Maps between almost Kähler manifolds and framed φ-manifolds

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Abstract

In this paper we study the $\pm(J,\varphi)$- holomorphic maps between an almost Kähler manifold and a metric framed φ-manifold. We prove that any $\pm(J,\varphi)$-holomorphic map is a harmonic map with the minimum energy in its homotopy class and we prove that a (J,φ)-holomorphic map between a Kähler manifold and a cosymplectic manifold is weakly stable and we calculate the kernel of Jacobi operator of such a map.

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1 Introduction

Let M be an m-dimensional smooth manifold endowed with a tensor field φ of type $(1,1)$, satisfying the algebraic condition

$$\varphi^3 + \varphi = 0. \tag{1.1}$$

The geometric structure on M defined by φ is called a φ-structure of rank r if the rank r of φ is constant on M and, in this case, M is called a φ-manifold. It follows easily that r is an even number.

If M is a φ-manifold and if there are $m-r$ vector fields ξ_i and $m-r$ differential 1-forms η_i satisfying

$$\varphi^2 = -I + \sum_{i=1}^{m-r} \eta_i \otimes \xi_i, \tag{1.2}$$

$$\eta_i(\xi_j) = \delta_j^i, \tag{1.3}$$

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where $i, j = 1, 2, ..., m-r$, M is said to be globally framed or to have a framed φ-structure. In this case M is called a globally framed φ-manifold or, simply, a framed φ-manifold. From (1.2) and (1.3), one obtains by some algebraic computations

$$\varphi\xi_i = 0, \quad \eta_i \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{1.4}$$

If $m = 2n+1$ and $rank\ \varphi = 2n$ one obtains an almost contact structure on M.

Let M be an m-dimensional globally framed φ-manifold with structure tensors (φ, ξ_i, η_i) with $rank\ \varphi = r$, and consider the manifold $M \times \mathbb{R}^{m-r}$. We denote a vector field on $M \times \mathbb{R}^{m-r}$ by $(X, \sum_{i=1}^{m-r} f_i \frac{\partial}{\partial t^i})$ where X is tangent to M, $\{t^1, ..., t^{m-r}\}$ are the usual coordinates on $\mathbb{R}^{m-r}$ and $\{f_1, ..., f_{m-r}\}$ are functions on $M \times \mathbb{R}^{m-r}$. Define an almost complex structure on $M \times \mathbb{R}^{m-r}$ by

$$J(X, \sum_{a=1}^{m-r} f_i \frac{\partial}{\partial t^i}) = (\varphi X - \sum_{i=1}^{m-r} f_i \xi_i, \sum_{i=1}^{m-r} \eta_i(X) \frac{\partial}{\partial t^i}).$$

It is easy to check that $J^2 = -I$. If J is integrable we say that the framed φ-structure is normal. A framed φ-structure is normal if the tensor field S of type (1,2) defined by

$$S = N_\varphi + \sum_{i=1}^{m-r} d\eta_i \otimes \xi_i, \tag{1.5}$$

vanishes, (see [5]), where

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], X, Y \in \chi(M), \tag{1.6}$$

is the Nijenhuis tensor field of φ.

If g is a (semi-)Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{m-r} \eta_i(X)\eta_i(Y), \tag{1.7}$$

then we say that $(\varphi, \xi_i, \eta_i, g)$ is a metric framed φ-structure and M is called a metric framed φ-manifold.

The metric g is called an associated (semi-)Riemannian metric.

The fundamental 2-form Ω of the considered metric framed φ-manifold M, is defined just like in the case of the almost Hermitian and almost contact metric manifold, by $\Omega = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$.

The framed φ-manifold M with structure tensors (φ, ξ_i, η_i) is called a $\mathcal{C}$-manifold if it is normal, $d\Omega = 0$ and $d\eta_i = 0$, $i = 1, ..., m-r$, (see [2]).

If on an almost contact manifold (M, φ, ξ, η) it is defined an associated Riemannian metric g then $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold. If on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ we have $\Omega = d\eta$, where Ω is the fundamental 2-form on M, then we say that $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold.If for an almost contact metric structure (φ, ξ, η, g) which is normal we have $d\eta = 0$ and $d\Omega = 0$ then $(N, \varphi, \xi, \eta, g)$ is called a cosymplectic manifold.

In [1] it is proved the following two results

Lemma 1.1. *For an almost contact metric structure (φ, ξ, η, g), the covariant derivative of φ is given by*

$$2g((\nabla_X\varphi)Y, Z)) = 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z) + g(N_\varphi(Y, Z), \varphi X)+$$

$$+((L_{\varphi Y}\eta)(Z) - (L_{\varphi Z}\eta)(Y))\eta(X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y).$$

where L denote the Lie derivative.

Remark 1.2. In Lemma 1.1 the author use the "alt" convention for calculus of $d\Omega$ and $d\eta$.

Theorem 1.3. *An almost contact metric structure (φ, ξ, η, g) is cosymplectic if and only if φ is parallel.*

Just like in [1] one obtains

Lemma 1.4. *If $(M, \varphi, \xi_i, \eta_i, g)$ is a metric framed φ-manifold, where $i = 1, ...n$, then*

$$2g((\nabla_X\varphi)Y, Z)) = 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z) + g(N_\varphi(Y, Z), \varphi X)+$$

$$+2\sum_{i=1}^{n}[d\eta_i(\varphi Y, X)\eta_i(Z) - d\eta_i(\varphi Z, X)\eta_i(Y)] + \sum_{i=1}^{n}[d\eta_i(\varphi Y, Z) + d\eta_i(Y, \varphi Z)]\eta_i(X).$$

Remark 1.5. If $(M, \varphi, \xi_i, \eta_i, g)$ is a normal metric framed φ-manifold, where $i = 1, ...n$, then

$$2g((\nabla_X\varphi)Y, Z)) = 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z)+$$

$$+2\sum_{i=1}^{n}[d\eta_i(\varphi Y, X)\eta_i(Z) - d\eta_i(\varphi Z, X)\eta_i(Y)].$$

Remark 1.6. It is easy to see that if $(M, \varphi, \xi_i, \eta_i, g)$ is a $\mathcal{C}$-manifold then φ is parallel.

Concerning the harmonic maps between Riemannian manifolds, we should recall some notions and results as they are presented in [8].

Let $f : M \to N$ be a smooth map between two Riemannian manifolds (M, g) and (N, h). Let $f^{-1}(TN)$ be the induced bundle over M of TN defined as follows, denote by $\pi : TN \to N$ the projection. Then

$f^{-1}TN = \{(x, u) \in M \times TN, \pi(u) = f(x), x \in M\} = \bigcup_{x\in M} T_{f(x)}N$.

The set of all C^∞-sections of $f^{-1}TN$, denoted by $\Gamma(f^{-1}TN)$ is

$\Gamma(f^{-1}TN) = \{V : M \to TN, C^\infty\text{-map}, V(x) \in T_{f(x)}N, x \in M\}$.

Denote by ∇^M, ∇^N, the Levi-Civita connections on (M, g) and (N, h) respectively. Then, for a smooth map f between (M, g) and (N, h), we define the induced connection $\widetilde{\nabla}$ on the induced bundle $f^{-1}TN$ as follows, for $X \in \chi(M), V \in \Gamma(f^{-1}TN)$, define $\widetilde{\nabla}_X V \in \Gamma(f^{-1}TN)$ by $\widetilde{\nabla}_X V = \nabla^N_{f_*X}V$.

Then the connection $\widetilde{\nabla}$ and the metric h are compatible, that is, for $V_1, V_2 \in \Gamma(f^{-1}TN), X \in \chi(M)$ we have

$$X(h(V_1, V_2)) = h(\widetilde{\nabla}_X V_1, V_2) + h(V_1, \widetilde{\nabla}_X V_2).$$

In [4] the authors study the maps on $\mathcal{C}$-manifolds, introduce a Lichnerowitz type invariant on f-structures and prove that a structure-preserving map is an absolute minimum of the energy functional in its homotopy class.

In the second and the third sections of this paper we give some similarly results (sometimes with the complete proofs) for the maps between an almost Kähler manifold and a metric framed φ-manifold. In the last section we prove that a (J,φ)-holomorphic map between a Kähler manifold and a cosymplectic manifold is weakly stable and we calculate the kernel of the Jacobi operator of the inclusion map $i : M \hookrightarrow M \times \mathbb{R}$, where (M, g, J) is a Kähler manifold.

2 $\pm(J,\varphi)$- holomorphic maps

Let $(N, \varphi, \xi_i, \eta_i)$ be a framed φ-manifold and let TN be its tangent bundle.

Let $T^C N$ be the complexification of TN. Then φ can be uniquely extended to a complex linear endomorphism of $T^C N$, denoted also by φ, which satisfies (1.2). The eigenvalues of φ are i, 0, -i. Consider the usual decomposition

$$T^C N = T'N \oplus T^0 N \oplus T''N$$

of $T^C N$ in the eigenbundles corresponding to the eigenvalues i, 0, -i of φ.

Let M be an almost complex manifold with the almost complex structure J. Then the complexification of the tangent space $T^C M$ can be decomposed into a direct sum of the eigenspaces of J

$$T^C M = T'M \oplus T''M$$

corresponding to the eigenvalues i, -i.

Definition 2.1. Let $f : M \to N$ be a smooth map between the almost complex manifold (M, J) and the framed φ-manifold (N, φ, ξ, η). We define

$$\partial f : T'M \to T'N$$
$$\partial\overline{f} \ : T'M \to T''N$$
$$\partial_0 f : T'M \to T_0 N$$
$$\overline{\partial} f : T''M \to T'N$$
$$\overline{\partial}\,\overline{f} : T''M \to T''N$$
$$\overline{\partial}_0 f : T''M \to T_0 N$$

by

$$df\mid_{T'M} = \partial f + \partial\overline{f} + \partial_0 f$$
$$df\mid_{T''M} = \overline{\partial} f + \overline{\partial}\,\overline{f} + \overline{\partial}_0 f.$$

For $X \in TM,\ X' = \frac{1}{2}(X - iJX) \in T'M,\ X'' = \frac{1}{2}(X + iJX) \in T''M$, we have

$$\partial f(X') = \frac{1}{4}(f_*X - if_*JX - \partial_0 f(X') - i\varphi f_*X - \varphi f_*JX), \tag{2.1}$$

$$\partial\overline{f}(X') = \frac{1}{4}(f_*X - if_*JX - \partial_0 f(X') + i\varphi f_*X + \varphi f_*JX). \tag{2.2}$$

Definition 2.2. Let $f : M \to N$ be a smooth map. If $f_* J = \varphi f_*$, then f is called a (J, φ)-holomorphic map (or a $+(J, \varphi)$-holomorphic map); if $f_* J = -\varphi f_*$, then f is called a (J, φ)-antiholomorphic map (or a $-(J, \varphi)-$holomorphic map).

Proposition 2.1. *Let $f : M \to N$ be a smooth map. Then f is a $-(J, \varphi)$-holomorphic map if and only if $\partial f = 0$ and $\partial_0 f = 0$.*

Proof. If $f : M \to N$ is a $-(J, \varphi)-$holomorphic map then we have for $X \in TM$, $X' = \frac{1}{2}(X - iJX) \in T^{'}M$, $X'' = \frac{1}{2}(X + iJX) \in T''M$, using (1.4) $\partial f(X') = -\frac{1}{4}\partial_0 f(X')$.

Since $\partial f \in T'N$, $\partial_0 f \in T^0 N$, then $\partial f(X') = 0$ and $\partial_0 f(X') = 0$.

Conversely, if $\partial f = 0$, $\partial_0 f = 0$ we have

$$\frac{1}{4}(f_* X - i f_* JX - i\varphi f_* X - \varphi f_* JX) = 0,$$

for $X \in \chi(M)$.

Then $f_* J = -\varphi f_*$.

Similarly, one obtains

Proposition 2.2. *Let $f : M \to N$ be a smooth map. Then f is a $+(J, \varphi)$-holomorphic map if and only if $\partial \overline{f} = 0$ and $\partial_0 f = 0$.*

3 Harmonicity of $\pm(J, \varphi)$-holomorphic maps

Assume that (M, g, J) is an almost Hermitian manifold, i.e. $g(JX, JY) = g(X, Y)$ for any vector fields X, Y on M, and $(N, \varphi, \xi_i, \eta_i, h)$ is a metric framed φ-manifold. Choosing a local Hermitian frame field $\{e_i, Je_i\}$ in M, we have the corresponding holomorphic orthonormal frame field $\varepsilon_i = \frac{\sqrt{2}}{2}(e_i - iJe_i)$ and the anti-holomorphic frame field $\overline{\varepsilon}_i = \frac{\sqrt{2}}{2}(e_i + iJe_i)$. For a smooth map $f : M \to N$ we have, after a direct computation

$$(3.1) \qquad | \partial f |^2 + \frac{1}{2} | \partial_0 f |^2 = \frac{1}{4}[h(f_* e_i, f_* e_i) + h(f_* Je_i, f_* Je_i) + +2h(f_* Je_i, \varphi f_* e_i)],$$

$$(3.2) \qquad | \partial \overline{f} |^2 + \frac{1}{2} | \partial_0 f |^2 = \frac{1}{4}[h(f_* e_i, f_* e_i) + h(f_* Je_i, f_* Je_i) - -2h(f_* Je_i, \varphi f_* e_i)].$$

Definition 3.1. We call

$$| \partial f |^2 + \frac{1}{2} | \partial_0 f |^2 = e'(f)$$

and

$$| \overline{\partial} f |^2 + \frac{1}{2} | \partial_0 f |^2 = e''(f)$$

partial energy densities. If M is compact,

$$E'(f) = \int_M e'(f) * 1, \quad E''(f) = \int_M e''(f) * 1, \quad E(f) = E'(f) + E''(f),$$

where $E'(f)$ and $E''(f)$ are called partial energies, and $*1$ is the volume form on M.

Obviously f is a $+(J,\varphi)$-holomorphic map if and only if $E'(f) = 0$ and f is a $-(J,\varphi)$-holomorphic map if and only if $E''(f) = 0$.

Let ω the fundamental 2-form of the almost Kähler manifold (M, g, J), that is the 2-form defined by $\omega(X,Y) = g(X, JY)$ for any $X, Y \in \chi(M)$ which satisfies $d\omega = 0$ and Ω the fundamental 2-form of the metric framed φ-manifold, $(N, \varphi, \xi_i, \eta_i, h)$, defined by $\Omega(X,Y) = h(X, \varphi Y)$, for any $X, Y \in \chi(N)$, which satisfies $d\Omega = 0$. Consider

$$K(f) = E'(f) - E''(f) = \int_M h(f_* Je_i, \varphi f_* e_i) * 1, \tag{3.3}$$

where $\{e_i, Je_i\}$ is a local Hermitian frame field.

Since

$$\omega(e_i, e_j) = h(Je_i, e_j) = 0,$$
$$\omega(Je_i, Je_j) = -h(e_i, Je_j) = 0,$$
$$\omega(e_i, Je_j) = h(Je_i, Je_j) = \delta_{ij},$$

then

$$\langle\ f^*\Omega, \omega\rangle = f^*\Omega(e_i, e_j)\omega(e_i, e_j) + f^*\Omega(e_i, Je_j)\omega(e_i, Je_j) +$$
$$+ f^*\Omega(Je_i, Je_j)\omega(Je_i, Je_j) = \Omega(f_* e_i, f_* Je_j) = h(\varphi f_* e_i, f_* Je_i).$$

Substituting it into (3.3), one obtains

$$K(f) = \int_M \langle f^*\Omega, \omega\rangle * 1.$$

Proposition 3.1. *$K(f)$ is a homotopy invariant.*

Proof. Let $f_t : M \times [0,1] \to N$ be a family of one - parameter maps. Then $\frac{d}{dt} f_t^*\Omega$ is an exact form, i.e. there exist $\theta_t = f_t^* i(f_{t*}\frac{\partial}{\partial t})\Omega$, where $i(X)\Omega$ denote the interior product of the vector X with the 2-form Ω, such that $\frac{d}{dt} f_t^*\Omega = d\theta_t$, (see[9]).

It follows that

$$\frac{d}{dt}K(f_t) = \frac{d}{dt}\int_M \langle\ f_t^*\Omega, \omega\rangle * 1 = \int_M \langle \frac{d}{dt} f_t^*\Omega, \omega\rangle * 1 =$$
$$= \int_M \langle d\theta_t, \omega\rangle * 1 = \int_M \langle \theta_t, \delta\omega\rangle * 1 =$$
$$= \int_M \langle \theta_t, - * d * \omega\rangle * 1 = -\int_M \langle \theta_t, *d(\omega)^{n-1}\rangle * 1 = 0.$$

Theorem 3.2. *Let M be an almost Kähler manifold, and let N be a metric framed φ-manifold with the fundamental 2-form Ω satisfying $d\Omega = 0$. If M is compact, then any $\pm(J,\varphi)$-holomorphic map is a harmonic map with the minimum energy in its homotopy class.*

Proof. It is sufficient to consider the $+(J,\varphi)$-holomorphic case.

Let f be a $+(J,\varphi)$-holomorphic map. Then $E''(f)=0$. Let $f_t : M\times[0,1]\to N$, and $f_0=f$.

Then

$$E(f_0)=E'(f_0)+E''(f_0)=E'(f_0)-E''(f_0)=K(f_0)=K(f_t)\leq E(f_t).$$

Corollary 3.3. *Let M and N be as above, $f_0 : M\to N$ a $+(J,\varphi)$-holomorphic map, and $f_1 : M\to N$ a $-(J,\varphi)$-holomorphic map. Then f_0 and f_1 can not be homotopic unless they are constant maps.*

Proof. If f_0 is homotopic to f, then

$$E(f_0)=E'(f_0)+E''(f_0)=E'(f_0)-E''(f_0)=K(f_0)=$$

$$=K(f_1)=E'(f_1)-E''(f_1)=-E'(f_1)-E''(f_1)=-E(f_1).$$

Thus $E(f_0)=E(f_1)=0$.

4 Stability of (J,φ)-holomorphic maps between a Kähler manifold and a cosymplectic manifold

Theorem 4.1. *Let (M,g,J) be a compact Kähler manifold with $\dim M=2m$, and let (N,φ,ξ,η,h) be a cosymplectic manifold. Let $f : M\to N$ be a (J,φ)-holomorphic map. Then*

$$\int_M h(J_fV,V)*1=\frac{1}{2}\int_M h(DV,DV)*1+\int_M tr(\eta\otimes\eta)(\widetilde{\nabla}V,\widetilde{\nabla}V)*1,$$

where $V\in\Gamma(f^{-1}TN)$, and J_f is the Jacobi operator of f defined by

$$J_fV=-\sum_{i=1}^{m}(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}-\widetilde{\nabla}_{\nabla_{e_i}e_i})V-\sum_{i=1}^{m}R^N(V,f_*e_i)f_*e_i, V\in\Gamma(f^{-1}TN),$$

where R^N denote the curvature tensor on N. For each $V\in\Gamma(f^{-1}TN)$, DV is an element of $\Gamma(f^{-1}TN\otimes T^*M)$ defined by

$$DV(X)=\widetilde{\nabla}_{JX}V-\varphi\widetilde{\nabla}_XV, X\in\chi(M),$$

and $(\eta\otimes\eta)(\widetilde{\nabla}V,\widetilde{\nabla}V)$ is defined by

$$(\eta\otimes\eta)(\widetilde{\nabla}V,\widetilde{\nabla}V)(X,Y)=(\eta\otimes\eta)(\widetilde{\nabla}_XV,\widetilde{\nabla}_YV),$$

where $X,Y\in\chi(M)$. Then

1) f is weakly stable, that is, each eigenvalue of J_f is nonnegative.

2) $\ker J_f=\{V\in\Gamma(f^{-1}TN), DV=0\}$.

Proof. We have, for an orthonormal basis $\{e_1, ..., e_m, Je_1, ..., Je_m\}$ in M

$$h(DV, DV) = \sum_{i=1}^{m}\{h(DV(e_i), DV(e_i)) + h(DV(Je_i), DV(Je_i))\} =$$

$$= \sum_{i=1}^{m}\{h(\widetilde{\nabla}_{Je_i}V - \varphi\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}V - \varphi\widetilde{\nabla}_{e_i}V) +$$

$$+ h(-\widetilde{\nabla}_{e_i}V - \varphi\widetilde{\nabla}_{Je_i}V, -\widetilde{\nabla}_{e_i}V - \varphi\widetilde{\nabla}_{Je_i}V).$$

Since $h(\varphi X, \varphi Y) = h(X, Y) - \eta(X)\eta(Y)$ we have

$$h(DV, DV) + \eta(\widetilde{\nabla}_{e_i}V)\eta(\widetilde{\nabla}_{e_i}V) + \eta(\widetilde{\nabla}_{Je_i}V)\eta(\widetilde{\nabla}_{Je_i}V) =$$

$$= 2\sum_{i=1}^{m}\{h(\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{e_i}V) - 2h(\varphi\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}V) + h(\widetilde{\nabla}_{Je_i}V, \widetilde{\nabla}_{Je_i}V)\}.$$

Thus

$$\int_M [h(J_f V, V) - \frac{1}{2}h(DV, DV) - tr(\eta \otimes \eta)(\widetilde{\nabla}V, \widetilde{\nabla}V)] * 1 = \tag{4.1}$$

$$= \int_M \sum_{i=1}^{m}\{-h(R^N(V, f_*e_i)f_*e_i, V) - h(R^N(V, f_*Je_i)f_*Je_i, V) +$$

$$+ 2h(\varphi\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}V)\} * 1,$$

since

$$\int_M h(J_f V, V) * 1 = \int_M \sum_{i=1}^{m}\{h(\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{e_i}V) + h(\widetilde{\nabla}_{Je_i}V, \widetilde{\nabla}_{Je_i}V) -$$

$$- h(R^N(V, f_*e_i)f_*e_i, V) - h(R^N(V, f_*Je_i)f_*Je_i, V)\} * 1, \text{(see[8])}.$$

Next, we shall prove that

$$A = R^N(V, f_*e_i)f_*e_i + R^N(V, f_*Je_i)f_*Je_i = \varphi R^N(f_*e_i, f_*Je_i)V. \tag{4.2}$$

Since $(N, \varphi, \xi, \eta, h)$ is a cosymplectic manifold we have $\nabla^N\varphi = 0$, that is $\nabla^N_X\varphi Y = \varphi\nabla^N_X Y$, for any $X, Y \in \chi(N)$, (see[1]).

From this and from (J, φ)-holomorphicity of f we have

$$A = -\varphi R^N(V, f_*e_i)f_*Je_i + \varphi R^N(V, f_*Je_i)f_*e_i =$$

$$= \varphi R^N(f_*e_i, V)f_*Je_i + \varphi R^N(V, f_*Je_i)f_*e_i = \varphi R^N(f_*e_i, f_*Je_i)V,$$

where we used the formulas, for $X, Y, Z \in \chi(N)$

$$R^N(X, Y)Z + R^N(Y, X)Z = 0,$$

$$R^N(X, Y)Z + R^N(Y, Z)X + R^N(Z, X)Y = 0.$$

From (4.1) and (4.2) we have

$$(4.3)\qquad \int_M [h(J_f V, V) - \frac{1}{2}h(DV, DV) - tr(\eta\otimes\eta)(\widetilde{\nabla}V, \widetilde{\nabla}V)] * 1 =$$

$$= \int_M \sum_{i=1}^{m}\{-h(\varphi R^N(f_*e_i, f_*Je_i)V, V) + 2h(\varphi\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}V)\} * 1.$$

To complete the proof, it is sufficient to show the second integral in (4.3) vanishes. To do this we show

$$(4.4)\qquad \int_M \sum_{i=1}^{m} -h(\varphi R^N(f_*e_i, f_*Je_i)V, V) * 1 =$$

$$= \int_M \sum_{i=1}^{m}\{h(\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}\varphi V) - h(\widetilde{\nabla}_{Je_i}V, \widetilde{\nabla}_{e_i}\varphi V)\} * 1.$$

Then the integral in (4.3) coincides with

$$\int_M \sum_{i=1}^{m}\{h(\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}\varphi V) - h(\widetilde{\nabla}_{Je_i}V, \widetilde{\nabla}_{e_i}\varphi V) + 2h(\varphi\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}V)\} * 1$$

which vanishes because $\widetilde{\nabla}_X\varphi V = \varphi\widetilde{\nabla}_X V$ since $(N, \varphi, \xi, \eta, h)$ is cosymplectic and $h(\varphi X, Y) = -h(X, \varphi Y)$.

Equation (4.4) can be derived as follows, since $[e_i, Je_i] = \nabla^M_{e_i}Je_i - \nabla^M_{Je_i}e_i$, one obtains

$$(4.5)\qquad -h(\varphi R^N(f_*e_i, f_*Je_i)V, V) = h(\varphi R^N(f_*e_i, f_*Je_i)V, \varphi V) =$$

$$= h(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{Je_i}V - \widetilde{\nabla}_{Je_i}\widetilde{\nabla}_{e_i}V - \widetilde{\nabla}_{[e_i, Je_i]}V, \varphi V) =$$

$$= e_i(h(\widetilde{\nabla}_{Je_i}V, \varphi V)) - h(\widetilde{\nabla}_{Je_i}V, \widetilde{\nabla}_{e_i}\varphi V) -$$

$$-Je_i(h(\widetilde{\nabla}_{e_i}V, \varphi V)) + h(\widetilde{\nabla}_{e_i}V, \widetilde{\nabla}_{Je_i}\varphi V) - h(\widetilde{\nabla}_{\nabla_{e_i}Je_i}V, \varphi V) + h(\widetilde{\nabla}_{\nabla_{Je_i}e_i}V, \varphi V),$$

since $\widetilde{\nabla}$ and h are compatible.

We define a smooth function s on M by

$$(4.6)\qquad s = \sum_{i=1}^{m}\{e_i(h(\widetilde{\nabla}_{Je_i}V, \varphi V)) - Je_i(h(\widetilde{\nabla}_{e_i}V, \varphi V)) -$$

$$-h(\widetilde{\nabla}_{\nabla_{e_i}Je_i}V, \varphi V) + h(\widetilde{\nabla}_{\nabla_{Je_i}e_i}V, \varphi V)\}.$$

Let X be a vector field on M defined by $g(X, Y) = h(\widetilde{\nabla}_{JY}V, \varphi V)$, for any $Y \in \chi(M)$.

Then we have

$$divX = \sum_{i=1}^{m}\{g(e_i, \nabla^M_{e_i}X) + g(Je_i, \nabla^M_{Je_i}X)\} = \sum_{i=1}^{m}\{e_i(g(e_i, X)) - g(\nabla^M_{e_i}e_i, X) +$$

$$+Je_i(g(Je_i,X))-g(\nabla^M_{Je_i}Je_i,X)\}=\sum_{i=1}^{m}\{e_i(h(\widetilde{\nabla}_{Je_i}V,\varphi V))-$$

$$-Je_i(h(\widetilde{\nabla}_{e_i}V,\varphi V))-h(\widetilde{\nabla}_{\nabla_{e_i}Je_i}V,\varphi V)+h(\widetilde{\nabla}_{\nabla_{Je_i}e_i}V,\varphi V)\}=s.$$

Thus

$$\int_M s*1=0. \tag{4.7}$$

From (4.5),(4.6) and (4.7) we have

$$\int_M\sum_{i=1}^{m}-h(\varphi R^N(f_*e_i,f_*Je_i)V,V)*1=\int_M\sum_{i=1}^{m}\{h(\widetilde{\nabla}_{e_i}V,\widetilde{\nabla}_{Je_i}\varphi V)-$$

$$h(\widetilde{\nabla}_{Je_i}V,\widetilde{\nabla}_{e_i}\varphi V)\}*1.$$

Then, one obtains

$$\int_M[h(J_fV,V)-\frac{1}{2}h(DV,DV)-tr(\eta\otimes\eta)(\widetilde{\nabla}V,\widetilde{\nabla}V)]*1=$$

$$=\int_M\sum_{i=1}^{m}\{h(\widetilde{\nabla}_{e_i}V,\widetilde{\nabla}_{Je_i}\varphi V)-h(\widetilde{\nabla}_{Je_i}V,\widetilde{\nabla}_{e_i}\varphi V)+2h(\varphi\widetilde{\nabla}_{e_i}V,\widetilde{\nabla}_{Je_i}V)\}*1=0.$$

Just as above, using Lemma 1.4, we can prove the following

Theorem 4.2. *Let (M,g,J) be a compact Kähler manifold with $\dim M=2m$, and let $(N,\varphi,\xi_i,\eta_i,h)$, where $i=1,...,n$, be a $\mathcal{C}$-manifold. Let $f:M\to N$ be a (J,φ)-holomorphic map. Then*

$$\int_M h(J_fV,V)*1=\frac{1}{2}\int_M h(DV,DV)*1+$$

$$+\int_M\sum_{i=1}^{n}tr(\eta_i\otimes\eta_i)(\widetilde{\nabla}V,\widetilde{\nabla}V)*1,$$

where $V\in\Gamma(f)^{-1}TN$, J_f is the Jacobi operator of f, and DV is defined as in Theorem 4.1. Then

1) f is weakly stable, that is, each eigenvalue of J_f is nonnegative.

2) $\ker J_f=\{V\in\Gamma(f^{-1}TN),DV=0\}$.

Let (M,g,J) be a 2n-dimensional Kähler manifold with local coordinates $\{x^1,...,x^{2n}\}$ and let t be the coordinate on $\mathbb{R}$. Then on $M\times\mathbb{R}$ set $\eta=dt$ and $\xi=\frac{\partial}{\partial t}$. We define the metric G on $M\times\mathbb{R}$ by

$$G((X,a\frac{\partial}{\partial t}),(Y,b\frac{\partial}{\partial t}))=g(X,Y)+(\eta\otimes\eta)((X,a\frac{\partial}{\partial t}),(Y,b\frac{\partial}{\partial t}))$$

for any $X,Y\in\chi(M)$ and $a,b:M\times\mathbb{R}\to\mathbb{R}$. In local coordinates G has the expression

$$G=g_{ij}dx^idx^j+dtdt$$

where g_{ij} are the components of g.

On $M \times \mathbb{R}$ we define the tensor field φ of type (1,1) by $\varphi\xi = 0$ and $\varphi(X, 0) = JX$ for $X \in \chi(M)$. Then (φ, ξ, η, G) is an almost contact metric structure on $M \times \mathbb{R}$. Let Ω be the fundamental 2-form on $(M \times \mathbb{R}, \varphi, \xi, \eta, G)$ defined by $\Omega(\widetilde{X}, \widetilde{Y}) = G(\widetilde{X}, \varphi\widetilde{Y})$ for any $\widetilde{X}, \widetilde{Y} \in \chi(M \times \mathbb{R})$.

By a straightforward computation one obtains $d\Omega = 0$ and $d\eta = 0$.

Since (M, g, J) is a Kähler manifold we have $N_J = 0$, where N_J is the Nijenhuis tensor of J.

Let N_φ be the Nijenhuis tensor of φ. Then one obtains $N_\varphi((X, 0), (Y, 0)) = N_J(X, Y)$ and $N_\varphi((X, 0), (0, \frac{\partial}{\partial t})) = 0$ for any $X, Y \in \chi(M)$. Then $N_\varphi = 0$ and since $d\Omega = 0$ and $d\eta = 0$ one obtains that $(M \times \mathbb{R}, \varphi, \xi, \eta, G)$ is a cosymplectic manifold.

Let $i : M \hookrightarrow M \times \mathbb{R}$ be the inclusion map. By the definition of φ we have $i_* J = \varphi i_*$ and then i is a (J, φ)-holomorphic map and, from Theorem 3.2, a harmonic map.

If we denote with ∇ and ∇' the Levi-Civita connections of M and $M \times \mathbb{R}$ respectively, we have

$$\nabla'_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \Gamma'^k_{ij} \frac{\partial}{\partial x_k} + \Gamma'^0_{ij} \frac{\partial}{\partial t},$$

$$\nabla'_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x_i} = \Gamma'^k_{i0} \frac{\partial}{\partial x_k} + \Gamma'^0_{i0} \frac{\partial}{\partial t},$$

$$\nabla'_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = \Gamma'^k_{00} \frac{\partial}{\partial x_k} + \Gamma'^0_{00} \frac{\partial}{\partial t},$$

where Γ'^k_{ij}, Γ'^0_{ij}, Γ'^k_{i0}, Γ'^k_{00}, Γ'^0_{00} are the Christoffel symbols of ∇'.

One obtains

$$\Gamma'^k_{ij} = \Gamma^k_{ij}, \quad \Gamma'^0_{ij} = \Gamma'^k_{00} = \Gamma'^k_{i0} = \Gamma'^0_{00} = 0,$$

where Γ^k_{ij} are the Christoffel symbols of ∇. Hence for a vector field $V' = (V, f\frac{\partial}{\partial t}) \in \Gamma(i^{-1}T(M \times \mathbb{R}))$, where $V \in \chi(M)$ and $f : M \to \mathbb{R}$ is a smooth function, we have

$$DV(X) = \widetilde{\nabla}_{JX} V' - \varphi \widetilde{\nabla}_X V' =$$

$$= \nabla'_{i_* JX}(V, f\frac{\partial}{\partial t}) - \varphi \nabla'_{i_* X}(V, f\frac{\partial}{\partial t}) =$$

$$= \nabla_{JX} V - J\nabla_X V + JX(f)\frac{\partial}{\partial t}$$

for any $X \in \chi(M)$, since (M, g, J) is a Kähler manifold and $(M \times \mathbb{R}, \varphi, \xi, \eta, G)$ is a cosymplectic manifold.

Then $DV = 0$ if and only if V is a holomorphic vector field, (see[8]), and f is a constant. Thus, using Theorem 4.1, we have

Proposition 4.3. *If (M, g, J) is a Kähler manifold and $(M \times \mathbb{R}, \varphi, \xi, \eta, G)$ is the cosymplectic manifold obtained as above then the inclusion map $i : M \hookrightarrow M \times \mathbb{R}$ is weakly stable. Moreover $Ker(J_i) = \{V' = (V, a\frac{\partial}{\partial t}) \in \Gamma(i^{-1}T(M \times \mathbb{R}))$, where V is a holomorphic vector field on M and $a \in \mathbb{R}$ is a constant$\}$.*

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