BULLETIN *of the*
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **23** (2000) 25-32

On Köthe-Toeplitz Duals of Generalized Difference Sequence Spaces

MIKAIL ET AND AYHAN ESI
Department of Mathematics, Fırat University, 23169-Elazig Turkiye (TR)
e-mail: mikailet@hotmail.com
e-mail: aesi23@hotmail.com

Abstract. In this paper, we define the sequence spaces $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_o)$, $(m \in \boldsymbol{N})$, and give some topological properties, inclusion relations of these sequence spaces, compute their continuous and Köthe-Toeplitz duals. The results of this paper, in a particular case, include the corresponding results of Kızmaz [5] , Çolak [1], [2], Et-Çolak [4], and Çolak *et al.* [3].

1. Introduction

Let ℓ_∞, c, and c_0 be the linear spaces of bounded, convergent, and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\| x \|_\infty = \sup_k |x_k|$$

where $k \in \boldsymbol{N} = \{1, 2, \cdots\}$, the set of positive integers.

Kızmaz [5] defined the sequence spaces

$$\ell_\infty(\Delta) = \{ \boldsymbol{x} = (x_k) : \Delta \boldsymbol{x} \in \ell_\infty \},$$
$$c(\Delta) = \{ \boldsymbol{x} = (x_k) : \Delta \boldsymbol{x} \in c \},$$
$$c_0(\Delta) = \{ \boldsymbol{x} = (x_k) : \Delta \boldsymbol{x} \in c_0 \}$$

where $\Delta \boldsymbol{x} = (\Delta x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaces with norm

$$\| x \|_1 = | x_1 | + \| \Delta \boldsymbol{x} \|_\infty .$$

Then Çolak [1] defined the sequence space $\Delta_v(X) = \{ \boldsymbol{x} = (x_k) : \Delta_v x_k \in X \}$, where $(\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and X is any sequence space, and investigated some topological properties of this space.

Recently Et and Çolak [4] generalized the above sequence spaces to the following sequence spaces.

$$\ell_\infty(\Delta^m) = \{ \boldsymbol{x} = (x_k) : \Delta^m \boldsymbol{x} \in \ell_\infty \},$$
$$c(\Delta^m) = \{ \boldsymbol{x} = (x_k) : \Delta^m \boldsymbol{x} \in c \},$$
$$c_0(\Delta^m) = \{ \boldsymbol{x} = (x_k) : \Delta^m \boldsymbol{x} \in c_0 \}$$

where $m \in \boldsymbol{N}$, $\Delta^0 \boldsymbol{x} = (x_k), \Delta\boldsymbol{x} = (x_k - x_{k+1}), \Delta^m \boldsymbol{x} = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, and

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v} .$$

These are Banach spaces with norm

$$\| \boldsymbol{x} \|_\Delta = \sum_{i=1}^{m} | x_i | + \left\| \Delta^m \boldsymbol{x} \right\|_\infty .$$

It is trivial that $c_0(\Delta^m) \subset c_0(\Delta^{m+1}), c(\Delta^m) \subset c(\Delta^{m+1}), \ell_\infty(\Delta^m) \subset \ell_\infty(\Delta^{m+1})$, and $c_0(\Delta^m) \subset c(\Delta^m) \subset \ell_\infty(\Delta^m)$ are satisfied and strict [4]. For convenience we denote these spaces $\Delta^m(\ell_\infty) = \ell_\infty(\Delta^m), \Delta^m(c) = c(\Delta^m)$, and $\Delta^m(c_0) = c_0(\Delta^m)$.

Throughout the paper we write $\sum_k$ for $\sum_{k=1}^{\infty}$ and $\lim_n$ for $\lim_{n\to\infty}$.

Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. Now we define

$$\begin{aligned} \Delta_v^m(\ell_\infty) &= \left\{ \boldsymbol{x} = (x_k) : \Delta_v^m \ \boldsymbol{x} \in \ell_\infty \right\} \\ \Delta_v^m(c) &= \left\{ \boldsymbol{x} = (x_k) : \Delta_v^m \ \boldsymbol{x} \in c \right\} \\ \Delta_v^m(c_0) &= \left\{ \boldsymbol{x} = (x_k) : \Delta_v^m \ \boldsymbol{x} \in c_0 \right\} \end{aligned} \tag{1.1}$$

where

$$m \in \boldsymbol{N},\ \Delta_v^0\ \boldsymbol{x} = (v_k x_k),\ \Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1}),\ \Delta_v^m\ x_k = (\Delta_v^{m-1}\ x_k - \Delta_v^{m-1}\ x_{k+1}),$$

and so that

$$\Delta_v^m\ x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i}\ x_{k+i}$$

It is trivial that $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_0)$ are linear spaces. If we take $(v_k)=(1,1,\cdots)$ and $m=1$ in (1.1), then we obtain $\Delta(\ell_\infty)$, $\Delta(c)$ and $\Delta(c_0)$. Also if we take $m=1$ and $(v_k)=(1,1,\cdots)$ in (1.1), then we obtain $\Delta_v(\ell_\infty)$, $\Delta_v(c)$ and $\Delta_v(c_0)$, and $\Delta^m(\ell_\infty)$, $\Delta^m(c)$ and $\Delta^m(c_0)$, respectively.

2. Main results

Theorem 2.1. *The sequence spaces* $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ *and* $\Delta_v^m(c_0)$ *are Banach spaces normed by*

$$\|\boldsymbol{x}\|_v=\sum_{i=1}^{m}|x_i v_i|+\left\|\Delta_v^m\boldsymbol{x}\right\|_\infty . \tag{2.1}$$

Proof. Omitted.

Let X stand for ℓ_∞, c and c_0 and let us define the operator

$$D:\Delta_v^m(X)\to\Delta_v^m(X)$$

by $D\boldsymbol{x}=(0,0,\cdots,x_{m+1},x_{m+2},\cdots)$, where $\boldsymbol{x}=(x_1,x_2,x_3,\cdots)$. It is trivial that D is a bounded linear operator on $\Delta_v^m(X)$. Furthermore the set

$$D\left[\Delta_v^m(X)\right]=D\Delta_v^m(X)=\left\{\boldsymbol{x}=(x_k):\boldsymbol{x}\in\Delta_v^m(X),x_1=x_2=\cdots=x_m=0\right\}$$

is a subspace of $\Delta_v^m(X)$ and $\|x\|_v=\left\|\Delta_v^m x\right\|_\infty$ in $D\Delta_v^m(X)$. $D\Delta_v^m(X)$ and X are equivalent as topological space since

$$\Delta_v^m:D\Delta_v^m(X)\to X\text{ , defined by }\Delta_v^m\boldsymbol{x}=y=(\Delta_v^m x_k) \tag{2.2}$$

is a linear homeomorphism [7].

Let X' and $[D\Delta_v^m(X)]'$ denote the continuous duals of X and $D\Delta_v^m(X)$, respectively. It can be shown that

$$T:[D\Delta_v^m(X)]'\to X'\ ,\ \ f_\Delta\to f_\Delta^0(\Delta_v^m)^{-1}=f$$

is a linear isometry. So $[D\Delta_v^m(X)]'$ is equivalent to X' [7].

Corollary 2.2.

(i) $\Delta_v^m(c)$ *and* $\Delta_v^m(c_0)$ *are closed subspaces of* $\Delta_v^m(\ell_\infty)$,

(ii) $\Delta_v^m(c)$ *and* $\Delta_v^m(c_0)$ *are separable spaces,*

(iii) $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ *and* $\Delta_v^m(c_0)$ *are BK-spaces with the same norm as in (2.1),*

(iv) $\Delta_v{}^m(\ell_\infty)$, $\Delta_v{}^m(c)$ *and* $\Delta_v{}^m(c_0)$ *are not sequence algebras.*

3. Dual spaces

In this section we give Köthe-Toeplitz duals of $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_0)$. Now we give the following lemmas.

Lemma 3.1. $x \in \Delta_v^m(\ell_\infty)$ *if and only if*

(i) $\sup_k k^{-1}\left|\Delta_v^{m-1} x_k\right| < \infty$.

(ii) $\sup_k \left|\Delta_v^{m-1} x_k - k(k+1)^{-1}\Delta_v^{m-1} x_{k+1}\right| < \infty$.

Proof. Omitted.

Lemma 3.2. $\sup_k k^{-i}\left|\Delta_v x_k\right| < \infty$ *implies* $\sup_k k^{-(i+1)}\left|v_k x_k\right| < \infty$ *for all* $i \in N$.

Proof. Omitted.

Lemma 3.3. $\sup_k k^{-i}\left|\Delta_v^{m-i} x_k\right| < \infty$ *implies* $\sup_k k^{-(i+1)}\left|\Delta_v^{m-(i+1)} x_k\right| < \infty$ *for all* $i,\ m \in N$ *and* $1 \le i < m$.

Proof. If $\Delta_v x_k$ is replaced with $\Delta_v^{m-i} x_k$ in Lemma.3.2, the result is immediate.

Lemma 3.4. $\sup_k k^{-1}\left|\Delta_v^{m-1} x_k\right| < \infty$ *implies* $\sup_k k^{-m}\left|v_k x_k\right| < \infty$.

Proof. For $i=1$ in Lemma.3.3, we obtain $\sup_k k^{-1}\left|\Delta_v^{m-1} x_k\right| < \infty$ implies $\sup_k k^{-2}\left|\Delta_v^{m-2} x_k\right| < \infty$. Again, for $i=2$ in Lemma 3.3, we obtain $\sup_k k^{-2}\left|\Delta_v^{m-2} x_k\right| < \infty$ implies $\sup_k k^{-3}\left|\Delta_v^{m-3} x_k\right| < \infty$. Continuing this procedure, for $i=m-1$, we arrive $\sup_k k^{-(m-1)}\left|\Delta_v x_k\right| < \infty$ implies $\sup_k k^{-m}\left|v_v x_k\right| < \infty$.

Lemma 3.5. *$x \in \Delta_v^m(\ell_\infty)$ implies $\sup_k k^{-m} |v_k x_k| < \infty$.*

Proof. Proof follows from Lemma.3.1 and Lemma.3.4.

Definition 3.6. [6] *Let X be a sequence space and define*

$$X^\alpha = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty, \text{for all } \boldsymbol{x} \in X \right\},$$

then X^α is called Köthe-Toeplitz dual of X. If $X \subset Y$, then $Y^\alpha \subset X^\alpha$. It is clear that $X \subset (X^\alpha)^\alpha = X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$ then X is called an α-space. In particular, an α-space is a Köthe space or a perfect sequence space.

Theorem 3.7. *Let $U_1 = \{ a = (a_k) : \sum_k k^m |a_k v_k^{-1}| < \infty \}$ and*

$$U_2 = \left\{ a = (a_k) : \sup_k k^{-m} |a_k v_k| < \infty \right\}, \quad \textit{then}$$

i) $(\Delta_v^m(\ell_\infty))^\alpha = (\Delta_v^m(c))^\alpha = (\Delta_v^m(c_0))^\alpha = U_1$

ii) $(\Delta_v^m(\ell_\infty))^{\alpha\alpha} = (\Delta_v^m(c))^{\alpha\alpha} = (\Delta_v^m(c_0))^{\alpha\alpha} = U_2$

Proof. Omitted.

Corollary 3.8. *$\Delta_v{}^m(\ell_\infty), \Delta_v{}^m(c)$ and $\Delta_v^m(c_o)$ are not perfect.*

Corollary 3.9. *If we take $(v_k) = (1, 1, \cdots)$ and $m = 1$, in Theorem 3.7, then we obtain for $X = \ell_\infty$ or c.*

(i) $(\Delta^m(X))^\alpha = \left\{ a = (a_k) : \sum_k k^m |a_k| < \infty \right\},$

(ii) $(\Delta^m(X))^{\alpha\alpha} = \left\{ a = (a_k) : \sup_k k^{-m} |a_k| < \infty \right\},$

(iii) $(\Delta_v(X))^\alpha = \left\{ a = (a_k) : \sum_k k \left| a_k v_k^{-1} \right| < \infty \right\}.$

Corollary 3.10. *If we take $v = (k^m)$ in Theorem 3.7, then we obtain*

(i) $(\Delta_v^m(\ell_\infty))^\alpha = (\Delta_v^m(c))^\alpha = (\Delta_v^m(c_o))^\alpha = \ell_1,$

(ii) $(\Delta_v^m(\ell_\infty))^{\alpha\alpha} = (\Delta_v^m(c))^{\alpha\alpha} = (\Delta_v^m(c_o))^{\alpha\alpha} = \ell_\infty.$

4. Inclusions theorems

In this section we give inclusion relation of these spaces. Firstly, we note that $\Delta_v^m(X)$ and $\Delta^m(X)$ overlap but neither one contains the other, for $X = \ell_\infty, c$ and c_0. For example, we choose, $\boldsymbol{x} = (k^m)$ and $v = (k)$, then $\boldsymbol{x} \in \Delta^m(\ell_\infty)$, but $\boldsymbol{x} \notin \Delta_v^m(\ell_\infty)$, conversely if we choose $\boldsymbol{x} = (k^{m+1})$ and $v = (k^{-1})$ then $\boldsymbol{x} \notin \Delta^m(\ell_\infty)$, but $\boldsymbol{x} \in \Delta_v^m(\ell_\infty)$.

Theorem 4.1.

(i) $\Delta_v^m(X) \subset \Delta_v^{m+1}(X)$ *and the inclusion is strict, for* $X = \ell_\infty, c$ *and* c_0,

(ii) $\Delta_v^m(c_0) \subset \Delta_v^m(c) \subset \Delta_v^m(\ell_\infty)$ *and the inclusion is strict.*

Proof.

(i) We give the proof for $X = \ell_\infty$ only. Let $\boldsymbol{x} \in \Delta_v^m(\ell_\infty)$. Since

$$\left|\Delta^{m+1} x_k v_k\right| \le \left|\Delta^m x_k v_k - \Delta^m x_{k+1} v_{k+1}\right| \le \left|\Delta^m x_k v_k\right| + \left|\Delta^m x_{k+1} v_{k+1}\right|$$

we obtain $\boldsymbol{x} \in \Delta_v^{m+1}(\ell_\infty)$. This inclusion is strict since the sequence $\boldsymbol{x} = (k^m)$ belongs to $\Delta_v^{m+1}(\ell_\infty)$, but does not belong to $\Delta_v^m(\ell_\infty)$, where $v = (k)$.

(ii) Proof is trivial.

Theorem 4.2. *Let* $u = (u_k)$ *and* $v = (v_k)$ *be any fixed sequences of nonzero complex numbers, then*

(i) *If* $\sup_k k^m \left|v_k^{-1} u_k\right| < \infty$, *then* $\Delta_v^m(\ell_\infty) \subset \Delta_u^m(\ell_\infty)$,

(ii) *If* $k^m \left|v_k^{-1} u_k\right| \to \ell (k \to \infty)$, *for some* ℓ, *then* $\Delta_v^m(c) \subset \Delta_u^m(c)$,

(iii) *If* $k^m \left|v_k^{-1} u_k\right| \to 0 \, (k \to \infty)$, *then* $\Delta_v^m(c_0) \subset \Delta_u^m(c_0)$.

Proof.

(i) $\sup_k k^m \left|v_k^{-1} u_k\right| < \infty$ and assume that $\boldsymbol{x} \in \Delta_v^m(\ell_\infty)$. Since

$$\left|\Delta_u^m(x)\right| = \left|\Delta^{m-1}(\Delta_u(x))\right| = \left|\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\Delta(x_{k+i}\,u_{k+i})\right|$$

$$\leq \sum_{i=0}^{m-1}\binom{m-1}{i}\Big[\left|(k+i)^m\left|v_{k+i}^{-1}u_{k+i}\right|(k+i)^{-m}\left|v_{k+i}x_{k+i}\right|\right.$$

$$\left. +\ (k+i+1)^m\left|v_{k+i+1}^{-1}\,u_{k+i+1}\right|(k+i+1)^{-m}\left|v_{k+i+1}\,x_{k+i+1}\right|\ \right|\Big]$$

we obtain $\boldsymbol{x}\in\Delta_u^m(\ell_\infty)$. If we take $v=(1,1,\cdots)$ and $u=(1,1,\cdots)$ in Theorem 4.2, then we have the corollaries, respectively.

Corollary 4 3.

(i) *If* $\sup_k k^m|v_k| < \infty$, *then* $\Delta^m(\ell_\infty)\subset\Delta_v^m(\ell_\infty)$,

(ii) *If* $k^m|v_k|\to\ell\,(k\to\infty)$, *for some* ℓ, *then* $\Delta^m(c)\subset\Delta_v^m(c)$,

(iii) *If* $k^m|v_k|\to 0\ (k\to\infty)$, *then* $\Delta^m(c_0)\subset{\Delta_v}^m(c_0)$.

Corollary 4 4.

(i) *If* $\sup_k k^m\left|v_k^{-1}\right| < \infty$, *then* $\Delta_v^m(\ell_\infty)\subset\Delta^m(\ell_\infty)$,

(ii) *If* $k^m\left|v_k^{-1}\right|\to\ell\,(k\to\infty)$, *for some* ℓ, *then* $\Delta_v^m(c)\subset\Delta^m(c)$,

(iii) *If* $k^m\left|v_k^{-1}\right|\to 0\,(k\to\infty)$, *then* $\Delta_v^m(c_0)\subset\Delta^m(c_0)$.

If we take $x=(k^m)$ in [3], then we obtain the following sequence spaces.

i) $v_\infty = \{\,v=(v_k):\sup_k k^m\,|v_k| < \infty\,\}$,

ii) $v_c = \{\,v=(v_k):\ k^m\,|v_k|\to\ell\,(k\to\infty\,\}$, for some $\ell\,\}$,

iii) $v_0=\{\,v=(v_k):k^m\,|v_k|\to 0\,(k\to\infty)\}$,

i′) $v_\infty^{-1}=\{\,v=(v_k):\sup_k k^m\,|v_k^{-1}| < \infty\}$,

ii′) $v_c^{-1}=\{v=(v_k):k^m\,|v_k^{-1}|\to\ell\,(k\to\infty)$, for some $\ell\,\}$,

iii′) $v_o^{-1}=\{\,v=(v_k):k^m\,|v_k^{-1}|\to 0\,(k\to\infty)\}$.

It is trivial that the sequence spaces v_∞, v_c and v_0 are BK-spaces with the norm $\|v\|=\sup_k k^m\,|v_k\,|$. The η-duals of these sequence spaces are also readily obtained by [3] , where $\eta=\alpha$, β and γ.

Theorem 4.5. Let X stand for v_∞, v_c and v_0, then $X \cap X^{-1} = \emptyset$.

Proof. We give the proof for $X = v_\infty$ only. Let $v \in v_\infty \cap v_\infty^{-1}$ and $v_k \neq 0$ for all k, then there are constants $M_1, M_2 > 0$ such that $k^m \, | v_k | \leq M_1$ and $k^m \, | v_k^{-1} | \leq M_2$ for all $k \in N$. This implies $k^{2m} \leq M_1 \, M_2$ for all k, a contradiction, since $m \geq 1$.

Theorem 4.6. $\ell_\infty \cap \Delta_v^m (c) = \ell_\infty \cap \Delta_v^m (c_0)$.

Proof. Let $\boldsymbol{x} \in \ell_\infty \cap \Delta_v^m (c)$. Then $\boldsymbol{x} \in \ell_\infty$ and $\Delta^{m-1} x_k v_k - \Delta^{m-1} x_{k+1} v_{k+1} \to \ell \, (k \to \infty)$, $\Delta^{m-1} x_k v_k - \Delta^{m-1} x_{k+1} v_{k+1} = \ell + \varepsilon_k$, $(\varepsilon_k \to 0, k \to \infty)$. This implies that

$$\ell = n^{-1} \Delta^{m-1} x_1 v_1 - n^{-1} \Delta^{m-1} x_{n+1} v_{n+1} + n^{-1} \sum_{k=1}^{n} \varepsilon_k .$$

This yields $\ell = 0$ and $x \in \ell_\infty \cap \Delta_v^m (c_0)$.

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Keywords and phrases: difference sequence spaces, Köthe-Toeplitz dual.

1991 Mathematics Subject Classification: 40A05, 40C05, 46A45.