Inequalities between volume, center of mass, circumscribed radius, order, and mean curvature

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Abstract

By applying the spectral decomposition of a submanifold of a Euclidean space, we derive several sharp geometric inequalities which provide us some best possible relations between volume, center of mass, circumscribed radius, inscribed radius, order, and mean curvature of the submanifold. Several of our results sharpen some well-known geometric inequalities.

1 Introduction

Let M be a compact Riemannian manifold (without boundary) and denote by Δ the Laplace operator of M acting as a differential operator on $C^\infty M$, the space of all smooth functions on M. We can define a metric on $C^\infty M$ by $(f,g) = \int_M fg dA$, where dA is the volume form of M. It is well-known that Δ is a self-adjoint differential operator on $C^\infty M$ which has an infinite discrete sequence of eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots \nearrow +\infty$. For each $i \in \mathbb{N}$ the eigenspace V_i corresponding to the eigenvalue λ_i has finite dimension. The eigenspaces are mutually orthogonal and their sum is dense in $C^\infty M$. So one can make a spectral decomposition $f = f_0 + \sum_{i=1}^{\infty} f_i$, for each real differentiable function f on M, where

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f_0 is a constant and $\Delta f_i = \lambda_i f_i$ for $i \geq 0$. The set $T(f) = \{i \in \mathbb{N}_0 | f_i \neq 0\}$ is called the type of f; f is said to be of finite type if $T(f)$ is a finite set. Otherwise, f is said to be of infinite type. The smallest element in $T(f)$ is called the lower order of f, denoted by l.o.(f); the supremum of $T(f)$ is called the upper order of f, denoted by u.o.(f). The function f is said to be of k-type if $T(f)$ contains exactly k elements.

For an isometric immersion $x : M \to E^m$ of M into a Euclidean m–space, we put

$$x = (x_1, \ldots, x_m)$$

where x_A is the A–th Euclidean coordinate function of M in E^m. For each x_A we have

$$x_A - (x_A)_0 = \sum_{t=p_A}^{q_A} (x_A)_t, \quad A = 1, \ldots, m,$$

where $p_A = l.o.(x_A)$ and $q_A = u.o.(x_A)$. Put

$$p = \inf_A \{p_A\}, \quad q = \sup_A \{q_A\}$$

where A ranges among all A such that $x_A - (x_A)_0 \neq 0$. It is easy to see that p and q are well–defined geometric invariants such that p is a positive integer and q is either an integer $\geq p$ or ∞. We call the pair $[p, q]$ the *order* of the submanifold; in particular, p and q are called the lower order and the upper order of M, respectively. We define $T(x)$ by $T(x) = \{t \in \mathbb{N}_0 : x_t \neq 0\}$. The submanifold M is said to be of *k–type* if $T(x)$ contains exactly k elements and M is *of finite type* if $T(x)$ contains finitely many elements (cf. [1,10] for details). By using the above notation we have the following spectral decomposition of x in vector form:

$$x = x_0 + \sum_{t=p}^{q} x_t = x_0 + \sum_{t \in T(x)} x_t. \tag{1.1}$$

For simplicity, we put $c = |x_0|$.

In [1,2] the first-named author used the spectral decomposition (1.1) to obtain some geometric inequalities. In this article, we obtain further geometric inequalities also by utilizing the spectral decomposition. Our results provide several best possible relations between the volume, the center of mass, the circumscribed radius, the inscribed radius, the order, and the mean curvature of the submanifold.

2 Volume, circumscribed radius, and total mean curvature

Let $B_u(R)$ denote the open ball in E^m centered at $u \in E^m$ and with radius R. Moreover, we denote the mean curvature vector of M in E^m by H.

First we mention the following easy lemma.

Lemma 1 *If M is a compact submanifold in E^m which is contained in the closed ball $\overline{B_0(R)}$, we have*

- *(a)* $R^2 - c^2 > 0$,
- *(b)* $||x||^2 - c^2 V$ *is invariant under Euclidean motions, where* $||\,.\,||$ *denotes the* L^2*-norm and* V *denotes the volume of* M*.*

Proof From (1.1), we obtain

$$||x||^2 = \int_M \langle x, x\rangle \, dV = ||x_0||^2 + \sum_{t=p}^{q} ||x_t||^2. \tag{2.1}$$

Since $||x_0||^2 = c^2 V$, (2.1) implies the lemma.

■

By using the spectral decomposition (1.1), we may obtain the following.

Theorem 2 *Let* M *be a compact* n*-dimensional submanifold of* E^m*.*

(i) If M *is contained in the closed ball* $\overline{B_0(R)}$*, then the mean curvature of* M *in* E^m *satisfies*

$$\int_M |H|^k dV \geq \frac{V}{(R^2 - c^2)^{k/2}}, \quad k = 2, 3, \cdots, n, \tag{2.2}$$

with any one of the equalities holding if and only if M *is contained in the hypersphere* $S_0(R)$ *and* M *is of 1-type.*

(ii) If M *is contained in* $E^m - B_0(r)$*, then*

$$\left(\frac{\lambda_p}{n}\right)^2 (r^2 - c^2)V \leq \int_M |H|^2 dV \leq \left(\frac{\lambda_q}{n}\right)^2 (R^2 - c^2)V, \tag{2.3}$$

where $[p, q]$ *is the order of* M *in* E^m*. Either equality of (2.3) holds if and only if* M *lies in a hypersphere centered at the origin and* M *is of 1-type.*

Proof For any integer $B \geq p$, we put

$$u^B = (||x_p||, ||x_{p+1}||, \ldots, ||x_B||),$$
$$v^B = (\lambda_p||x_p||, \lambda_{p+1}||x_{p+1}||, \ldots, \lambda_B||x_B||).$$

Then we have

$$\left\langle u^B, u^B \right\rangle = \sum_{t=p}^{B} ||x_t||^2, \quad \left\langle v^B, v^B \right\rangle = \sum_{t=p}^{B} \lambda_t^2 ||x_t||^2, \quad \left\langle u^B, v^B \right\rangle = \sum_{t=p}^{A} \lambda_t ||x_t||^2.$$

Thus, by the Schwartz inequality, we have

$$(\sum_{t=p}^{B} ||x_t||^2)(\sum_{t=p}^{B} \lambda_t^2 ||x_t||^2) \geq (\sum_{t=p}^{B} \lambda_t ||x_t||^2)^2. \tag{2.4}$$

On the other hand, we have (cf. [1])

$$\left\langle u^B, u^B \right\rangle \longrightarrow (x, x) - c^2 V,$$

$$\left\langle v^B, v^B \right\rangle \longrightarrow n^2 \int_M |H|^2 dV,$$

$$\left\langle u^B, v^B \right\rangle \longrightarrow -n \int_M < x, H > dV = nV$$

as $B \to \infty$. Thus, (2.4) yields

$$((x,x) - c^2 V) \int_M |H|^2 dV \geq V^2. \tag{2.5}$$

Since $(x,x) \leq R^2 V$, (2.5) implies (2.2) for $k = 2$.

Now, by using Hölder's inequality, we find

$$\left(\int_M |H|^{2r} dV\right)^{\frac{1}{r}} \left(\int_M dV\right)^{\frac{1}{s}} \geq \int_M |H|^2 dV \geq \frac{V}{R^2 - c^2}$$

with $\frac{1}{r} + \frac{1}{s} = 1, r, s > 1$. Let $r = \frac{k}{2}$, we obtain inequality (2.2).

It is easy to see that the equality sign of (2.2) holds for some $k = 2, \cdots, n$, if and only if M is contained in the hypersphere $S_0(R)$ and M is of 1-type.

For (2.2), we consider

$$((x,x) - c^2 V) n^2 \int_M |H|^2 dV \geq n^2 V^2 = (\sum_{t \geq p} \lambda_t ||x_t||^2)^2$$

$$\geq \lambda_p^2 (\sum_{t \geq p} ||x_t||^2)^2 = \lambda_p^2 ((x,x) - c^2 V)^2,$$

which implies the first inequality of (2.2). The second inquality of (2.2) can be proved in a similar way. The remaining part can be verified easily.

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Theorem 2 implies immediately the following.

Corollary 3 *Let M be a compact n-dimensional submanifold of E^m. If M is contained in the closed ball $\overline{B_0(R)}$, then*

$$\max |H|^2 \geq \frac{1}{R^2 - c^2}, \tag{2.6}$$

with equality holding if and only if M is a minimal submanifold of a hypersphere of E^m.

Furthermore, if M has constant mean curvature and M is contained in $E^m - B_0(r)$, then

$$\left(\frac{\lambda_p}{n}\right)^2 (r^2 - c^2) \leq |H|^2 \leq \left(\frac{\lambda_q}{n}\right)^2 (R^2 - c^2), \tag{2.7}$$

with either equality holding if and only if M is of 1-type.

Remark 2.1. Since the center of mass of a submanifold in E^m is different from the center of the circumscribed sphere in general, inequalities (2.2) and (2.6) improve the main result of [8] and inequality (14) of [7, page 210], respectively.

By applying the spectral decomposition (1.1), we may also obtain the following inequalities.

Proposition 4 *Let M be a compact n-dimensional submanifold of E^m.*

(i) If M is contained in $E^m - B_0(r)$, then

$$(r^2 - c^2)\lambda_1 \leq n \tag{2.8}$$

where the equality hold if and only if M is contained in the hypersphere $S_0(r)$ and M is of 1-type.

(ii) Furthermore, if M is contained the closed ball $\overline{B_0(R)}$, then

$$\lambda_p(r^2 - c^2) \leq n \leq \lambda_q(R^2 - c^2), \tag{2.9}$$

where $[p, q]$ is the order of M. Either equality of (2.9) *holds if and only if M is contained in a hypersphere centered at the origin and M is of 1-type.*

Proof This proposition follows easily from the following two equations:

$$nV = \sum_{t \geq p} \lambda_t ||x_t||^2 \geq \lambda_p((x, x) - c^2 V) \geq \lambda_1(r^2 - c^2)V$$

and

$$nV = \sum_{t \geq p} \lambda_t ||x_t||^2 \leq \lambda_q((x, x) - c^2 V) \leq \lambda_q(R^2 - c^2)V.$$

As immediate consequences of Proposition 4 we have the following.

Corollary 5 *Let M be the ellipsoid in E^3 defined by*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{e^2} = 1, \quad a \leq b \leq e. \tag{2.10}$$

Then the first nonzero eigenvalue of the Laplacian of M satisfies

$$\lambda_1 \leq 2/a^2. \tag{2.11}$$

The equality sign of (2.11) holds if and only if M is a sphere, i.e., $a = b = e$.

Corollary 6 *Let M be the anchor ring in E^3 defined by*

$$x(u, v) = ((a + \epsilon \cos u) \cos v, (a + \epsilon \cos u) \cos v, \epsilon \sin u), \quad \epsilon < a. \tag{2.12}$$

Then the first nonzero eigenvalue of the Laplacian of M satisfies

$$\lambda_1 < \frac{2}{(a - \epsilon)^2}. \tag{2.13}$$

As a generalization of Corollary 6, we give the following estimate of λ_1 for closed tubes.

Theorem 7 *Let σ be a closed curve in E^3. Denote by σ_0 the center of mass of σ and by $T_\epsilon(\sigma)$ the tube around σ with a radius ϵ. If σ is contained in $E^3 - B_{\sigma_0}(r)$ and $\epsilon < r$, then the first nonzero eigenvalue λ_1 of the Laplacian of $T_\epsilon(\sigma)$ satisfies*

$$(r-\epsilon)^2\lambda_1 < 2. \tag{2.14}$$

Proof Without loss of generality, we may assume that $\sigma : [0,\ell] \to E^3$ is a unit speed closed curve in E^3 whose center of mass is at the orgin of E^3. Denote by T, N, B the Frenet frame of σ. Then the tube $T_\epsilon(\sigma)$ is given by

$$x(s,\theta) = \sigma(s) + \epsilon\cos\theta\, N + \epsilon\sin\theta\, B. \tag{2.15}$$

From (2.15) we find

$$dV = \epsilon(1-\epsilon\kappa\cos\theta)d\theta ds, \tag{2.16}$$

where κ is the curvature function of σ. By using (2.15) and (2.16) we have

$$\begin{aligned} x_0 &= \frac{\epsilon}{V}\int_0^\ell\int_0^{2\pi} x(s,\theta)(1-\epsilon\kappa\cos\theta)d\theta ds \\ &= \frac{\epsilon}{V}\left(2\pi\int_0^\ell \sigma(s)ds - \epsilon\pi\int_0^\ell H_\sigma ds\right), \end{aligned} \tag{2.17}$$

where H_σ is the mean curvature vector of σ. Since $\int_0^\ell H_\sigma ds = 0$, (2.17) implies that the center of mass of the tube coincides with the center of mass of σ. Therefore, by applying Proposition 4, we obtain $(r-\epsilon)^2\lambda_1 \le 2$. If $(r-\epsilon)^2\lambda_1 = 2$, then the tube is of 1-type which is impossible. So, we obtain (2.14).

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Remark 2.2. Proposition 4 improves Theorem 9.1 of [1, page 307].

Remark 2.3. By using Proposition 4, one may obtain similar result for tubes around compact submanifolds in E^m in a similar way.

Remark 2.4. In the same spirit, one may restate Theorems 2 and 3 of [2] as the following.

(i) Let M be a compact n-dimensional submanifold of E^m. If M is contained in the closed ball $\overline{B_0(R)}$, then

$$\int_M |H|^2 dV \ge \frac{1}{n^2}\{n(\lambda_1+\lambda_2) + (c^2 - R^2)\lambda_1\lambda_2\}V \tag{2.18}$$

where the equality holds if and only if M is contained in the hypersphere $S_0(R)$ and either M is of 1-type with order $[1,1]$ or $[2,2]$, or M is of 2-type and of order $[1,2]$.

(ii) Let M be a compact n-dimensional submanifold of E^m. If M is contained in $E^m - B_0(r)$ and M is of finite type with order $[p,q]$, then

$$\int_M |H|^2 dV \le \frac{1}{n^2}\{n(\lambda_p+\lambda_q) + (c^2 - r^2)\lambda_p\lambda_q\}V \tag{2.19}$$

where the equality holds if and only if M is contained in the hypersphere $S_0(R)$ and M is either of 1-type or of 2-type.

3 Mean curvature of submanifolds of non-negative kind

Let $x : M \to E^m$ be a finite type isometric immersion whose spectral decomposition is given by

$$x = x_0 + \sum_{t \in T(x)} x_t. \tag{4.1}$$

The immersion is said to be *of non-negative kind* if $\langle x_i(u), x_j(u) \rangle \geq 0$, for any $i, j \in T(x)$ and $u \in M$.

For submanifolds of non-negative kind, we have the following best possible pointwise estimate of the mean curvature.

Theorem 8 *Let $x : M \to E^m$ be a (non-minimal) submanifold of non-negative kind. Then, at each point of M, we have*

$$|H|^2 \geq (\frac{\lambda_p}{n})^2 \langle x - x_0, x - x_0 \rangle , \tag{3.2}$$

where p is the lower order of M.

The equality of (3.2) holds identically if and only if M is a minimal submanifold of a hypersphere of E^m.

Proof Let M be a submanifold of non-negative kind whose spectral decomposition is given by (3.1). If M is non-minimal, then

$$x - x_0 = \sum_{i \in T(x)} x_i, \tag{3.3}$$

where $T(x)$ is finite non-empty set. Thus, we have

$$\langle x_i(u), x_j(u) \rangle \geq 0, \quad i, j \in T(x), \quad u \in M, \tag{3.4}$$

$$-nH = \sum_{i \in T(x)} \lambda_i x_i. \tag{3.5}$$

Thus we get

$$\langle x - x_0, x - x_0 \rangle = \sum_{i,j \in T(x)} \langle x_i, x_j \rangle , \tag{3.6}$$

$$-n \langle x - x_0, H \rangle = \sum_{i,j \in T(x)} \lambda_i \langle x_i, x_j \rangle , \tag{3.7}$$

Since p is the lower order of the submanifold, (3.3)–(3.7) and the Schwartz inequality imply

$$n^2 \sum_{i,j \in T(x)} \langle x_i, x_j \rangle |H|^2 \geq (\sum_{i,j \in T(x)} \lambda_i \langle x_i, x_j \rangle)^2 \geq \lambda_p^2(\sum_{i,j \in T(x)} \langle x_i, x_j \rangle)^2 \tag{3.8}$$

which implies (3.2).

If the equality sign of (3.2) holds identically, then M is of 1-type. Since M is assumed to be a non-minimal submanifold of E^m, a well-known result of Takahashi [9] implies that M is a minimal submanifold of a hypersphere of E^m.

The converse of this is easy to verify.

■

As an immediate consequence of Theorem 8, we obtain

Corollary 9 *Let $x : M \to E^m$ be a compact submanifold of non-negative kind. Then, at each point of M, we have*

$$|H|^2 \geq (\frac{\lambda_1}{n})^2 \langle x - x_0, x - x_0 \rangle \,, \tag{3.9}$$

with equality holding identically if and only if M is a minimal submanifold of a hypersphere of E^m, immersed by eigenfunctions of the first eigenvalue λ_1 of the Laplacian of M.

Now, we recall the notions of orthogonal, pointwise orthogonal, and linearly submanifolds introduced in [4] (see, also [3]).

Let $x : M \to E^m$ be an isometric immersion whose spectral decomposition is given by (3.1). For each $i \in T(x)$, denote by E_i the subspace of E^m spanned by $\{x_i(p),\, p \in M\}$. The submanifold M is said to be *orthogonal* (respectively, *linearly independent*) if the subspaces E_i, $i \in T(x)$, are mutually orthogonal (respectively, *linearly independent*). The submanifold is said to be *pointwise orthogonal* if, for each point $u \in M$, the vectors $\{x_i(u),\ i \in T(x)\}$ are mutually orthogonal.

There exist many examples of orthogonal submanifolds and linearly independent submanifolds in E^m (cf.[3,4,5] details). Moreover, the class of pointwise orthogonal submanifolds is much wider than the class of orthogonal submanifolds. For example, every mass-symmetric spherical 2-type submanifold is pointwise orthogonal, although it is not orthogonal in general (cf. [3]).

Since pointwise orthogonal submanifolds are of non-negative kind, Theorem 8 and Corollary 9 imply immediately the following.

Corollary 10 *Let $x : M \to E^m$ be a non-minimal, pointwise orthogonal submanifold. Then, at each point of M, we have*

$$|H|^2 \geq (\frac{\lambda_p}{n})^2 \langle x - x_0, x - x_0 \rangle \,, \tag{3.10}$$

where p is the lower order of M.

The equality of (3.10) holds identically if and only if M is of 1-type.

Corollary 11 *Let $x : M \to E^m$ be a compact, pointwise orthogonal submanifold. Then, at each point of M, we have*

$$|H|^2 \geq (\frac{\lambda_1}{n})^2 \langle x - x_0, x - x_0 \rangle \,, \tag{3.11}$$

with equality holding identically if and only if M is a minimal submanifold of a hypersphere of E^m, immersed by eigenfunctions of the first eigenvalue λ_1 of the Laplacian of M.

Remark 3.1. It was proved in [4,5] that an immersion $x : M \to E^m$ is linearly independent if and only if it satisfies the Dillen-Pas-Verstraelen condition introduced in [6], *i.e.*, $\Delta x = Ax + B$ for some $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$. It is easy to see that a linearly independent immersion is of non-negative kind if and only if each entry of A is non-negative.

4 Some additional inequalities

In this section we give the following inequalities.

Proposition 12 *Let M be a compact n-dimensional submanifold of E^m. Then*

$$(\frac{\lambda_q^2}{n})V \geq \int_M \langle dH, dH\rangle\, dV \geq (\frac{\lambda_p^2}{n})V, \tag{4.1}$$

where $[p, q]$ is the order of M. Either equality of (4.1) holds if and only if M is of 1-type.

Furthermore, if M is contained in the closed ball $\overline{B_0(R)}$ and lies in outside of the open ball $B_0(r)$, then we also have

$$\frac{\lambda_q{}^3}{n^2}(R^2 - c^2)V \geq \int_M \langle dH, dH\rangle\, dV \geq \frac{\lambda_p^3}{n^2}(r^2 - c^2)V, \tag{4.2}$$

where $[p, q]$ is the order of M in E^m. Either one of the equality signs of (4.2) holds if and only if M is lies on a hypersphere centered at the origin and M is of 1-type.

Proof Consider the L^2-inner product:

$$(dH, dH) = \int_M \langle dH, dH\rangle\, dV, \tag{4.3}$$

where d denotes the exterior differential operator acting on E^m-valued functions on M. Let δ denote the co-differential operator. Then we have

$$(dH, dH) = (\delta dH, H) = (\Delta H, H). \tag{4.4}$$

From (1.1), (4.3) and (4.4) we find

$$n^2 \int_M \langle dH, dH\rangle\, dV = \sum_{i,j\in T(x)} (\lambda_i^2 x_i, \lambda_j x_j). \tag{4.5}$$

Becuase $(x_i, x_j) = 0$ for $i \neq j$, we get

$$n^2 \int_M \langle dH, dH\rangle\, dV = \sum_{i\in T(x)} \lambda_i^3 (x_i, x_i). \tag{4.6}$$

On the other hand, we have

$$\sum_{i\in T(x)} \lambda_i^3(x_i,x_i) \geq \lambda_p^2 \sum_{i\in T(x)} \lambda_i(x_i,x_i) = -n\lambda_p^2 \int_M \langle x, H\rangle\, dV = n\lambda_p^2 V. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain the right inequality of (4.1). The left inequality of (4.1) can be obtained in a similar way.

For (4.2), we consider the following inequality:

$$\sum_{i\in T(x)} \lambda_i^3(x_i,x_i) \geq \lambda_p^3 \sum_{i\in T(x)} (x_i,x_i) = \lambda_p^3((x,x) - c^2V) \geq \lambda_p^3((r^2-c^2)V. \tag{4.8}$$

(4.6) and (4.8) implies the right inequality of (4.2). The left inequality of (4.2) can be proved in a similar way.

The remaining part of the Proposition can be verified easily.

∎

Remark 4.1. It is easy to see that $\langle dH, dH\rangle$ satisfies the following identity:

$$\langle dH, dH\rangle = ||DH||^2 + ||A_H||^2, \tag{4.9}$$

where D and A denote the normal connection and the shape operator of M in E^m, respectively.

Remark 4.2. By using the similar argument as given in Proposition 12, we can also obtain the following inequalities for a compact submanifold in E^m:

$$(\frac{\lambda_q^3}{n})V \geq \int_M \langle \Delta H, \Delta H\rangle\, dV \geq (\frac{\lambda_p^3}{n})V, \tag{4.10}$$

Either equality of (4.1) holds if and only if M is of 1-type.

Furthermore, if M is contained in the closed ball $\overline{B_0(R)}$ and lies outside of the open ball $B_0(r)$, then we also have

$$\left(\frac{{\lambda_q}^2}{n}\right)^2 (R^2-c^2)V \geq \int_M \langle \Delta H, \Delta H\rangle\, dV \geq \left(\frac{\lambda_p^2}{n}\right)^2 (r^2-c^2)V, \tag{4.11}$$

where $[p,q]$ is the order of M in E^m. Either one of the equality signs of (4.11) holds if and only if M is lies in a hypersphere centered at the origin and M is of 1-type.

Remark 4.3. In [1, page 271], we have the following formula:

$$\Delta H = \Delta^D H + ||A_\xi||^2 H + a(H) + (\Delta H)^T, \tag{4.12}$$

where Δ^D is the Laplacian operator associated with the normal connection D, ξ a unit vector parallel to H, $a(H)$ the allied mean curvature vector, and $(\Delta H)^T$ the tangential component of ΔH.

For example, if M has parallel mean curvature vector, *i.e.*, $DH = 0$, then (4.12) reduces to

$$\Delta H = ||A_\xi||^2 H + a(H). \tag{4.13}$$

Thus, in this case, (4.10) yields

$$(4.14)\qquad (\frac{\lambda_q^3}{n})V \geq \int_M \{||A_\xi||^4|H|^2 + |a(H)|^2\}dV \geq (\frac{\lambda_p^3}{n})V,$$

$$(4.15)\qquad \left(\frac{{\lambda_q}^2}{n}\right)^2 (R^2 - c^2)V \geq \int_M \{||A_\xi||^4|H|^2 + |a(H)|^2\}dV \geq \left(\frac{\lambda_p^2}{n}\right)^2 (r^2 - c^2)V.$$

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