

Theorem 2.2 ([11])**.** *Let $\mu \in \mathscr{K}$. Then*

$$\int_{\mathbb{R}^d} u^2(x)\mu(dx) \le \|G_\beta \mu\|_\infty \mathscr{E}_\beta(u,u), \quad u \in \mathscr{D}(\mathscr{E}), \tag{2.7}$$

where $\mathscr{E}_\beta(u,u) = \mathscr{E}(u,u) + \beta \int_{\mathbb{R}^d} u^2 dx$.

Theorem 2.3. ([14, Theorem 10], [13, Theorem 2.7]) *If $\mu \in \mathscr{K}_\infty(1)$, then the embedding of $\mathscr{D}(\mathscr{E})$ into $L^2(\mu)$ is compact. If $d > \alpha$ and $\mu \in \mathscr{K}_\infty$, then the embedding of $\mathscr{D}_e(\mathscr{E})$ into $L^2(\mu)$ is compact.*

3 Construction of ground states

For $d \le \alpha$ (resp. $d > \alpha$), let μ be a non-trivial measure in $\mathscr{K}_\infty(1)$ (resp. $\mathscr{K}_\infty$). Define

$$\lambda(\theta) = \inf\left\{\mathscr{E}_\theta(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1\right\}, \quad \theta \ge 0. \tag{3.1}$$

Lemma 3.1. *The function $\lambda(\theta)$ is increasing and concave. Moreover, it satisfies $\lim_{\theta\to\infty}\lambda(\theta) = \infty$.*

Proof. It follows from the definition of $\lambda(\theta)$ that it is increasing. For $\theta_1, \theta_2 \ge 0$, $0 \le t \le 1$

$$\begin{aligned}\lambda(t\theta_1 + (1-t)\theta_2) &= \inf\left\{\mathscr{E}_{t\theta_1+(1-t)\theta_2}(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} \\ &\ge t \inf\left\{\mathscr{E}_{\theta_1}(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} + (1-t)\inf\left\{\mathscr{E}_{\theta_2}(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} \\ &= t\lambda(\theta_1) + (1-t)\lambda(\theta_2).\end{aligned}$$

We see from Theorem 2.2 that for $u \in \mathscr{D}(\mathscr{E})$ with $\int_{\mathbb{R}^d} u^2 d\mu = 1$, $\mathscr{E}_\theta(u,u) \ge 1/\|G_\theta\mu\|_\infty$. Hence we have

$$\lambda(\theta) \ge \frac{1}{\|G_\theta \mu\|_\infty}. \tag{3.2}$$

By the definition of the Kato class, the right hand side of (3.2) tends to infinity as $\theta \to \infty$. □

Lemma 3.2. *If $d \le \alpha$, then $\lambda(0) = 0$.*

Proof. Note that for $u \in \mathscr{D}(\mathscr{E})$

$$\lambda(0)\int_{\mathbb{R}^d} u^2 d\mu \le \mathscr{E}(u,u).$$

Since $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is recurrent, there exists a sequence $\{u_n\} \subset \mathscr{D}(\mathscr{E})$ such that $u_n \uparrow 1$ q.e. and $\mathscr{E}(u_n,u_n) \to 0$ ([5, Theorem 1.6.3, Theorem 2.1.7]). Hence if $\lambda(0) > 0$, then $\mu = 0$, which is contradictory. □

We see from Theorem 2.3 and Lemma 3.2 that if $d \le \alpha$, then there exist $\theta_0 > 0$ and $h \in \mathcal{D}(\mathcal{E})$ such that

$$\lambda(\theta_0) = \inf\left\{ \mathcal{E}_{\theta_0}(h,h) : \int_{\mathbb{R}^d} h^2 d\mu = 1 \right\} = 1.$$

We can assume that h is a strictly positive continuous function (e.g. Section 4 in [14]).
Let $M_t^{[h]}$ be the martingale part of the Fukushima decomposition ([5, Theorem 5.2.2]):

$$h(X_t) - h(X_0) = M_t^{[h]} + N_t^{[h]}. \tag{3.3}$$

Define a martingale by

$$M_t = \int_0^t \frac{1}{h(X_{s-})} dM_s^h$$

and denote by L_t^h the unique solution of the Doléans-Dade equation:

$$Z_t = 1 + \int_0^t Z_{s-} dM_s. \tag{3.4}$$

Then we see from the Doléans-Dade formula that L_t^h is expressed by

$$\begin{aligned} L_t^h &= \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t \right) \prod_{0<s\le t} (1 + \Delta M_s) \exp(-\Delta M_s) \\ &= \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t \right) \prod_{0<s\le t} \frac{h(X_s)}{h(X_{s-})} \exp\left(1 - \frac{h(X_s)}{h(X_{s-})} \right). \end{aligned}$$

Here M_t^c is the continuous part of M_t and $\Delta M_s = M_s - M_{s-}$. By Itô's formula applied to the semi-martingale $h(X_t)$ with the function $\log x$, we see that L_t^h has the following expression:

$$L_t^h = e^{-\theta_0 t} \frac{h(X_t)}{h(X_0)} \exp(A_t^\mu). \tag{3.5}$$

Let $d > \alpha$ and suppose that $\theta_0 = 0$, that is,

$$\lambda(0) = \inf\left\{ \mathcal{E}(u,u) : \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\} = 1.$$

We then see from [14, Theorem 3.4] that there exists a function $h \in \mathcal{D}_e(\mathcal{E})$ such that $\mathcal{E}(h,h) = 1$. We can also assume that h is a strictly positive continuous function and satisfies

$$\frac{c}{|x|^{d-\alpha}} \le h(x) \le \frac{C}{|x|^{d-\alpha}}, \quad |x| > 1 \tag{3.6}$$

(see (4.19) in [14]). We define the MF L_t^h by

$$L_t^h = \frac{h(X_t)}{h(X_0)} \exp(A_t^\mu). \tag{3.7}$$

We denote by $\mathbf{M}^h = (\Omega, \mathbb{P}_x^h, X_t)$ the transformed process of $\mathbf{M}^\alpha$ by L_t^h,

$$\mathbb{P}_x^h(d\omega) = L_t^h(\omega) \cdot \mathbb{P}_x(d\omega).$$

Proposition 3.3. *The transformed process* $\mathbf{M}^h = (\mathbb{P}_x^h, X_t)$ *is Harris recurrent, that is, for a non-negative function* f *with* $m(\{x : f(x) > 0\}) > 0$*,*

$$\int_0^\infty f(X_t)dt = \infty \ \ \mathbb{P}_x^h\text{-}a.s., \tag{3.8}$$

where m *is the Lebesgue measure.*

Proof. Set $A = \{x : f(x) > 0\}$. Since $\mathbf{M}^h$ is an h^2dx-symmetric recurrent Markov process,

$$\mathbb{P}_x[\sigma_A \circ \theta_n < \infty, \ \forall n \geq 0] = 1 \ \text{ for q.e. } x \in \mathbb{R}^d \tag{3.9}$$

by [5, Theorem 4.6]. Moreover, since the Markov process $\mathbf{M}^h$ has the transition density function

$$e^{-\theta_0 t} \cdot \frac{p^\mu(t,x,y)}{h(x)h(y)}$$

with respect to h^2dx, (3.9) holds for all $x \in \mathbb{R}^d$ by [5, Problem 4.6.3]. Using the strong Feller property and the proof of [8, Chapter X, Proposition (3.11)], we see from (3.9) that $\mathbf{M}^h$ is Harris recurrent. □

We see from [14, Theorem 4.15] : If $\theta_0 > 0$, then $h \in L^2(\mathbb{R}^d)$ and $\mathbf{M}^h$ is positive recurrent. If $\theta_0 = 0$ and $\alpha < d \leq 2\alpha$, then $h \notin L^2(\mathbb{R}^d)$ $\mathbf{M}^h$ is null recurrent. If $\theta_0 = 0$ and $d \geq 2\alpha$, then $h \in L^2(\mathbb{R}^d)$ $\mathbf{M}^h$ is positive recurrent.

4 Penalization problems

In this section, we prove Theorem 1.1.

(1°) **Recurrent case** ($d \leq \alpha$)

Theorem 4.1. *Assume that* $d \leq \alpha$*. Then there exist* $\theta_0 > 0$ *and* $h \in \mathscr{D}(\mathscr{E})$ *such that* $\lambda(\theta_0) = 1$ *and* $\mathscr{E}_{\theta_0}(h,h) = 1$*. Moreover, for each* $x \in \mathbb{R}^d$

$$e^{-\theta_0 t}\mathbb{E}_x\left[e^{A_t^\mu}\right] \longrightarrow h(x)\int_{\mathbb{R}^d} h(x)dx \ \ as \ \ t \longrightarrow \infty.. \tag{4.1}$$

Proof. The first assertion follows from Theorem 2.3 and Lemma 3.2. Note that

$$e^{-\theta_0 t}\mathbb{E}_x\left[e^{A_t^\mu}\right] = h(x)\mathbb{E}_x^h\left[\frac{1}{h(X_t)}\right]$$

Then by [13, Corollary 4.7] the right hand side converges to $h(x)\int_{\mathbb{R}^d} h(x)dx$. □

Theorem 4.1 implies (1.7). Indeed,

$$\begin{aligned}
\frac{\mathbb{E}_x\left(\exp(A_t^\mu)|\mathscr{F}_s\right)}{\mathbb{E}_x\left(\exp(A_t^\mu)\right)} &= \frac{e^{-\theta_0 t}\mathbb{E}_x\left(\exp(A_t^\mu)|\mathscr{F}_s\right)}{e^{-\theta_0 t}\mathbb{E}_x\left(\exp(A_t^\mu)\right)} \\
&= \frac{e^{-\theta_0 s}\exp(A_s^\mu)e^{-\theta_0(t-s)}\mathbb{E}_{X_s}\left(\exp(A_{t-s}^\mu)\right)}{e^{-\theta_0 t}\mathbb{E}_x\left(\exp(A_t^\mu)\right)} \\
\longrightarrow & \ \ \frac{e^{-\theta_0 s}\exp(A_s^\mu)h(X_s)\int_{\mathbb{R}^d} h(x)dx}{h(x)\int_{\mathbb{R}^d} h(x)dx} = L_s^h \ \ as \ \ t \longrightarrow \infty.
\end{aligned}$$

We showed in [3, Theorem 2.6 (b)] that the transformed process $\mathbf{M}^h$ is recurrent. We see from this fact that L_t^h is martingale, $\mathbb{E}(L_t^h)=1$. Therefore Scheff's lemma leads us to Theorem 1.1 (i) (e.g. [9]).

(2°) **Transient case** ($d>\alpha$)

If $\lambda(0)<1$, there exist $\theta_0>0$ and $h\in\mathscr{D}(\mathscr{E})$ such that $\lambda(\theta_0)=1$ and $\mathscr{E}_{\theta_0}(h,h)=1$. Then we can show the equation (4.1) in the same way as above. If $\lambda(0)>1$, then A_t^μ is gaugeable (see Theorem 4.1 below), that is,

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}_x\left[e^{A_\infty^\mu}\right]<\infty,$$

and thus

$$\lim_{t\to\infty}\mathbb{E}_x\left[e^{A_t^\mu}\right]=\mathbb{E}_x\left[e^{A_\infty^\mu}\right].$$

Hence for any $s\geq 0$ and any $\mathscr{F}_s$-measurable bounded function Z

$$\begin{aligned}\frac{\mathbb{E}_x\left[Ze^{A_t^\mu}\right]}{\mathbb{E}_x\left[e^{A_t^\mu}\right]}&=\frac{\mathbb{E}_x\left[Ze^{A_s^\mu}\mathbb{E}_{X_s}\left[e^{A_{t-s}^\mu}\right]\right]}{\mathbb{E}_x\left[e^{A_t^\mu}\right]}\\ &\longrightarrow\frac{\mathbb{E}_x\left[Ze^{A_s^\mu}\mathbb{E}_{X_s}\left[e^{A_\infty^\mu}\right]\right]}{\mathbb{E}_x\left[e^{A_\infty^\mu}\right]}=\frac{1}{h(x)}\mathbb{E}_x\left[Ze^{A_s^\mu}h(X_s)\right]=\mathbb{E}_x^h[Z]\end{aligned}$$

as $t\to\infty$.
In the remainder of this section, we consider the case when $\lambda(0)=1$. It is known that a measure $\mu\in\mathscr{K}_\infty$ is Green-bounded,

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{d\mu(y)}{|x-y|^{d-\alpha}}<\infty. \tag{4.2}$$

To consider the penalisation problem for μ with $\lambda(0)=1$, we need to impose a condition on μ.

Definition 4.2. (I) A measure $\mu\in\mathscr{K}$ is said to be *special* if

$$\sup_{x\in\mathbb{R}^d}\left(|x|^{d-\alpha}\int_{\mathbb{R}^d}\frac{d\mu(y)}{|x-y|^{d-\alpha}}\right)<\infty. \tag{4.3}$$

We denote by $\mathscr{K}_\infty^S$ the set of special measures.
(II) A PCAF A_t is said to be *special* with respect to $\mathbf{M}^h$, if for any positive Borel function g with $\int_{\mathbb{R}^d}g\,dx>0$

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}_x^h\left[\int_0^\infty\exp\left(-\int_0^t g(X_s)ds\right)dA_t\right]<\infty.$$

A Kato measure with compact support belongs to $\mathscr{K}_\infty^S$. The set $\mathscr{K}_\infty^S$ is contained in $\mathscr{K}_\infty$,

$$\mathscr{K}_\infty^S\subset\mathscr{K}_\infty. \tag{4.4}$$

Indeed, since for any $R>0$

$$M(\mu):=\sup_{x\in\mathbb{R}^d}\left(|x|^{d-\alpha}\int_{\mathbb{R}^d}\frac{d\mu(y)}{|x-y|^{d-\alpha}}\right)\geq R^{d-\alpha}\sup_{x\in B(R)^c}\int_{\mathbb{R}^d}\frac{d\mu(y)}{|x-y|^{d-\alpha}},$$

we have

$$\sup_{x\in\mathbb{R}^d}\int_{B(R)^c}\frac{d\mu(y)}{|x-y|^{d-\alpha}} = \sup_{x\in B(R)^c}\int_{B(R)^c}\frac{d\mu(y)}{|x-y|^{d-\alpha}} \le \frac{M(\mu)}{R^{d-\alpha}} \longrightarrow 0,\ \ R\to\infty.$$

Lemma 4.3. *Let B_t be a PCAF. Then*

$$\mathbb{E}_x\left[\int_0^\infty e^{(A_t^\mu - B_t)}dA_t^\mu\right] = h(x)\mathbb{E}_x^h\left[\int_0^\infty e^{-B_t}\frac{dA_t^\mu}{h(X_t)}\right].$$

Proof. We have

$$\begin{aligned} h(x)\mathbb{E}_x^h\left[\int_0^s e^{-B_t}\frac{dA_t^\mu}{h(X_t)}\right] &= \mathbb{E}_x\left[e^{A_s^\mu}h(X_s)\int_0^s e^{-B_t}\frac{dA_t^\mu}{h(X_t)}\right] \\ &= \mathbb{E}_x\left[\int_0^s e^{A_s^\mu}h(X_s)e^{-B_t}\frac{dA_t^\mu}{h(X_t)}\right]. \end{aligned}$$

Put $Y_t = e^{A_s^\mu}h(X_s)e^{-B_t}/h(X_t)$. Then since Y_t is a right continuous process, its optional projection is equal to $\mathbb{E}_x[Y_t|\mathscr{F}_t]$ (e.g. [7, Theorem 7.10]). Hence the right hand side equals

$$\mathbb{E}_x\left[\int_0^s \mathbb{E}_x\left[Y_t|\mathscr{F}_t\right]dA_t^\mu\right] = \mathbb{E}_x\left[\int_0^s e^{A_t^\mu}e^{-B_t}\frac{1}{h(X_t)}\mathbb{E}_{X_t}\left[e^{A_{s-t}^\mu}h(X_{s-t})\right]dA_t^\mu\right].$$

Since $\mathbb{E}_{X_t}\left[e^{A_{s-t}^\mu}h(X_{s-t})\right] = h(X_t)$, the right hand side equals

$$\mathbb{E}_x\left[\int_0^s e^{A_t^\mu - B_t}dA_t^\mu\right].$$

Hence the proof is completed by letting $s\to\infty$. □

The next theorem was proved in [15].

Theorem 4.1. ([15]) *Suppose $d>\alpha$. For $\mu = \mu^+ - \mu^- \in \mathscr{K}_\infty - \mathscr{K}_\infty$, let $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^+}$. Then the following conditions are equivalent:*

(i) $\sup_{x\in\mathbb{R}^d}\mathbb{E}_x[e^{A_\infty^\mu}]<\infty$.

(ii) *There exists the Green function $G^\mu(x,y)<\infty$ $(x\neq y)$ of the operator $-\frac{1}{2}(-\Delta)^{\alpha/2}+\mu$ such that*

$$\mathbb{E}_x\left[\int_0^\infty e^{A_t^\mu}f(X_t)dt\right] = \int_{\mathbb{R}^d}G^\mu(x,y)f(y)dy.$$

(iii) $\inf\left\{\mathscr{E}(u,u)+\int_{\mathbb{R}^d}u^2d\mu^- : \int_{\mathbb{R}^d}u^2d\mu^+ = 1\right\}>1.$

We see from (4.19) in [14] that if one of the statements in Theorem 4.1 holds, then $G^\mu(x,y)$ satisfies

$$G(x,y)\le G^\mu(x,y)\le CG(x,y). \tag{4.5}$$

Lemma 4.4. *If $\mu \in \mathcal{K}_\infty^S$, then $\displaystyle\int_0^t \frac{dA_s^\mu}{h(X_s)}$ is special with respect to $\mathbf{M}^h$.*

Proof. We may assume that g is a bounded positive Borel function with compact support. Note that by Lemma 4.3

$$\begin{aligned} & \mathbb{E}_x^h\left[\int_0^\infty \exp\left(-\int_0^t g(X_s)ds\right)\frac{dA_t^\mu}{h(X_t)}\right] \\ = & \frac{1}{h(x)}\mathbb{E}_x\left[\int_0^\infty \exp\left(A_t^\mu - \int_0^t g(X_s)ds\right)dA_t^\mu\right] \\ = & \frac{1}{h(x)}G^{\mu-g\cdot dx}\mu(x). \end{aligned}$$

If the measure μ satisfies $\lambda(0)=1$, then $\mu - g\cdot dx \in \mathcal{K}_\infty - \mathcal{K}_\infty$ satisfies Theorem 4.1 (iii), and $G^{\mu-g\cdot dx}(x,y)$ is equivalent with $G(x,y)$ by (4.5). Therefore the equation (3.6) implies that (4.3) is equivalent to that $\sup_{x\in\mathbb{R}^d}\left\{(1/h(x))G^{\mu-g\cdot dx}\mu(x)\right\}<\infty$. □

We note that by Lemma 4.3

$$\mathbb{E}_x\left[e^{A_t^\mu}\right] = 1 + \mathbb{E}_x\left[\int_0^t e^{A_s^\mu}dA_s^\mu\right] = 1 + h(x)\mathbb{E}_x^h\left[\int_0^t \frac{dA_s^\mu}{h(X_s)}\right].$$

Thus for a finite positive measure ν,

$$\mathbb{E}_\nu\left[e^{A_t^\mu}\right] = \nu(\mathbb{R}^d) + \langle \nu, h\rangle \mathbb{E}_{\nu^h}^h\left[\int_0^t \frac{dA_s^\mu}{h(X_s)}\right] \tag{4.6}$$

where $\nu^h = h\cdot\nu/\langle\nu,h\rangle$. For a positive smooth function k with compact support, put

$$\psi(t) = \mathbb{E}_x^h\left[\int_0^t k(X_s)ds\right].$$

Then $\lim_{t\to\infty}\psi(t)=\infty$ by the Harris recurrence of $\mathbf{M}^h$. Moreover,

$$\lim_{t\to\infty}\frac{\psi(t+s)}{\psi(t)} = 1. \tag{4.7}$$

Indeed,

$$\begin{aligned} \psi(t+s) &= \mathbb{E}_x^h\left[\int_0^t k(X_u)du\right] + \mathbb{E}_x^h\left[\mathbb{E}_{X_t}^h\left[\int_0^s k(X_u)du\right]\right] \\ &\le \psi(t) + \|k\|_\infty s, \end{aligned}$$

and

$$1 \le \frac{\psi(t+s)}{\psi(t)} \le 1 + \frac{\|k\|_\infty s}{\psi(t)}.$$

We see from [4, Lemma 4.4] that the Revuz measure of A_t^μ is $h^2\mu$ as a PCAF of $\mathbf{M}^h$. Since by (4.6)

$$\frac{1}{\psi(t)}\mathbb{E}_\nu\left[e^{A_t^\mu}\right] = \frac{\nu(\mathbb{R}^d)}{\psi(t)} + \langle\nu,h\rangle\frac{\mathbb{E}_{\nu^h}^h\left[\int_0^t (1/h(X_s))dA_s^\mu\right]}{\mathbb{E}_x^h\left[\int_0^t k(X_s)ds\right]}$$

and $\int_0^t (1/h(X_s))dA_s^\mu$ and $\int_0^t k(X_s)ds$ are special with respect to $\mathbb{M}^h$, we see from Chacon-Ornstein type ergodic theorem in [2, Theorem 3.18] that

$$\frac{1}{\psi(t)}\mathbb{E}_\nu\left[e^{A_t^\mu}\right] \longrightarrow \langle \nu, h\rangle \cdot \frac{\langle \mu, h\rangle}{\int_{\mathbb{R}^d} kh^2 dx} \tag{4.8}$$

as $t \to \infty$. Note that $\langle \mu, h\rangle < \infty$ by (3.6) and (4.2).
For a bounded $\mathscr{F}_s$-measurable function Z, define a positive finite measure ν by

$$\nu(B) = \mathbb{E}_x\left[Ze^{A_s^\mu}; X_s \in B\right], \ \ B \in \mathscr{B}(\mathbb{R}^d).$$

Then by the Markov property,

$$\mathbb{E}_x\left[Ze^{A_t^\mu}\right] = \mathbb{E}_\nu\left[e^{A_{t-s}^\mu}\right].$$

Therefore

$$\begin{aligned}
\lim_{t\to\infty}\frac{\mathbb{E}_x\left[Ze^{A_t^\mu}\right]}{\mathbb{E}_x\left[e^{A_t^\mu}\right]} &= \lim_{t\to\infty}\frac{\mathbb{E}_x\left[Ze^{A_t^\mu}\right]/\psi(t)}{\mathbb{E}_x\left[e^{A_t^\mu}\right]/\psi(t)} \\
&= \lim_{t\to\infty}\frac{(\psi(t-s)/\psi(t))\mathbb{E}_\nu\left[e^{A_{t-s}^\mu}\right]/\psi(t-s)}{\mathbb{E}_x\left[e^{A_t^\mu}\right]/\psi(t)}.
\end{aligned}$$

By (4.7) and (4.8), the right hand side equals

$$\frac{(\langle \nu, h\rangle\langle \mu, h\rangle)/\int_{\mathbb{R}^d} kh^2 dx}{(h(x)\langle \mu, h\rangle)/\int_{\mathbb{R}^d} kh^2 dx} = \frac{\langle \nu, h\rangle}{h(x)} = \frac{1}{h(x)}\mathbb{E}_x\left[Ze^{A_s^\mu}h(X_s)\right] = \mathbb{E}_x^h[Z]. \tag{4.9}$$

Remark 4.5. We suppose that $d > \alpha$ and $\lambda(0) = 1$. If $d > 2\alpha$, then $h \in L^2(\mathbb{R}^d)$ on account of (3.6). Hence $\mathbf{M}^h$ is an ergodic process with the invariant probability measure h^2dx, and thus for a smooth function k with compact support,

$$\frac{\psi(t)}{t} = \frac{1}{t}\mathbb{E}_x^h\left[\int_0^t k(X_s)ds\right] \longrightarrow \int_{\mathbb{R}^d} gh^2 dx.$$

Hence we see that for $\mu \in \mathscr{K}_\infty^S$

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_x\left[e^{A_t^\mu}\right] = h(x)\langle \mu, h\rangle. \tag{4.10}$$

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