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A NOTE ON THE BALLISTIC LIMIT OF RANDOM MOTION IN A RANDOM POTENTIAL

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Abstract
It has been shown that certain types of random walks in random potentials and Brownian motion in Poissonian potentials undergo a phase transition from sub-ballistic to ballistic behavior when the strength of the underlying drift is increased. The ballistic behavior has been manifested by indicating a limiting area for the normalized motion. In the present article, we provide a refined description of this limiting area with a further development for the case of rotation invariant Poissonian potentials.

1 Introduction and preliminaries

In this article, we simultaneously study a discrete and a continuous model of a random motion evolving under the influence of a nonnegative random potential. In the *discrete setting*, we consider a symmetric nearest-neighbor random walk $(S_t)_{t\in\mathbb{N}}$ on $\mathbb{Z}^d$ with start at the origin. The random potential is supposed to be an i.i.d. collection $(V_x)_{x\in\mathbb{Z}^d}$ of nonnegative, nondegenerate random variables with existing dth moment. For convenience, we additionally assume $\operatorname{essinf} V_x = 0$. In the *continuous setting*, we consider a symmetric brownian motion $(S_t)_{t\in\mathbb{R}^+}$ on $\mathbb{R}^d$ with start at the origin. The potential $(V_x)_{x\in\mathbb{R}^d}$ is supposed to be obtained by translating a fixed nonnegative shape function W at the points of a Poissonian cloud $\omega = \sum_i \delta_{x_i}$ on $\mathbb{R}^d$, with fixed intensity $\nu > 0$:

$$V_x(\omega) \stackrel{\text{def}}{=} \sum\nolimits_i W(x - x_i) \quad \text{for} \quad \omega = \sum\nolimits_i \delta_{x_i} ,$$

where ω runs over the set of cloud configurations (i.e. simple pure point Radon measures on $\mathbb{R}^d$) and the nonnegative function W is chosen bounded, measurable, compactly supported and nondegenerate.
We write P for the probability and E for the expectation associated to the random motion. The random potential is supposed to be independent of P. Its probability and expectation are

denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively. Thereby, if not explicitly indicated, we do not distinguish between the discrete and the continuous setting.
The influence of the potential to the random motion is manifested by introducing the so-called *path measures* for finite paths. To any drift $h \in \mathbb{R}^d$ and with ω being a concrete realization of the potential, they are given by

$$dQ^h_{t,\omega} = \frac{1}{Z^h_{t,\omega}} \exp\left(h \cdot S_t - \sum_{k=1}^{t} V_{S_k}(\omega)\right) dP\,, \quad t \in \mathbb{N}\,,$$

in the discrete and

$$dQ^h_{t,\omega} = \frac{1}{Z^h_{t,\omega}} \exp\left(h \cdot S_t - \int_0^t V_{S_u}(\omega)\, du\right) dP\,, \quad t \in \mathbb{R}^+,$$

in the continuous setting. The normalizations $Z^h_{t,\omega}$ are called *partition functions*, and they are given by

$$Z^h_{t,\omega} = E\left[\exp\left(h \cdot S_t - \sum_{k=1}^{t} V_{S_k}(\omega)\right)\right], \quad t \in \mathbb{N}\,,$$

in the discrete and

$$Z^h_{t,\omega} = E\left[\exp\left(h \cdot S_t - \int_0^t V_{S_u}(\omega)\, du\right)\right], \quad t \in \mathbb{R}^+,$$

in the continuous setting. Again, if not explicitly mentioned, we will not distinguish between the discrete and the continuous version.
In this definitions, the path measures and the partition functions are random objects again, depending on the concrete realization of the potential. This approach is called the *quenched setting*. It is also convenient to consider the *annealed setting*, in which the objects are averaged with respect to the environment. The two settings, at least as far as they are studied here, can be treated in a very similar way. In particular, all the proofs we perform are identical for both settings. To avoid repeatings, we thus restrict ourselves to the quenched setting.
A particle moving under the path measures underlies two contrary influences. On the one hand, it is convenient to stay in regions of small potentials and thus to dislocate only slowly. On the other hand, the particle is urged to develop in the direction of the drift h, where the strength of the urge of course increases with the size of h. As a consequence, there is a transition from a *sub-ballistic phase* for small drifts, where the displacement of S_t from the origin is of order $o(t)$, and a *ballistic phase* for large drifts, where the displacement of S_t from the origin grows of order $O(t)$.
This was first established for the continuous setting by Sznitman (see Chapter 5 of [2]) and then adapted to the discrete setting by Zerner and Flury in [3], [1]. In the underlying method, the velocity of dislocation is obtained from a large deviation principle for S_t/t under the path measures $Q^h_{t,\omega}$, of which the rate function is connected to the exponential growth rate of the partition function $Z^h_{t,\omega}$. The transition in the velocity of dislocation then is (partially) brought back to a transition in the growth rate of $Z^h_{t,\omega}$.
More detailed, it is shown that there exists a family $(\alpha_\lambda)_{\lambda \geq 0}$ of (nonrandom) norms on $\mathbb{R}^d$, such that, on set of full $\mathbb{P}$-measure, S_t/t under $Q^0_{t,\omega}$ satisfies a large deviation principle with

convex rate function

$$I(x) \stackrel{\mathrm{def}}{=} \sup_{\lambda\geq 0}\left(\alpha_\lambda(x)-\lambda\right), \qquad x\in\mathbb{R}^d.$$

On the same set of full $\mathbb{P}$-measure, by an application of Varadhan's lemma, one then obtains that

$$\lim_{t\to\infty}\frac{\log Z^h_{t,\omega}}{t} = \sup_{x\in\mathbb{R}^d}\left(h\cdot x - I(x)\right), \tag{1}$$

and that S_t/t under $Q^h_{t,\omega}$ satisfies a large deviation principle with rate function

$$I_h(y) \stackrel{\mathrm{def}}{=} I(y) - h\cdot y + \sup_{x\in\mathbb{R}^d}\left(h\cdot x - I(x)\right).$$

It then is clear that, again on the same set of full $\mathbb{P}$-measure,

$$\mathrm{dist}\left(\frac{S_t}{t},\, M_h\right)\to 0 \quad \text{in } Q^h_{t,\omega} \text{ probability as } t\to\infty, \tag{2}$$

where the limiting area

$$M_h \stackrel{\mathrm{def}}{=} \left\{y\in\mathbb{R}^d : \, h\cdot y - I(y) = \sup_{x\in\mathbb{R}^d}\left(h\cdot x - I(x)\right)\right\}$$

is a compact set.
The norms $(\alpha_\lambda)_{\lambda\geq 0}$ are the so-called *Lyapunov functions* or *Lyapunov norms*. In the discrete setting, they are characterized by

$$\alpha_\lambda(x) = -\lim_{n\to\infty}\frac{1}{n}\log E\left[\exp\left(-\sum_{k=1}^{H_{nx}}\left(\lambda + V_{S_k}(\omega)\right)\right);\, H_{nx}<\infty\right], \tag{3}$$

where $H_{nx} = \inf\{k\in\mathbb{N}_0 : \, S_k = nx\}$ denotes the first hitting time of $nx\in\mathbb{Z}^d$. In the continuous setting, the they can be expressed by

$$\alpha_\lambda(x) = -\lim_{n\to\infty}\frac{1}{n}\log E\left[\exp\left(-\int_0^{H_{nx}}\left(\lambda + V_{S_u}(\omega)\right)du\right);\, H_{nx}<\infty\right], \tag{4}$$

where $H_{nx} = \inf\{u\in\mathbb{R}^+ : \, \|nx - S_u\|\leq 1\}$ denotes the first entrance time to the closed unit ball around $nx\in\mathbb{R}^d$. In both settings, the limits exist $\mathbb{P}$-almost surely and in $L^1(\mathbb{P})$ and the map $\lambda\mapsto\alpha_\lambda(x)$, to any fixed $x\in\mathbb{R}^d\setminus\{0\}$, is continuous, concave and strictly increasing with $\lim_{\lambda\to\infty}\alpha_\lambda(x)=\infty$ (Proposition 4 and page 272 in [3] and Theorem 2.5 and Proposition 2.9 in Chapter 5 of [2]).
The transition from sub-ballistic to ballistic behavior is specified by the dual norms

$$\alpha^*_\lambda(\ell) \stackrel{\mathrm{def}}{=} \sup_{x\neq 0}\left(\frac{\ell\cdot x}{\alpha_\lambda(x)}\right), \quad \ell\in\mathbb{R}^d.$$

It is easily seen that $(\alpha^*_\lambda)_{\lambda\geq 0}$ is indeed a family of norms again. Also, to any fixed $\ell\in\mathbb{R}^d\setminus\{0\}$, the map $\lambda\mapsto 1/\alpha^*_\lambda(\ell)$ inherits the property of being continuous, concave and strictly increasing with $\lim_{\lambda\to\infty} 1/\alpha^*_\lambda(\ell)=\infty$.

Still according to the references mentioned above, we finally have the following characterization of the phase transition:

$$\text{For } \alpha_0^*(h) < 1, \ \sup_{x\in\mathbb{R}^d} (h\cdot x - I(x)) = 0 \text{ and } M_h = \{0\}. \tag{5}$$

$$\text{For } \alpha_0^*(h) > 1, \ \sup_{x\in\mathbb{R}^d} (h\cdot x - I(x)) > 0 \text{ and, as a consequence, } 0 \notin M_h. \tag{6}$$

The aim of this article is to improve the picture of the ballistic phase by the following refined description of the limiting area M_h.

Proposition 1. *For $h \in \mathbb{R}^d$ with $\alpha_0^*(h) > 1$, we have*

$$y \in M_h \quad \Longleftrightarrow \quad \begin{cases} h\cdot y = \alpha_{\lambda(h)}(y)\,, \\ I(y) = \alpha_{\lambda(h)}(y) - \lambda(h)\,, \end{cases} \tag{7}$$

where $\lambda(h) > 0$ is the unique number with $\alpha^*_{\lambda(h)}(h) = 1$.

The condition $h\cdot y = \alpha_{\lambda(h)}(y)$ indicates the possible directions for $S(t)$ to evolve. That is, the sites in M_h lie in the half-lines from the origin to one of the points in which the hyperplane $\{x \in \mathbb{R}^d : \, h{\cdot}x = 1\}$ touches the closed unit ball $\{x \in \mathbb{R}^d : \, \alpha_{\lambda(h)}(x) \le 1\}$, where the existence of such a touching point is guaranteed by the choice of $\lambda(h)$. The condition $I(y) = \alpha_{\lambda(h)}(y) - \lambda(h)$, on the other hand, specifies the possible speed of the dislocation in such a direction (i.e. the possible distance from the origin of $y \in M_h$ in a certain half-line). To make this point more clear, we consider the right and the left derivatives

$$\dot\alpha^+_\lambda(x) \stackrel{\text{def}}{=} \lim_{\varepsilon\downarrow 0} \frac{\alpha_{\lambda+\varepsilon}(x) - \alpha_\lambda(x)}{\varepsilon} \quad \text{and} \quad \dot\alpha^-_\lambda(x) \stackrel{\text{def}}{=} \lim_{\varepsilon\downarrow 0} \frac{\alpha_\lambda(x) - \alpha_{\lambda-\varepsilon}(x)}{\varepsilon}\,,$$

which exist by concavity. By the definition of I, we then have $I(y) = \alpha_{\lambda(h)}(y) - \lambda(h)$ if and only if $\dot\alpha^-_{\lambda(h)}(y) \ge 1 \ge \dot\alpha^+_{\lambda(h)}(y)$, which again, by scalar linearity inherited from $\alpha_{\lambda(h)}$, is equivalent to

$$\dot\alpha^-_{\lambda(h)}(e_y) \ge \frac{1}{\|y\|} \ge \dot\alpha^+_{\lambda(h)}(e_y)\,, \tag{8}$$

where $e_y \stackrel{\text{def}}{=} y/\|y\|$.
Observe also that (7) implies

$$\sup_{x\in\mathbb{R}^d} (h\cdot x - I(x)) = \lambda(h)\,, \tag{9}$$

which, since $\lambda(h) > 0$, then leads to the ballistic characterization in (6).
We now have a closer look at the continuous setting in the case where the shape function W is invariant under rotations, i.e. when

$$W(x) = W'(\|x\|) \quad \forall\, x \in \mathbb{R}^d, \tag{10}$$

for a suitable function W'. In that case, for any rotation φ on $\mathbb{R}^d$, we have

$$V_x(\omega) = V_{\varphi(x)}(\varphi(\omega)) \quad \forall\, x \in \mathbb{R}^d, \tag{11}$$

where $\varphi(\omega) \stackrel{\text{def}}{=} \sum_i \delta_{\varphi(x_i)}$ for $\omega = \sum_i \delta_{x_i}$, and we thus obtain

$$\begin{aligned} &\exp\Big(-\int_0^{H_{nx}} \big(\lambda + V_{S_u}\big)\,du\Big) 1_{\{H_{nx}<\infty\}}\, dP \otimes d\mathbb{P} \\ &\quad = \exp\Big(-\int_0^{H_{nx}} \big(\lambda + V_{\varphi(S_u)}\big)\,du\Big) 1_{\{H_{nx}<\infty\}}\, dP \otimes d\mathbb{P}\varphi^{-1} && (12)\\ &\quad = \exp\Big(-\int_0^{H_{n\varphi(x)}} \big(\lambda + V_{S_u}\big)\,du\Big) 1_{\{H_{n\varphi(x)}<\infty\}}\, dP\varphi^{-1} \otimes d\mathbb{P}\varphi^{-1} && (13)\end{aligned}$$

for any $x \in \mathbb{R}^d$. As a consequence, since $dP\varphi^{-1} = dP$ and $d\mathbb{P}\varphi^{-1} = d\mathbb{P}$, we see that the rotation invariance is transmitted to the Lyapunov norms $(\alpha_\lambda)_{\lambda\geq 0}$ by their representation in (4). That is, to any fixed λ, we have

$$\frac{\alpha_\lambda(x)}{\|x\|} = \text{const.} \quad \forall\, x \in \mathbb{R}^d \setminus \{0\}. \tag{14}$$

In that situation, we find an even simpler description of the limiting area M_h. To do this, we consider the derivatives

$$\partial_h^+ \lambda(h) \stackrel{\text{def}}{=} \lim_{t\downarrow 1} \frac{\lambda(th) - \lambda(h)}{t-1} \quad \text{and} \quad \partial_h^- \lambda(h) \stackrel{\text{def}}{=} \lim_{t\uparrow 1} \frac{\lambda(h) - \lambda(th)}{1-t}\,.$$

The limits exist by convexity, which e.g. is obtained from (9), and $\partial_h^{\pm}\lambda(h) = \nabla\lambda(h)\cdot h$ wherever $\lambda(h)$ is differentiable.

Corollary 2. *In the continuous setting with rotation invariant shape function W, for $h \in \mathbb{R}^d$ with $\alpha_0^*(h) > 1$, we have*

$$M_h = \Big\{ te_h :\, \frac{\partial_h^- \lambda(h)}{\|h\|} \leq t \leq \frac{\partial_h^+ \lambda(h)}{\|h\|} \Big\}, \tag{15}$$

where $e_h = h/\|h\|$.

Observe that the description of the limiting area in (15), by some further rotation invariance arguments, reduces to

$$M_h = \{\nabla\lambda(h)\} \tag{16}$$

wherever the gradient $\nabla\lambda(h)$ exists (i.e. where $\partial_h^+\lambda(h) = \partial_h^-\lambda(h)$).

Proof of Corollary 2. By the definition of $\alpha^*_{\lambda(h)}$ and the rotation property in (14), we have

$$\alpha^*_{\lambda(h)}(h) = \frac{h\cdot y}{\alpha_{\lambda(h)}(y)} \quad\Longleftrightarrow\quad y \in \{th : t \geq 0\}\,, \tag{17}$$

which, since $\alpha^*_{\lambda(h)}(h) = 1$, is equivalent to

$$\alpha_{\lambda(h)}(y) = h\cdot y \quad\Longleftrightarrow\quad y \in \{th : t \geq 0\}\,. \tag{18}$$

From Proposition 1 and in view of (8), we thus obtain

$$M_h = \Big\{ th :\, 1/\dot\alpha^-_{\lambda(h)}(h) \leq t \leq 1/\dot\alpha^+_{\lambda(h)}(h) \Big\}\,. \tag{19}$$

Moreover, since (18) in particular implies $\|h\|^2 = \alpha_{\lambda(h)}(h)$, we have

$$2\|h\|^2 = \nabla\big(\|h\|^2\big)\cdot h = \nabla\alpha_{\lambda(h)}(h)\cdot h = \alpha_{\lambda(h)}(h) + \dot\alpha^{\pm}_{\lambda(h)}(h)\,\partial^{\pm}_h\lambda(h)\,, \tag{20}$$

and therefore

$$\partial^{\pm}_h\lambda(h) = \frac{2\|h\|^2 - \alpha_{\lambda(h)}(h)}{\dot\alpha^{\pm}_{\lambda(h)}(h)} = \frac{\|h\|^2}{\dot\alpha^{\pm}_{\lambda(h)}(h)}\,, \tag{21}$$

which completes the proof. □

2 Proof of the main result

Let $D \overset{\text{def}}{=} \big\{x\in\mathbb{R}^d : I(x)<\infty\big\}$ denote the effective domain of I. In the continuous setting, as is shown in Lemma 4.1 in Chapter 5 of [2], we have $D=\mathbb{R}^d$. In the discrete setting, on the other hand, D is bounded and equals the closed unit ball of the 1-norm (see e.g. equation (66) in [3]). Thereby, I (as a convex rate function) is continuous when restricted to D.
In order to prove Proposition 1, the crucial point is to evaluate the derivatives

$$\partial^+_x I(x) \overset{\text{def}}{=} \lim_{t\downarrow 1}\frac{I(tx)-I(x)}{t-1}\,,\quad x\in D^\circ,$$
$$\partial^-_x I(x) \overset{\text{def}}{=} \lim_{t\uparrow 1}\frac{I(x)-I(tx)}{1-t}\,,\quad x\in D\setminus\{0\}\,,$$

where the existence of the limits is guaranteed by convexity. To this purpose, we introduce

$$\lambda^+_x \overset{\text{def}}{=} \inf\{\lambda\ge 0 : \dot\alpha^+_\lambda(x)<1\}\,,$$
$$\lambda^-_x \overset{\text{def}}{=} \sup\{\lambda> 0 : \dot\alpha^-_\lambda(x)>1\}\vee 0\,.$$

Observe that λ^+_x (respectively λ^-_x) is the largest (respectively smallest) transition point from nondecreasing to nonincreasing behavior for the map $\lambda\mapsto\alpha_\lambda(x)-\lambda$, and that consequently

$$\lambda^+_x = \max\{\lambda\ge 0 : I(x)=\alpha_\lambda(x)-\lambda\}\,, \tag{22}$$
$$\lambda^-_x = \min\{\lambda\ge 0 : I(x)=\alpha_\lambda(x)-\lambda\}\,, \tag{23}$$

for all $x\in D^\circ$.

Lemma 3. *We have*

$$\partial^+_x I(x) = \alpha_{\lambda^+_x}(x)\,,\quad x\in D^\circ, \tag{24}$$
$$\partial^-_x I(x) = \alpha_{\lambda^-_x}(x)\,,\quad x\in D\setminus\{0\}\,, \tag{25}$$

where $\alpha_\infty(x) \overset{\text{def}}{=} \lim_{\lambda\to\infty}\alpha_\lambda(x)=\infty$ *for* $x\neq 0$.

Proof. We start with (24). From the characterization of λ^+_x in (22), we obtain

$$\begin{aligned}\partial^+_x I(x) &= \lim_{t\downarrow 1}\frac{\alpha_{\lambda^+_{tx}}(tx)-\lambda^+_{tx}-\alpha_{\lambda^+_x}(x)+\lambda^+_x}{t-1}\\ &= \lim_{t\downarrow 1}\frac{\alpha_{\lambda^+_{tx}}(tx)-\alpha_{\lambda^+_{tx}}(x)}{t-1} - \lim_{t\downarrow 1}\frac{\lambda^+_{tx}-\lambda^+_x}{t-1} - \frac{\alpha_{\lambda^+_{tx}}(x)-\alpha_{\lambda^+_x}(x)}{t-1}\,.\end{aligned} \tag{26}$$

It is easily seen that $\lim_{t\downarrow 1}\lambda^+_{tx}=\lambda^+_x$ and

$$\lim_{t\downarrow 1}\frac{\alpha_{\lambda^+_{tx}}(tx)-\alpha_{\lambda^+_{tx}}(x)}{t-1}=\lim_{t\downarrow 1}\alpha_{\lambda^+_{tx}}(x)=\alpha_{\lambda^+_x}(x)\,. \tag{27}$$

What remains to show is that the second limit in (26) is zero. To this end, observe that the concavity of $\alpha_\lambda(x)$ in λ implies

$$\dot\alpha^+_{\lambda^+_x}(x)\big(\lambda^+_{tx}-\lambda^+_x\big)\ge\alpha_{\lambda^+_{tx}}(x)-\alpha_{\lambda^+_x}(x)\ge\dot\alpha^-_{\lambda^+_{tx}}(x)\big(\lambda^+_{tx}-\lambda^+_x\big)\,. \tag{28}$$

Observe also that $1\ge\dot\alpha^+_{\lambda^+_x}(x)$ and, since either $\lambda^-_{tx}=\lambda^+_{tx}$ (and thus $\dot\alpha^-_{\lambda^+_{tx}}(tx)=\dot\alpha^-_{\lambda^-_{tx}}(tx)\ge 1$) or $\lambda^-_{tx}<\lambda^+_{tx}$ and $\dot\alpha^-_\lambda(tx)=1$ for $\lambda^-_{tx}<\lambda\le\lambda^+_{tx}$,

$$\dot\alpha^-_{\lambda^+_{tx}}(x)=\frac{1}{t}\,\dot\alpha^-_{\lambda^+_{tx}}(tx)\ge\frac{1}{t}\,. \tag{29}$$

We therefore have

$$0\le\frac{\lambda^+_{tx}-\lambda^+_x}{t-1}-\frac{\alpha_{\lambda^+_{tx}}(x)-\alpha_{\lambda^+_x}(x)}{t-1}\le\big(\lambda^+_{tx}-\lambda^+_x\big)\,\frac{1-\frac{1}{t}}{t-1}\,, \tag{30}$$

where

$$\lim_{t\downarrow 1}\big(\lambda^+_{tx}-\lambda^+_x\big)\,\frac{1-\frac{1}{t}}{t-1}=\lim_{t\downarrow 1}\frac{\lambda^+_{tx}-\lambda^+_x}{t}=0\,. \tag{31}$$

That is, the second limit in (26) is zero, which completes the proof of (24).
It remains to show (25), which (by the convexity of I) though easily follows from (24) by

$$\partial^-_x I(x)=\lim_{t\uparrow 1}\partial^+_{tx}I(tx)=\lim_{t\uparrow 1}\alpha_{\lambda^+_{tx}}(tx)=\alpha_{\lambda^-_x}(x)\,, \tag{32}$$

where at the last step, we used $\lim_{t\uparrow 1}\lambda^+_{tx}=\lambda^-_x$. □

Before stepping to the proof of Proposition 1, we have a closer look at the limiting behavior of I on its effective domain D. In the continuous setting, we have

$$\lim_{s\to\infty}\frac{\partial^-_{sx}I(sx)}{s}=\infty \tag{33}$$

for all $x\in\mathbb{R}^d\setminus\{0\}$ as follows from Lemma 4.1 in Chapter 5 of [2]. In the discrete setting, the analogous behavior is not a priori guaranteed. There, in view of (25), it remains the possibility that

$$\lim_{t\uparrow 1}\frac{\partial^-_{ty}I(ty)}{t}=\partial^-_y I(y)<\infty \tag{34}$$

for $y\in\partial D$ if and only if $\lambda^-_y<\infty$ and

$$\dot\alpha^-_\lambda(y)=1\quad\forall\lambda>\lambda^-_y\,. \tag{35}$$

Proof of Proposition 1. The existence and uniqueness of $\lambda(h)$, as well as the property $\lambda(h) > 0$, are plain by the assumption $\alpha_0^*(h) > 1$ and the above mentioned properties of $\lambda \mapsto 1/\alpha_\lambda^*(\ell)$. It thus remains to show (7).

By scalar linearity and continuity, there exists e with $\|e\| = 1$, such that

$$1 = \alpha^*_{\lambda(h)}(h) = \sup_{x:\, \|x\|=1} \left(\frac{h \cdot x}{\alpha_{\lambda(h)}(x)} \right) = \frac{h \cdot e}{\alpha_{\lambda(h)}(e)} \,. \tag{36}$$

Since $\lambda \mapsto \alpha_\lambda(e)$ is strictly increasing, the right derivative $\dot\alpha^+_{\lambda(h)}(e)$ is strictly positive. We thus can set $z = e/\dot\alpha^+_{\lambda(h)}(e)$. By (36) and scalar linearity, we have

$$h \cdot z = \alpha_{\lambda(h)}(z) \,. \tag{37}$$

In addition, since $\lambda \mapsto \alpha_\lambda(z)$ is concave and $\dot\alpha^+_{\lambda(h)}(z) = 1$, the map $\lambda \mapsto \alpha_\lambda(z) - \lambda$ is nondecreasing for $\lambda \le \lambda(h)$ and nonincreasing for $\lambda > \lambda(h)$, such that

$$I(z) = \alpha_{\lambda(h)}(z) - \lambda(h) \,. \tag{38}$$

From (37) and (38), we thus obtain

$$\lambda(h) = h \cdot z - I(z) \le \sup_{x \in \mathbb{R}^d} \left(h \cdot x - I(x) \right) . \tag{39}$$

Suppose now $y \in M_h$ and therefore $y \ne 0$. If $y \in D^\circ$, we necessarily have

$$\partial_y^- I(y) \le h \cdot y \le \partial_y^+ I(y) \,. \tag{40}$$

Lemma 3 then implies that, by continuity, there exists $\lambda \in [\lambda_y^-, \lambda_y^+]$, such that

$$\alpha_\lambda(y) = h \cdot y \,. \tag{41}$$

If we are in the discrete setting and $y \in \partial D$, which is possible if $\partial_y^- I(y) < \infty$ only, there also exists $\lambda \in [\lambda_y^-, \infty)$ such that (41) is true. Moreover, from the bounds of λ, respectively from (35) if $y \in \partial D$ in the discrete setting, we obtain

$$I(y) = \alpha_\lambda(y) - \lambda \,. \tag{42}$$

Now, since $\lambda \mapsto \alpha_\lambda^*(h)$ is nonincreasing, (41) implies $\lambda \le \lambda(h)$. After all, we thus have shown

$$\lambda(h) \overset{(39)}{\le} \sup_{x \in \mathbb{R}^d} \left(h \cdot x - I(x) \right) \;=\; h \cdot y - I(y) \overset{(41)(42)}{=} \lambda \;\le\; \lambda(h) \,, \tag{43}$$

such that the left-to-right implication of (7) now is given by (41), (42) and $\lambda = \lambda(h)$, and the right-to-left implication follows from $\lambda(h) = \sup_{x \in \mathbb{R}^d} (h \cdot x - I(x))$. □

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