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A RELATION BETWEEN DIMENSION OF THE HARMONIC MEASURE, ENTROPY AND DRIFT FOR A RANDOM WALK ON A HYPERBOLIC SPACE

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Abstract

We establish in this paper an exact formula which links the dimension of the harmonic measure, the asymptotic entropy and the rate of escape for a random walk on a discrete subgroup of the isometry group of a Gromov hyperbolic space. This completes a result of [11], where only an upper bound for the dimension was proved.

Introduction

Let Γ be a discrete subgroup of the isometry group of a Gromov hyperbolic space X and μ a probability measure on Γ. The associated (right) random walk is the processus $(x_n)_{n\geq 0}$ defined by: $x_0 = e$ and, for $n \geq 1$, $x_n = h_1 \cdots h_n$, where $(h_i)_{i>0}$ is a sequence of independent random variables which take their values in Γ, with distribution μ. Under some additional hypotheses, a point o in X being fixed, the trajectory $(x_n o)_{n\geq 0}$ converges almost surely to a point x_∞ in the geometric boundary ∂X of X. The distribution ν of x_∞ is called the harmonic measure associated with the random walk.

The boundary ∂X is equipped with a family of metrics d_a, where a is a parameter. The aim of this paper is to established the following formula, which links the dimension of ν with respect to d_a, the asymptotic entropy $h(\mu)$ and the rate of escape $l(\mu)$ of the random walk (see Theorem 3.1 for a precise statement):

$$\dim\ \nu \ = \ \frac{1}{\log a}\ \frac{h(\mu)}{l(\mu)}\ . \tag{0.1}$$

This result completes the result obtained in [11], where only the bound from above in (0.1) was proved. The main goal of the above mentioned paper was to show that the harmonic measure could be singular with respect to the Hausdorff measure; we refer to it for additional comments on the question of the dimension of the harmonic measure.

The first result on this subject was obtained in [10] for a random walk on $SL(2,\mathbb{R})$ (which acts by isometries on the hyperbolic disc); a similar result was established in the case where X is a tree in [8]. In our context of a hyperbolic space, and under the assumptions that μ is symmetric and finitely supported, the following result on the pointwise dimension is established in [2]:

$$\lim_{r\to 0}\frac{\log \nu B_a(\xi,r)}{\log r} = \frac{1}{\log a}\,\frac{h(\mu)}{l(\mu)} \quad \nu - a.s. \tag{0.2}$$

The dimension involved in our main theorem (3.1) is the box dimension and in this sense our result is weaker than (0.2). Indeed when $\log \nu B_a(\xi,r)/\log r$ converges a.s. to a constant, all reasonable notions of dimension coincide (see [13] or [12]). The interest of our result however lies in the fact that our hypotheses are the weakest possible - i.e. the weakest under which (0.1) makes sense: we just assume that μ admits a finite first moment.

The paper is organized as follows. In Section 1 we recall briefly the needed notions on random walks, hyperbolic spaces and dimensions of measures. We prove also in this first section a covering result on ∂X which is necessary in order to compare the dimensions. In Section 2 we establish the lower bound for the dimension; which allows us to state our main result in Section 3.

1 Preliminaries and notations

1.1 Hyperbolic space

The notion of Gromov hyperbolic space goes back to [7]. We refer to [4] and [6] for the definitions and properties that we state in this paragraph.

Definition

Let (X,d) be a Gromov hyperbolic space. We denote by $B(x,R)$ the open ball with center x and radius R in (X,d). Recall that the Gromov product (w.r.t. a point $w\in X$) on X^2 is defined by:

$$(x|y)_w = \frac{1}{2}[d(w,x)+d(w,y)-d(x,y)]\ . \tag{1.1}$$

The fact that (X,d) is δ-hyperbolic, for $\delta \geq 0$, is characterized by the fact that for every $w\in X$, for every $(x,y,z)\in X^3$,

$$(x|z)_w \geq \min\{(x|y)_w,(y|z)_w\} - \delta\ . \tag{1.2}$$

We assume moreover that (X,d) is proper and geodesic; we fix a base point o in X and denote by $(.|.)$ the Gromov product with respect to o.

Boundary

The space X is compactified by its geometric boundary ∂X. The action of the isometry group on X extends to a continuous action on the boundary. We also denote by $(.|.)$ the extension of the Gromov product to ∂X; on $X\cup\partial X$, (1.2) becomes:

$$(x|z)_w \geq \min\{(x|y)_w,(y|z)_w\} - 2\delta\ . \tag{1.3}$$

Two distinct points of $X \cup \partial X$ can be joined by a geodesic. We shall also use the fact that for $x, y \in X \cup \partial X$, if $(x_n)_{n\geq 0}$ converges to x and $(y_n)_{n\geq 0}$ converges to y, then

$$(x|y) - 2\delta \leq \liminf_{n\to\infty} (x_n|y_n) \leq (x|y) \ . \tag{1.4}$$

On this boundary a family of metrics d_a, $1 < a < a_0$, is defined, which have the following property: there exists a constant $\lambda > 1$ such that $\forall x, y \in \partial X$

$$\lambda^{-1} a^{-(x|y)} \leq d_a(x, y) \leq \lambda a^{-(x|y)} \ . \tag{1.5}$$

We denote by $B_a(\xi, r)$ the open balls in $(\partial X, d_a)$; and, for $r > 0$, $\log_a r = \frac{\log r}{\log a}$.

Discrete subgroups

Let Γ be a group of isometries acting properly discontinuously on X. In particular, each orbit of Γ in X is discrete. Denote by L_Γ the limit set of Γ. The group Γ is said to be elementary if L_Γ consists of at most two elements. If G is non-elementary, L_Γ is uncountable. The exponential growth rate of Γ (in X) is:

$$v(\Gamma) = \limsup_{R\to\infty} \frac{\log \ Card \ \{g \in \Gamma \mid d(o, go) \leq R\}}{R} \ .$$

Another property we shall need is quasi-convex-cocompactness of Γ, which admits the following characterization ([3]): there exists $C > 0$ such that every geodesic radius σ joining o to a point in L_Γ stay at a distance lower than C from Γo.
Note that if $v(\Gamma)$ is finite and in addition Γ is quasi-convex-cocompact, then the Hausdorff dimension (with respect to the metric d_a) of L_Γ is equal to $v(\Gamma)/\log a$ (see [3]).
As important examples of quasi-convex-cocompact discrete groups, let us quote finitely generated hyperbolic groups acting on their Cayley graph (w.r.t. a finite set of generators) equipped with the word-length metric, or discrete convex cocompact groups of isometries acting on Riemaniann manifold with pinched sectional curvature.

1.2 Random walk

Definition

Given a countable group Γ and a probability measure μ on Γ, the random walk associated to μ is defined as the homogeneous Markov processus on Γ with initial state e and transition probability $p(g, h) = \mu(g^{-1}h)$. Given a sequence $(h_n)_{n\geq 1}$ of independent random variables with law μ, one defines a realization $\boldsymbol{x} = (x_n)_{n\geq 0}$ of this processus by $x_0 = e$ and for $n > 0$, $x_n = h_1 \cdots h_n$. The sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ is called the trajectory of the random walk. We denote by $\mathbb{P}$ the law of this trajectory in $\Gamma^{\mathbb{N}}$. The distribution of the position x_n of the trajectory at time n is the convolution product $\mu^n = \mu * \cdots * \mu$, where μ appears n times.

Asymptotic entropy

Recall that the entropy of a probability measure m with countable support is defined by:

$$H(m) = \sum_g - m(g) \log(m(g)) \ .$$

The sequence $(H(\mu^n))_n$ is subadditive so, assuming that the entropy of μ is finite, one can define the asymptotic entropy ([1]) of the random walk by

$$h(\mu) = \lim_{n\to+\infty} \frac{1}{n} H(\mu^n) \ . \tag{1.6}$$

Noting that

$$H(\mu^n) = -\int \log(\mu^n(x_n)) d\mathbb{P} \ ,$$

and using Kingman subadditive ergodic theorem, one gets ([5]) a $\mathbb{P}$-almost sure convergence:

$$-\frac{1}{n}\log(\mu^n(x_n)) \longrightarrow h(\mu) \ . \tag{1.7}$$

Rate of escape

Consider now a group Γ acting by isometries on the metric space (X, d). Assume that μ admits a finite first moment, that is to say

$$\sum_{g\in\Gamma} \mu(g) \ d(o, go) < \infty \ .$$

The rate of escape (also called drift) $l(\mu)$ of the random walk is defined, using once again a subadditivity property, by

$$l(\mu) = \lim_{n\to+\infty} \frac{1}{n} d(o, x_n o) \quad \mathbb{P} - a.s. \ . \tag{1.8}$$

Note that under the assumption that $v(\Gamma) < +\infty$, if μ admits a finite first moment, then it has also a finite entropy (see [9]), and so the asymptotic entropy of the random walk is well defined.

> **From now on, and unless otherwise specified,** Γ is a non-elementary group acting by isometries on a proper geodesic δ-hyperbolic space (X, d). We fix a metric d_a on the boundary. We assume that Γ acts properly discontinuously, is quasi-convex-cocompact, and that its exponential growth rate is finite. Moreover μ is a fixed probability measure on Γ, such that its support generates Γ as a semigroup and which admits a finite first moment. We consider the random walk $(x_n)_{n\geq 0}$ associated with μ ; and we adopt the above notations.

Asymptotic behaviour of the random walk

Under the above hypotheses, the behaviour of the paths of the random walk is described by the following:

Theorem 1.1. *([9]) The trajectory $(x_n o)_{n\geq 0}$ converges $\mathbb{P}$-almost surely to an element x_∞ in ∂X. More precisely, for $\mathbb{P}$-almost every trajectory $(x_n o)_{n\geq 0}$, there exists a geodesic radius σ such that*

$$d(x_n o, \sigma(l(\mu) n)) = o(n) \ .$$

We denote by bnd the map (defined on a set of $\mathbb{P}$-measure 1) from $\Gamma^{\mathbb{N}}$ to ∂X which associates x_∞ to $\boldsymbol{x} = (x_n)_{n\geq 0}$, and by ν the distribution of x_∞, which we call the harmonic measure. In particular we have $\nu = bnd(\mathbb{P})$.
Note that under these hypotheses, we have $l(\mu) > 0$; as well as $h(\mu) > 0$ (see [9]).
The support of ν is clearly included in the limit set L_Γ. It can be seen, using the minimality of L_Γ and the fact that the measure ν is μ-stationary, that the support of ν is in fact equal to L_Γ (see e.g. [11]).

1.3 Dimension of measures

The Hausdorff dimension of a probability measure m on ∂X is defined by

$$\dim_H m = \inf\{\dim_H A \mid m(A) = 1\} \ .$$

Note that this dimension is an invariant of the type of the measure m. Now let us define other notions of dimension of a measure we shall use and explore the relations between them (see [12] for an extensive presentation of this subject).

The first is the so-called box dimension. We denote by $N(A, r)$ the minimal number of balls of radius r needed to cover a set $A \subset \partial X$. The lower and upper box dimensions of A are defined respectively as

$$\underline{\dim}_B A = \liminf_{r\to 0} \frac{\log N(A,r)}{\log 1/r} \quad \text{and} \quad \overline{\dim}_B A = \limsup_{r\to 0} \frac{\log N(A,r)}{\log 1/r} \ ,$$

and the lower upper box dimensions of a probability measure m as

$$\underline{\dim}_B m = \lim_{\eta\to 0} \inf\{\underline{\dim}_B A \mid m(A) \geq 1-\eta\}$$
$$\overline{\dim}_B m = \lim_{\eta\to 0} \inf\{\overline{\dim}_B A \mid m(A) \geq 1-\eta\} \ .$$

When the upper and lower box dimensions coincide, we denote it by $\dim_B m$.

We focus now on the pointwise dimension. The lower and upper pointwise dimensions are defined for every ξ in X respectively by:

$$\underline{\dim}_P m(\xi) = \liminf_{r\to 0} \frac{\log mB_a(\xi,r)}{\log r} \quad \text{and} \quad \overline{\dim}_P m(\xi) = \limsup_{r\to 0} \frac{\log mB_a(\xi,r)}{\log r} \ ,$$

($\dim_P m(\xi)$ if both coincide).

In order to compare the box and pointwise dimensions, we shall need a covering property on the limit set. Recall that a cover of a set A is said to have finite multiplicity M if each point of A lies in at most M elements of the cover.
For a point $\xi \in \partial X$, we denote by σ_ξ a (unit speed) geodesic ray joining o to ξ. For the rest of this paragraph, we omit to precise that the properties are satisfied for r small enough. The following lemma establishes a link between shadows and balls on ∂X.

Lemma 1.2. *For ξ, ξ' in ∂X; for every $R_1 \in \mathbb{R}$, $R_2 > 0$, if*

$$d\left(\sigma_\xi(-\log_a r + R_1), \sigma_{\xi'}(-\log_a r + R_1)\right) < R_2 \ ,$$

then we have

$$d_a(\xi,\xi') < \ r\times\lambda a^{-R_1+R_2/2+4\delta} \ .$$

Conversely assume that $d_a(\xi,\xi') < r$*; then, for every* $R_1\in\mathbb{R}$*,*

$$d\left(\sigma_\xi(-\log_a r+R_1),\sigma_{\xi_i}(-\log_a r+R_1)\right) < 2\log_a\lambda+4|R_1|+8\delta \ .$$

Proof. If $d\left(\sigma_\xi(-\log_a r+R_1),\sigma_{\xi'}(-\log_a r+R_1)\right) < R_2$, then by (1.1),

$$(\sigma_\xi(-\log_a r+R_1)|\sigma_{\xi'}(-\log_a r+R_1)) > -\log_a r+R_1-R_2/2 \ .$$

Besides, by (1.4), we have

$$(\xi|\sigma_\xi(-\log_a r+R_1)) \geq \liminf_{n\to+\infty}(\sigma_\xi(n)|\sigma_\xi(-\log_a r+R_1)) = -\log_a r+R_1 \ ,$$

as well as $(\sigma_{\xi'}(-\log_a r+R_1)|\xi') \geq -\log_a r+R_1$. Using (1.3), this implies

$$(\xi|\xi') > -\log_a r+R_1-R_2/2-4\delta \ ,$$

and the result follows from (1.5).
Conversely, if $d_a(\xi,\xi') < r$, we have $\lambda^{-1}a^{-(\xi|\xi')} < r$, and so $(\xi|\xi') > -\log_a r-\log_a\lambda$. Using (1.3), this implies, for $R_1\in\mathbb{R}$,

$$(\sigma_\xi(-\log_a r+R_1)|\sigma_{\xi'}(-\log_a r+R_1)) > -\log_a r-\log_a\lambda-|R_1|-4\delta \ ,$$

and so

$$\begin{aligned} d\left(\sigma_\xi(-\log_a r+R_1),\sigma_{\xi'}(-\log_a r+R_1)\right) < {} & 2\left(-\log_a r+R_1\right) \\ & -2\left(-\log_a r-\log_a\lambda-|R_1|-4\delta\right) \ , \end{aligned}$$

which gives the result. □

Proposition 1.3. *There exists a constant* $M>0$ *such that every subset* A *of* L_Γ *admits, for every* $r>0$*, a covering by balls centered in* A*, of radius* r*, with multiplicity* M*.*

Proof. Let A be a subset of L_Γ and $r>0$. We set

$$T = -\log_a r+\log_a\lambda+C+4\delta \ ,$$

where the constant C comes from the caracterization of quasi-convex-cocompactness (see par. 1.1). Since Γ is quasi-convex-cocompact, for each $\xi\in A$, there exists $g_{r,\xi}\in\Gamma$ such that

$$d\left(\sigma_\xi(T),g_{r,\xi}o\right) \leq C \ .$$

Denote by $\{g_i \ , \ i\in I\}$ the set of $g_{r,\xi}$, $\xi\in A$, and for each $i\in I$ choose $\xi_i\in A$ such that $g_{r,\xi_i}=g_i$.
Now, for a $\xi\in A$, take $i\in I$ such that $g_{r,\xi}=g_i$. We have

$$d\left(\sigma_{\xi_i}(T),\sigma_\xi(T)\right) \leq 2C \ .$$

So by the use Lemma 1.2,

$$d_a(\xi,\xi_i) < r\times\lambda a^{-(\log_a\lambda+C+4\delta)+2C/2+4\delta} = r \ ,$$

and $\xi \in B_a(\xi_i, r)$. Therefore $\{B_a(\xi_i, r), i \in I\}$, is a covering of A.
Now let us prove that this covering has finite multiplicity (not depending on r).
For $\xi \in \partial X$, let be $i \in I$ such that $\xi \in B_a(\xi_i, r)$. We have, by the second part of Lemma 1.2,

$$d\left(\sigma_\xi(T), \sigma_{\xi_i}(T)\right) < 2\log_a \lambda + 4|\log_a \lambda + C + 4\delta| + 8\delta ,$$

and so

$$d\left(\sigma_\xi(T), g_i\right) \leq d\left(\sigma_\xi(T), \sigma_{\xi_i}(T)\right) + d\left(\sigma_{\xi_i}(T), g_i\right) < D ,$$

where D is a constant (i.e. does not depend on r but only on C, δ and a). But the number of elements in Γ that are at a distance lower than D from $\sigma_\xi(T)$ is bounded by a constant M, so that the covering just constructed has finite multiplicity M. $\square$

This covering property of the boundary has the following consequence (see [12]):

Proposition 1.4. *Assume that for a probability measure m on L_Γ, there exists d such that $\overline{\dim}_P m(\xi) \leq d$ for m-almost every ξ. Then $\overline{\dim}_B m \leq d$.*

We have also (see [12]) $\dim_H m \leq \underline{\dim}_B m$.

2 Lower bound for the dimension

Proposition 2.1. *The harmonic measure ν associated to the random walk satisfies*

$$\underline{\dim}_B \nu \geq \frac{1}{\log a} \frac{h(\mu)}{l(\mu)} .$$

In this section, in order to simplify the notations, we write d for the dimension $\underline{\dim}_B \nu$ and l and h for respectively $l(\mu)$ and $h(\mu)$. We first need the following lemma, which is a direct consequence of the definition of this dimension.

Lemma 2.2. *For every $\eta > 0$, there exists a sequence $(r_k)_{k>0}$ which converges to zero and such that for every $k > 0$, there exists a set $E_{\eta,k} \subset \partial X$ which satisfies:*

$$\log \ Card \ E_{\eta,k} \leq (d + 1/k) \log(1/r_k) ,$$

and

$$\nu\left(\bigcup_{\xi \in E_{\eta,k}} B_a(\xi, r_k)\right) \geq 1 - \eta .$$

Now let us prove Proposition 2.1. Fix $\epsilon > 0$, $\eta > 0$ and set

$$\Omega_\epsilon^n = \left\{\boldsymbol{x} \mid \mu^n(x_n) \leq e^{-n(h-\epsilon)} , \ d\left(\sigma_{bnd(\boldsymbol{x})}(nl), x_n o\right) \leq n\epsilon\right\} ,$$

where $\sigma_{bnd(\boldsymbol{x})}$ denotes a geodesic joining o to $x_\infty = bnd(\boldsymbol{x})$. In view of the behaviour of the random walk (Theorem 1.1 and (1.7)), there exists $N_{\epsilon,\eta}$, such that for every $n \geq N_{\epsilon,\eta}$, $\mathbb{P}(\Omega_\epsilon^n) \geq 1 - \eta$. Then we take $(r_k)_{k>0}$ as in Lemma 2.2, set $n_k = [(-\log_a r_k)/l]$ (integer part) and, with the notations of this lemma,

$$\Pi_{\epsilon,\eta}^k = \Omega_\epsilon^{n_k} \bigcap bnd^{-1}\left(\bigcup_{\xi \in E_{\eta,k}} B_a(\xi, r_k)\right) .$$

We immediately get, for k such that $n_k \geq N_{\epsilon,\eta}$, $\mathbb{P}(\Pi^k_{\epsilon,\eta}) \geq 1-2\eta$, and

$$\mathbb{P}\left(\Pi^k_{\epsilon,\eta}\right) \leq \sum_{\xi \in E_{\eta,k}} \mathbb{P}\left(\Omega^{n_k}_\epsilon \cap bnd^{-1}\left(B_a(\xi, r_k)\right)\right) \ . \tag{2.1}$$

Let us fix $\xi \in E_{\eta,k}$ and assume that $\boldsymbol{x} \in \Omega^{n_k}_\epsilon \cap bnd^{-1}(B_a(\xi, r_k))$. We have $d(bnd(\boldsymbol{x}), \xi) < r_k$, so that, by Lemma 1.2,

$$d\left(\sigma_{bnd(\boldsymbol{x})}(-\log_a r_k), \sigma_\xi(-\log_a r_k)\right) \leq 2\log_a \lambda + 8\delta \ .$$

Besides

$$d\left(\sigma_{bnd(\boldsymbol{x})}(n_k l), x_{n_k} o\right) \leq n_k \epsilon \ ,$$

and, since $|n_k l - (-\log_a r_k)| \leq l$, this yields

$$d\left(\sigma_\xi(n_k l), x_{n_k} o\right) \leq n_k \epsilon + 2l + 2\log_a \lambda + 8\delta \ ,$$

and $d\left(\sigma_\xi(n_k l), x_{n_k} o\right) \leq 2n_k\epsilon$ for k big enough.
So we get, for a $\xi \in E_{\eta,k}$,

$$\mathbb{P}\left(\Omega^{n_k}_\epsilon \cap bnd^{-1}\left(B_a(\xi, r_k)\right)\right) \leq \mathbb{P}\left\{\boldsymbol{x} \mid x_{n_k} \in B\left(\sigma_\xi(n_k l), 2n_k\epsilon\right) \ , \ \mu^{n_k}(x_{n_k}) \leq e^{-n_k(h-\epsilon)}\right\} \ ,$$

and since the distribution of x_{n_k} under $\mathbb{P}$ is μ^{n_k}, this implies

$$\begin{aligned}\mathbb{P}\left(\Omega^{n_k}_\epsilon \cap bnd^{-1}\left(B_a(\xi, r_k)\right)\right) &\leq \mu^{n_k}\left\{g \in \Gamma \mid g \in B\left(\sigma_\xi(n_k l), 2n_k\epsilon\right) \ , \ \mu^{n_k}(g) \leq e^{-n_k(h-\epsilon)}\right\} \\ &\leq Card\ B\left(\sigma_\xi(n_k l), 2n_k\epsilon\right) \times e^{-n_k(h-\epsilon)} \ .\end{aligned}$$

Now we use the fact that, since $v(\Gamma) < +\infty$, there exists a constant $V > 0$ such that for every $x \in X$ and every $R > 0$,

$$Card\left\{g \in \Gamma \mid d(x, go) \leq R\right\} \leq V^R \ .$$

So the above inequality, (2.1) and the fact that $Card\ E_{\eta,k} \leq (1/r_k)^{d+1/k}$ give

$$1 - 2\eta \leq \mathbb{P}\left(\Pi^k_{\epsilon,\eta}\right) \leq \left(\frac{1}{r_k}\right)^{d+1/k} V^{2n_k\epsilon}\ e^{-n_k(h-\epsilon)} \ ,$$

therefore

$$\frac{\log(1-2\eta)}{n_k} \leq \frac{(\log_a r_k)/l}{n_k}\ l(d+1/k) + \frac{2n_k\epsilon}{n_k}\log V - (h-\epsilon) \ .$$

Finaly we note that $\lim_{k\to\infty} \frac{(\log_a r_k)/l}{n_k} = 1$; so, doing $k \to \infty$, and since ϵ and η are arbitrary, this achieves the proof.

3 Conclusion

It was shown in [11] that in our context, the harmonic measure satisfies:

$$\overline{\dim}_P \nu(\xi) \leq \frac{1}{\log a}\ \frac{h(\mu)}{l(\mu)} \quad \nu - a.s. \ .$$

So, taking into account the result of Section 2 and Proposition 1.4, we get the following

Theorem 3.1. *Under our hypotheses, the harmonic measure satisfies:*

$$\dim_B \nu = \frac{1}{\log a} \frac{h(\mu)}{l(\mu)} .$$

Remark 3.2. *Under our hypotheses, it is known that* $h(\mu) > 0$ *(see [9]); so as a consequence of Theorem 3.1 we have* $\dim_B \nu > 0$.

References

[1] A. Avez, *Entropie des groupes de type fini*, C. R. Acad. Sci. Paris, Sér. A, vol. 275 (1972), p 1363-1366.

[2] S. Blachère, P. Haïssinsky, and P. Mathieu, *Harmonic measures versus quasiconformal measures for hyperbolic groups*, preprint (2007).

[3] M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific Journal of Mathematics, vol.159 (1993), p 241-270. MR1214072

[4] M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie et théorie des groupes : les groupes hyperboliques de Gromov*, Lecture Notes in Math. 1441, Springer, 1990. MR1075994

[5] Y. Derriennic, *Quelques applications du théorème ergodique sous-additif*, Asterisque 74 (1980), p 183-201. MR0588163

[6] E. Ghys, P. De La Harpe (eds.), *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Birkhäuser, Basel, 1990. MR1086648

[7] M. Gromov, *Hyperbolic groups*, Essays in Group Theory (S.M. Gersten, ed.), MSRI Publ., vol. 8, Springer, New York, 1987, p 75-263. MR0919829

[8] V. A. Kaimanovich, *Hausdorff dimension of the harmonic measure on trees*, Ergod. Th. & Dynam. Sys. (1998), p 631-660. MR1631732

[9] V. A. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*, Annals of Mathematics, vol. 152 (2000), p 659-692. MR1815698

[10] F. Ledrappier, *Une relation entre entropie, dimension et exposant pour certaines marches aléatoires*, C. R. Acad. Sci. Paris Sér. I Math., vol. 296 (1983), p. 369-372. MR0699165

[11] V. Le Prince, *Dimensional properties of the harmonic measure for a random walk on a hyperbolic group*, Trans. of the AMS 359 (2007), p 2881-2898. MR2286061

[12] Ya. B. Pesin, *Dimension theory in dynamical systems*, Chicago Lect. Notes in Math. (1997). MR1489237

[13] L. S. Young, *Dimension, entropy and Lyapunov exponents*, Ergod. Th. & Dynam. Sys. (1982), p 109-124. MR0684248