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Edited by: Mirela Ştefănescu and Klaus W. Roggenkamp

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Preface

Here you are some papers of the participants at the Euroconference on Algebra, i: August 2000, held at the Ovidius University of Constantza.

The aim of these proceedings is to give the readers an idea about the scientifi activities developed in the afternoons of the conference.

The lectures given in the mornings have been published in NATO Science Series II, Mathematics, Physics and Chemistry, vol. 28 appeared at Kluwer Academi Publishers in 2001.

Six of the papers have continued ideas presented in the morning sessions, namel; they are concerned with representation theory of groups and algebras - (Gudivok & ChukhraJ, Antipov & Antipova & Kemer, Kirichenko, Malinin, Rump, Williams).

Almost all the other papers present some new properties of group algebras, Li superalgebras, groups, special modules and rings.

K. W. Roggenkamp's paper is concerned with general ideas on the rôle of universitie in our days.

Papers have been included into the volume following the opinion of referees.

We address our thanks for the good job done by Dr. Viviana Ene in preparation o the volume for being published.

And again we are grateful to the University of Constantza and the local people ii organizing committee who have done the sessions running smoothly.

Mirela Ştefănescı

K.W. Roggenkam]

CONTENTS

CONJUGACY CLASSES OF THE GROUP OF UNITS IN GROUP ALGEBRAS OF FINITE p-GROUPS

A.A. Bovdi and C. Polcino Milies

Dedicated to Professor L.G. Kovács on his 65th birthday

Abstract

Let $\mathbb{F}_{p^n} G$ be the group algebra of a finite p-group G over a field $\mathbb{F}_{p^n}$ with p^n elements, and $V(\mathbb{F}_{p^n} G)$ the subgroup of units of augmentation 1. We investigate the conjugacy classes of $V(\mathbb{F}_{p^n} G)$, showing that p^{2n} divides the order of the conjugacy class $|Ca|$ for a noncentral element $a \in \mathbb{F}_{p^n} G$ and $V(\mathbb{F}_{p^n} G)$ contains a conjugacy class of order p^{2n}, if a nonabelian finite p-group G has a factor group G/H such that its centre is of index p^2.

Introduction*.

Let $\mathbb{F}_{p^n} G$ be the group algebra of a finite p-group G over a field $\mathbb{F}_{p^n}$ with p^n elements, and $V(\mathbb{F}_{p^n} G)$ the subgroup of units of augmentation 1. The group $V(\mathbb{F}_{p^n} G)$ is always a finite p-group of order $p^{n(|G|-1)}$ and it coincides with $1 + A(\mathbb{F}_{p^n} G)$, where the augmentation ideal $A(\mathbb{F}_{p^n} G)$ of $\mathbb{F}_{p^n} G$ is nilpotent. One the hardest and most important problems for modular group algebras consists of describing the structure of $V(\mathbb{F}_{p^n} G)$, which has a complicated structure, even for a fairly simple p-group G.

Note that the Lie structure of the associated Lie algebra of $\mathbb{F}_{p^n} G$ reflects well the characteristics of the group of units and there is a close relationship between the properties of these two structures. Moreover, the Lie methods and the Jennings'theory made possible to characterize the group of units $V(\mathbb{F}_{p^n} G)$ under different group-theoretical assumptions and to determine the exponent and the nilpotency class of $V(\mathbb{F}_{p^n} G)$ for some classes of p-groups G[2].

Recently, Bovdi and Patay [4] described the centre of $V(\mathbb{F}_{p^n} G)$. In this paper we study conjugacy classes of $V(\mathbb{F}_{p^n} G)$ for a finite p-group G. We improve the result of Rao and Sandling [8] and we prove that p^{2n} divides the order $|C_a|$ for every noncentral element $a \in \mathbb{F}_{p^n} G$, showing that $V(\mathbb{F}_{p^n} G)$ contains a conjugacy class of order p^{2n}, if the nonabelian finite p-group G has a factor group G/H such that its centre is of index p^2.

CONJUGACY CLASSES

Coleman [5] has shown that, if G is a finite p-group and C_u is a conjugacy class in $V(\mathbb{F}_{p^n} G)$ that contains an element from G then $C_u \cap G$ is a conjugacy class in G. Now we will give a conjugacy class C_u containing no elements u from G such that $|C_u| > 1$.

We freely use the fact that if $\varepsilon(x)$ denotes the sum of coefficients of the element $x \in \mathbb{F}G$, then $\varepsilon : \mathbb{F}G \to \mathbb{F}$ is a homomorphism. Also we use the notation $\widehat{M}$ to indicate the sum of all elements of a finite subset $M \subset G$ in $\mathbb{F}G$.

Let c be the nilpotency class of a finite p-group G. We denote by G_{c-1} the $(c-1)$-th term of the lower central series of G. If $g \in G_{c-1} \backslash G_c$, then the map $h \to (g,h)$ is a homomorphism of G onto the central subgroup $W_g = \{(g,h)|h \in G\}$ and the conjugacy class C_g of G coincides with gW_g.

Proposition. *Let G be a finite p-group and let $g_1, g_2, ..., g_s$ be elements of the subset $G_{c-1} \backslash G_c$ not pairwise conjugate in G. If $\mathbb{F}$ is a field of characteristic p and $\zeta(G)$ is the centre of G, then the conjugacy classes*

$$\left\{ C_{g_i + z_{g_i}} \mid 0 \neq z_{g_i} \in (\mathbb{F}\zeta(G))\widehat{W}_{g_i}, \quad i = 1, 2, ..., s \right\}$$

of $V(\mathbb{F}G)$ are pairwise different and these conjugacy classes do not contain elements from G.

Proof. Let $g \in G_{c-1} \backslash G_c$. First, we will prove that, for any $0 \neq z \in (\mathbb{F}\zeta(G))\widehat{W}_g$, the element $g + z$ is not conjugate to elements of G in $V(\mathbb{F}G)$.

Suppose that $y^{-1}(g + z)y = g_1 \in G$. It is easy to see that $\widehat{W}_g$ and $g\widehat{W}_g$ are central elements of the group algebra, hence $g_1\widehat{W}_g = y^{-1}(g + z)\widehat{W}_g y = $
$= y^{-1}g\widehat{W}_g y = g\widehat{W}_g$. It follows that $g_1 = gw$, for some $w \in W$ and the elements g and g_1 are conjugate in G. Therefore, we can actually choose $y \in V(\mathbb{F}G)$ such that $y^{-1}(g + z)y = g$. Set $H = C_G(g)$. Clearly, $Supp(z) \subseteq H$ and

$y = x_0 + x_1 u_1 + \ldots + x_s u_s$, where $x_i \in \mathbb{F}H$ and $1, u_1, \ldots, u_s$ are element in distinct cosets of H. Then

$$(g+z)x_0 + (g+z)x_1 u_1 + \ldots + (g+z)x_s u_s =$$

$$= x_0 g + x_1 g(g, u_1^{-1})u_1 + \ldots + x_s g(g, u_s^{-1})u_s.$$

Since (g, u_i^{-1}) is central and belongs to H, from the previous equality obtain that

$$x_0 z = 0 \text{ and } x_i(1 + g^{-1}z - (g, u_1^{-1})) = 0$$

for $i = 1, 2, \ldots, s$. Recall that if $\varepsilon(x) \neq 0$, then x is a unit. Since $(g, u_i^{-1}) \in$ and $Supp(g^{-1}z) \subseteq g^{-1}G_t \neq G_t$, it follows that $1 + g^{-1}z - (g, u_i^{-1}) \neq 0$ and (1) we obtain $\varepsilon(x_i) = 0$ for $i \geq 1$. Similarly, $\varepsilon(x_0) = 0$ and $\varepsilon(y) = 0$, which impossible.

Let $g_i, g_j \in G_{c-1} \backslash G_c$ and $0 \neq z_k, v_k \in (\mathbb{F}\zeta(G))\widehat{W_{g_k}}$. Suppose that

$$y^{-1}(g_i + z_i)y = g_j + v_j$$

for some z_i and v_j. If $i = j$, then $y^{-1}(g_i + (z_i - v_i))y = g_i$ and by the previc statement $v_i = z_i$. Finally, assume that $g_i, g_j \in G_{c-1} \backslash G_c$ are not conjugate G. Then

$$g_i \widehat{W_{g_i}} = y^{-1}(g_i + z_i)y \widehat{W_{g_i}} = (g_j + v_j)\widehat{W_{g_i}} = g_j \widehat{W_{g_i}} + v_j \widehat{W_{g_i}}.$$

Therefore, $g_j \widehat{W_{g_i}}$ is a central element in $\mathbb{F}G$. It follows that $(g_i, g)\widehat{W_{g_i}} =$ for any $g \in G$ and $W_{g_j} \subseteq W_{g_i}$. By symmetry we obtain that $W_{g_i} = W_{g_j}$. T $v_j \widehat{W_{g_i}} = 0$, $g_i \widehat{W_{g_i}} = g_j \widehat{W_{g_i}}$ and $i = j$, which is impossible.

As it is well-known [3] that, if G is an extension of a finite group M of order by the dihedral group D_6 of order 6, then $V(\mathbb{F}_2 G)$ contains a conjug class of order 2. Rao and Sandling [8] proved that if G is a finite p-group, t p^2 divides the order of every non singleton conjugacy class of $V(\mathbb{F}_p G)$. In f the question of whether there exists a conjugacy class of order p^{2n} in $V(\mathbb{F}_p$ is still open. We will show that there exists $u \in \mathbb{F}_{p^n} G$ such that a conjug class C_u is of order p^{2n}, if G has a factor group G/H such that its cent of index p^2. We consider the following three cases in which the centraliz some element has this property.

Case 1. Let G be a finite 2-group and G has a factor group G/H commutator subgroup of order 2 and its centre is of index 4. Then there elements $a, b \in G$ such that $(a, b) = c \notin H$. Let L/H be the centraliz $\{aH, bH\}$ in G/H. Then L is normal in G and its index equals 4. The elei $\widehat{H}$ is central in $\mathbb{F}_{2^n} G$ and

$$za\widehat{H} = a\widehat{H}z \text{ and } zb\widehat{H} = b\widehat{H}z,$$

for every $z \in \mathbb{F}_{2^n} L$. Choose any transversal ω of L in G'. Then $\widehat{L} = \widehat{\omega}(1+c)\widehat{H}$ and $y = 1 + a(\alpha\widehat{\omega} + \beta\widehat{\omega}c)\widehat{H}$ is a noncentral unit in $\mathbb{F}_{2^n} G$ for $\alpha \neq \beta$, where $\alpha, \beta \in \mathbb{F}_{2^n}$. By (2), the element $x = (x_0 + x_1 a) + (x_2 + x_3 a)b \in V(\mathbb{F}_{2^n} G)$, with $x_i \in \mathbb{F}_{2^n} L$, belongs to the centralizer of the element $y = 1 + a(\alpha\widehat{\omega} + \beta\widehat{\omega}c)\widehat{H}$ if and only if

$$(\alpha\widehat{\omega} + \beta\widehat{\omega}c)x_2(1+c)\widehat{H} = 0 \text{ and } (\alpha\widehat{\omega} + \beta\widehat{\omega}c)x_3(1+c)\widehat{H} = 0.$$

Since $\alpha + \beta \neq 0$, these conditions can be written as

$$\widehat{L}x_2 = 0 \text{ and } \widehat{L}x_3 = 0. \tag{3}$$

It follows that x_2 and x_3 are not units and, as it is well-known, we have that $\varepsilon(x_2) = \varepsilon(x_3) = 0$. Conversely, any elements $x_2, x_3 \in \mathbb{F}_{2^n} L$ with $\varepsilon(x_2) = \varepsilon(x_3) = 0$ satisfy condition (3).

We have proved that the unit $x = (x_0 + x_1 a) + (x_2 + x_3 a)b$ belongs to the centralizer of $y = 1 + a(\alpha\widehat{\omega} + \beta\widehat{\omega}c)\widehat{H}$ if and only if $\varepsilon(x_0 + x_1) = 1$ and $\varepsilon(x_2) = \varepsilon(x_3) = 0$. Therefore, we showed that the element $y \in \mathbb{F}_{2^n} G$ has the centralizer of order $2^{|G|-3}$ and the order of the conjugacy class C_y equals 2^{2n}.

Case 2. Suppose that $p > 2$ and there exist a factor group G/H with commutator subgroup of order p and elements $a, b \in G$ such that $P = <a, b, H>/H$ is a non-metacyclic subgroup of G/H.

According to Lemma 2.5 in [1], the elements a and b can be chosen in such a way that P has the following defining relations

$$(aH)^{p^m} = H, \ (bH)^{p^n} = H, \ b^{-1}abH = acH, \ (cH)^p = H, \ (a,c),(b,c) \in H,$$

where $c = (a,b)$ and p^m, p^n are invariants of the abelian group $P/<cH>$. It is clear that $c \notin H_1 = <a^p, b^p, H>$ and we can suppose that H is chosen in such a way that $H = H_1$. Again, let L/H be the centralizer of $\{aH, bH\}$ in G/H.

We set

$$y = (b-1)^{p-1}(a-1)^{p-1}\eta\widehat{\omega} \text{ and } \eta = (c-1)^{p-2}\widehat{H},$$

where ω is any transversal of HG_2 in L. The ideal $I(H)$ generated by the elements of the form $h-1$ with $h \in H$ is such that $\mathbb{F}_{p^n} G/\mathcal{I}(H) \cong \mathbb{F}_{p^n}[G/H]$. One verifies easily that any element $u \in V(\mathbb{F}_{p^n} G)$ can be written as

$$u = z + \sum_{\substack{i,j=0 \\ i+j \geq 1}}^{p-1} (a-1)^i(b-1)^j\omega_{ij}, \tag{4}$$

where $z, \omega_{ij} \in \mathbb{F}_{p^n} L$.

We wish to decide when u belongs to the centralizer $C_V(1 + y)$. This condition is satisfied if and only if the Lie commutator $[y, u] = 0$.

We will freely use the following facts:

1. the well-known identities $[u_1 u_2, u_3] = [u_1, u_3]u_2 + u_1[u_2, u_3]$ and $[u_1, u_2 u_3] = [u_1, u_2]u_3 + u_2[u_1, u_3]$, where $u_i \in \mathbb{F}_{p^n} G$;

2. if the subgroup W of G contains the commutator subgroup G', then in the product $u_1 u_2 ... u_s \widehat{W}$ every two elements u_i and u_j commute.

We first establish a number of results, which will be needed to compute the Lie commutator $[y, u]$.

Of course, $\alpha \widehat{H} = 0$ for every $\alpha \in \mathcal{I}(H)$. Since L/H in the centralizer of the elements aH and bH in G/H, we have $[w, (a - 1)^r (b - 1)^k] \in \mathcal{I}(H)$ for every $w \in \mathbb{F}_{p^n} L$ and

$$[w, (a - 1)^r (b - 1)^k \widehat{H}] = 0. \tag{5}$$

Recall that $\widehat{L} = \widehat{\omega}(c - 1)\eta = \widehat{\omega}(c - 1)^{p-1}\widehat{H}$. We claim that the element $(c - 1)^k \widehat{H}$ is central. Indeed, the element cH is central in G/H and $(c, g) \in H$ for all $g \in G$, thus

$$g^{-1}(c - 1)^k \widehat{H} g = (c(c, g) - 1)^k \widehat{H} = (c - 1)^k \widehat{H}. \tag{6}$$

Suppose that $[w, y] \neq 0$ for some $w \in \mathbb{F}_{p^n} L$. It is easy to see that $[w, \widehat{\omega}]\widehat{H} = w_1(c - 1)\widehat{H}$, where $w_1 \in \mathbb{F}_{p^n} L$. From (5) and (6) we have

$$[w, y] = [w, (b-1)^{p-1}(a-1)^{p-1}](c-1)^{p-2}\widehat{H}\widehat{\omega} + (b-1)^{p-1}(a-1)^{p-1}(c-1)^{p-2}\widehat{H}[w, \widehat{\omega}]$$

$$= (b - 1)^{p-1}(a - 1)^{p-1}(c - 1)^{p-1}\widehat{H}w_1 = (b - 1)^{p-1}(a - 1)^{p-1}w_1\widehat{H}, \tag{7}$$

where $W = <c, H>$. Since $G' \subseteq W$, we conclude that $[w, y]$ is central in $\mathbb{F}_{p^n} G$.

Let $i + j \geq 1$. Note that $a^p, b^p \in H$ so it follows that $(a-1)^p \widehat{H} = (b-1)^p \widehat{H} = 0$. Since $x\widehat{H}$ is central in $\mathbb{F}_{p^n} G$ for any $x \in \mathbb{F}_{p^n} G$, according to (7), we have

$$(1 - 1)^i(b - 1)^j[w, y] = (a - 1)^i(b - 1)^j(b - 1)^{p-1}(a - 1)^{p-1}w_1\widehat{H} = 0. \tag{8}$$

Clearly, $(b - 1)y = y(a - 1) = 0$ and (7)-(8) yields

$$[y, u] = [y, z] + \sum_{\substack{i + j \geq 1 i, j = 0}}^{p-1} [y, (a - 1)^i(b - 1)^j]w_{ij} =$$

$$= [y, z] + \sum_{i=1}^{p-1}[y, (a - 1)^i]w_{i0} + \sum_{i=1}^{p-1}[y, (b - 1)^i]w_{0i}. \tag{9}$$

Since $G' \subseteq L$, the element $x\widehat{L}$ is central for any $x \in \mathbb{F}_{p^n} G$ and, as a consequence of this argument and (6), we have

$$(a-1)(c-1)(b-1)^k(a-1)^{p-1}\eta\widehat{\omega} = (a-1)(b-1)^k(a-1)^{p-1}(c-1)^{p-1}\widehat{H}\widehat{\omega} =$$
$$= (a-1)(b-1)^k(a-1)^{p-1}\widehat{L} = (a-1)(a-1)^{p-1}\widehat{L}(b-1)^k = 0. \qquad (10)$$

It can easily be observed that

$$(a-1)(b-1)=(b-1)((a-1)(c-1)+(a-1)+(c-1))+(a-1)(c-1)+(c-1) \quad (11)$$

and, in view of (10), a routine calculation shows that

$$(a-1)y = (a-1)(b-1) \cdot (b-1)^{p-2}(a-1)^{p-1}\eta\widehat{\omega} =$$
$$= (b-1)(a-1)(b-1)^{p-2}(a-1)^{p-1}\eta\widehat{\omega} + ((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p-1}\widehat{L} =$$
$$= (b-1)\cdot(a-1)(b-1)\cdot(b-1)^{p-3}(a-1)^{p-1}\eta\widehat{\omega} + ((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p-1}\widehat{L} =$$
$$= (b-1)^2(a-1)(b-1)^{p-3}(a-1)^{p-1}\eta\widehat{\omega} + 2((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p-1}\widehat{L} =$$
$$= ... = (b-1)^{p-2}(a-1)(b-1)(a-1)^{p-1}\eta\widehat{\omega} + (p-2)((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p-1}\widehat{L} =$$
$$= -((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p-1}\widehat{L} \qquad (12)$$

and $[y, (a-1)] = -(a-1)y$. Since $G' \subseteq L$, the element $(a-1)y$ is central by (12) and for $k > 1$ we have

$$[y, (a-1)^k] = -(a-1)y(a-1)^{k-1} = ((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p-1}(a-1)^{k-1}\widehat{L} = 0.$$

As before, a similar argument shows that

$$(b-1)^{p-1}(a-1)^k(b-1)(a-1)^l(c-1)^{p-1}\widehat{\omega}\widehat{H} = 0$$

and

$$y(b-1) = -(b-1)^{p-1}((a-1)^{p-2}+(a-1)^{p-1})\widehat{L}.$$

Thus

$$[y, b-1] = y(b-1) = -(b-1)^{p-1}((a-1)^{p-1}+(a-1)^{p-2})\widehat{L},$$

hence $y(b-1)$ is central. Thus, for $k > 1$, we conclude

$$[y, (b-1)^k] = y(b-1)^k = (b-1)^{k-1}y(b-1) = 0.$$

Note that $\widehat{L}w_{kl} = \varepsilon(w_{kl})\widehat{L}$. By the preceding discussion, it follows that the Lie commutator (9) reduces easily to the form

$$[y, u] = [y, z] + \varepsilon(w_{10}((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p-1}\widehat{L} -$$

$$-\varepsilon(w_{01})(b-1)^{p-1}((a-1)^{p-1}+(a-1)^{p-2})\widehat{L}.$$

Now suppose that L/H is abelian. Then $[\widehat{\omega},z] \in \mathcal{I}(H)$ and

$$[y,z] = (b-1)^{p-1}(a-1)^{p-1}(c-1)^{p-2}\widehat{H}[\widehat{\omega},z] = 0.$$

We obtain that, if L/H is abelian, then u belongs to the centralizer $C_{\mathbb{F}_{p^n}G}(1+y)$ if and only if

$$\varepsilon(w_{10}) = 0 \text{ and } \varepsilon(w_{01}) = 0. \tag{13}$$

Since $|V(\mathbb{F}_{p^n}G)| = |\mathbb{F}_{p^n}|^{|G|-1}$ and by (13), the centralizer of $y \in \mathbb{F}_{p^n}G$ has the order $p^{n(|G|-3)}$ and the order of the conjugacy class C_y equals p^{2n}.

Case 3. Now assume that $p > 2$ and G has a factor group G/R with commutator subgroup of order p such that $G/R =< a,b,R >/R$ is a metacyclic group.

By the classification of metacyclic p-groups, which was obtained by Newman and Xu in [6], the elements a and b can be chosen such that P has the following presentation:

$$P = \Big\langle aR, bR \mid (aR)^{p^{r+s+k}} = R, (bR)^{p^{r+s+t}} = (aR)^{p^{r+s}}, \; b^{-1}abR = a^{1+p^r}R \Big\rangle,$$

where s,k are non-negative integers, $r \geq 1$ and $k \leq r$. Since the element $a^{p^r}H$ has order p, we conclude that $s+k=1$ and the group P splits [8].

Let $H =< b^p, R >$. Then G/H has the following presentation:

$$a^{p^{r+1}} \equiv 1 \pmod{H}, \; b^p \equiv 1 \pmod{H}, \; b^{-1}ab \equiv a^{1+p^r} \pmod{H}. \tag{14}$$

We set

$$y = (b-1)^{p-1}(a-1)^{p^r-1}(a^{p^r}-1)^{p-2}\widehat{H} = (b-1)^{p-1}(a-1)^{p^{r+1}-p^r-1}\widehat{H}.$$

In this situation, the identity (11) has the form

$$(a-1)(b-1) = (b-1)((a-1)^{p^r+1} + (a-1) + (a-1)^{p^r})+$$

$$+(a-1)^{p^r+1} + (a-1)^{p^r} \pmod{\mathcal{I}(H)}. \tag{15}$$

Let $W =< a^{p^r}, H >$. As in (9), it is easy to show that $(a^{p^r}-1)^i\widehat{H}$ is central in $\mathbb{F}_{p^n}G$.

Thus

$$(a-1)^{p^r+1}(b-1)^k(a-1)^{p^r-1}(a^{p^r}-1)^{p-2}\widehat{H} =$$

$$= (a-1)(b-1)^k(a-1)^{p^r-1}\widehat{W} = 0, \tag{16}$$

since $(a-1)\widehat{W}$ is central and $(a^{p^r}-1)\widehat{W}=0$.

We use the preceding method and computations as in (12) to establish

$$(a-1)y = (a-1)(b-1)\cdot(b-1)^{p-2}(a-1)^{p^r-1}(a^{p^r}-1)^{p-2}\widehat{H} = .$$

$$= (b-1)(a-1)(b-1)^{p-2}(a-1)^{p^{r+1}-p^r-1}\widehat{H} + ((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p^r-1}\widehat{W} =$$

$$= ... = (b-1)^{p-1}\widehat{W} - ((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p^r-1}\widehat{W},$$

by (16). Therefore,

$$(a-1)y = (b-1)^{p-1}\widehat{W} + ((b-1)^{p-1}-(b-1)^{p-2})(a-1)^{p^r-1}\widehat{W}. \qquad (17)$$

As $G' \subseteq W$, it follows that $(a-1)y$ is central in $\mathbb{F}_{p^n}G$ and, for $k > 1$, we have

$$(a-1)^k y = (a-1)y(a-1)^{k-1} = (b-1)^{p-1}(a-1)^{k-1}\widehat{W},$$

by (16). It is easy to see that $y(a-1) = (b-1)^{p-1}\widehat{W}$ and for $k \geq 1$

$$y(a-1)^k = (b-1)^{p-1}(a-1)^{k-1}\widehat{W}.$$

Therefore,

$$[y, a-1] = ((b-1)^{p-1}+(b-1)^{p-2})(a-1)^{p^r-1}\widehat{W} \qquad (18)$$

and $[y, (a-1)^k] = 0$, for $k > 1$.

Finally, as before,

$$(b-1)^{p-1}(a-1)^{p^r-i}(b-1)(a-1)^{p^r+j}(a^{p^r}-1)^{p-2}\widehat{H} =$$

$$(b-1)^{p-1}(a-1)^{p^r-i}(b-1)(a-1)^j\widehat{W} = 0.$$

It follows that

$$[y, b-1] = (b-1)^{p-1}(a-1)^{r^r-3}\cdot(a-1)(b-1)\cdot(a-1)(a^{p^r}-1)^{p-2}\widehat{H}+$$

$$+(b-1)^{p-1}((a-1)^{p^r-1}+(a-1)^{p^r-2})\widehat{W} = ... =$$

$$= -(b-1)^{p-1}((a-1)^{p^r-1}+(a-1)^{p^r-2})\widehat{W}, \qquad (19)$$

hence $y(b-1)$ is central and $[y, (b-1)^k] = 0$, for $k > 1$. It yields that, if $i+j > 1$, then

$$y(a-1)^i(b-1)^j = 0 \qquad (20)$$

One verifies easily that any element $u \in V(\mathbb{F}_{p^n}G)$ can be written as

$$u = \sum_{i=0}^{p^{r+1}-1}\sum_{j=0}^{p-1} w_{ij}(a-1)^i(b-1)^j + \alpha,$$

where $w_{ij} \in \mathbb{F}_{p^n}$ and $\alpha \in \mathcal{I}(H)$. As before, $\alpha\widehat{H} = 0$, thus the Lie commutator $[\alpha, y] = 0$. We determine when u belongs to the centralizer $C_V(1 + y)$. This condition is satisfied if and only if the Lie commutator $[y, u] = 0$. By (18)-(20), we have

$$[y, u] = w_{10}[y, a - 1] + w_{01}[y, b - 1] =$$

$$= w_{10}((b - 1)^{p-1} + ((b - 1)^{p-1} + (b - 1)^{p-2})(a - 1)^{p-1})\widehat{W} -$$

$$- w_{01}(b - 1)^{p-1}((a - 1)^{p-1} + (a - 1)^{p-2})\widehat{W}.$$

We conclude that u belongs to the centralizer $C_V(1 + y)$ if and only if

$$w_{10} = 0 \text{ and } w_{01} = 0.$$

Therefore, the centralizer has the order $|\mathbb{F}_{p^n}|^{|G|-3}$, so it follows that the order of the conjugacy class C_{1+y} is p^{2n}.

Theorem. *Let G be a finite p-group and $\mathbb{F}$ a field of characteristic p. If u is a noncentral element in $\mathbb{F}G$ then the conjugacy class $C_u = \{x^{-1}ux \mid x \in V(\mathbb{F}G)\}$ has the following properties:*
  (i) if $\mathbb{F}$ is an infinite field then C_u is infinite;
  (ii) if $|\mathbb{F}| = p^n$, then $p^{2^n} \mid |C_u|$.
Moreover, if a nonabelian finite p-group G has a factor group G/H such that its centre is of index p^2, then $V(\mathbb{F}_{p^n}G)$ has a conjugacy class of order $|\mathbb{F}_{p^n}|^2$.

Proof. Let $H = C_G(u)$ and $a \in G \backslash H$. Then $(1 - \alpha)^{-1}(a - \alpha) \in V(\mathbb{F}G)$ for $1 \neq \alpha \in \mathbb{F}$, $(1 - \alpha)(a - \alpha)^{-1}u(1 - \alpha)^{-1}(a - \alpha) = (a - \alpha)^{-1}u(a - \alpha)$ and the elements from the subset

$$\{(a - \alpha)^{-1}u(a - \alpha) \mid 1 \neq \alpha \in \mathbb{F}\}$$

are pairwise different for any a. Indeed, if $(a - \alpha)^{-1}u(a - \alpha) = (a - \beta)^{-1}u(a - \beta)$ then

$$u(a - \alpha)(a - \beta)^{-1} = u(a - \beta + \beta - \alpha)(a - \beta)^{-1} = u(1 + (\beta - \alpha)(a - \beta)^{-1})$$

and we obtain

$$(\alpha - \beta)(u(a - \alpha)^{-1} - (a - \alpha)^{-1}u) = 0.$$

Therefore, $ua = au$, which is impossible. Thus, if $\mathbb{F}$ is infinite, then C_u is also infinite and, if $\mathbb{F}$ is a finite field then $|C_u| \geq |\mathbb{F}|$.

Let $\mathbb{F}_{p^n}$ be a finite field of p^n elements and denote by $C_{\mathbb{F}_{p^n}G}(u)$ the centralizer of a noncentral element u in $\mathbb{F}_{p^n}G$ and $V = V(\mathbb{F}_{p^n}G)$. Since $C_{\mathbb{F}_{p^n}G}(u) = \mathbb{F}_{p^n} + C_V(u)$ and $|C_{\mathbb{F}_{p^n}G}(u)| = p^n \cdot |C_V(u)|$, we have

$$|V : C_V(u)| = |\mathbb{F}_{p^n}G : C_{\mathbb{F}_{p^n}G}(u)| = p^{kn}, \tag{21}$$

where k is the codimension of the linear subspace $C_{\mathbb{F}_{p^n} G}(u)$ of $\mathbb{F}_{p^n} G$ over $\mathbb{F}_{p^n}$.

Suppose that $C_{\mathbb{F}_{p^n} G}(u)$ has the codimension $k = 1$. Then $|C_u| = p^n$, whence it follows that

$$\{u, (a - \alpha)^{-1} u(a - \alpha) \mid 1 \neq \alpha \in \mathbb{F}_{p^n}\}$$

is the conjugacy class C_u of order p^n, for any $a \in G \backslash H$.

Clearly, if $g \in G \backslash H$, then $(g - \delta)^{-1} u(g - \delta) \in C_u$ for all $\delta \in C_{\mathbb{F}_{p^n} G}(u) \cap A(\mathbb{F}_{p^n} G)$, where $A(\mathbb{F}_{p^n} G)$ is the augmentation ideal. We will prove that

$$(g - \delta)^{-1} u(g - \delta) = g^{-1} u g \text{ for all } \delta \in C_{\mathbb{F}_{p^n} G}(u) \cap A(\mathbb{F}_{p^n} G).$$

Indeed, assume that $|\mathbb{F}_{p^n}| > 2$ and $(g - \delta)^{-1} u(g - \delta) = (g - \alpha)^{-1} u(g - \alpha)$ for some $\alpha \in \mathbb{F}_{p^n} \backslash \{0, 1\}$. Then, $(\alpha - \delta)(u(g - \delta)^{-1} - (g - \delta)^{-1} u) = 0$ and we have a contradiction, since $\alpha - \delta$ is a unit.

Therefore, we have $(g - \delta)^{-1} u(g - \delta) = g^{-1} u g$, for all $\delta \in C_{\mathbb{F}_{p^n} G}(u) \cap A(\mathbb{F}_{p^n} G)$. From this, is follows that $\delta(ug^{-1} - g^{-1}u) = 0$ for all $\delta \in C_{\mathbb{F}_{p^n} G}(u) \cap A(\mathbb{F}_{p^n} G)$. Let $\mathcal{I}(H)$ be the left ideal generated by $h - 1$ with $h \in H$. Clearly, $h - 1 \in C_{\mathbb{F}_{p^n} G}(u) \cap A(\mathbb{F}_{p^n} G)$ for all $h \in H$ and $g^{-1}u - ug^{-1}$ belongs to the right annihilator of the ideal $\mathcal{I}(H)$. It is well-known ([7], Lemma 3.1.2) that every element from the right annihilator has the following form

$$g^{-1}u - ug^{-1} = \widehat{H}(\gamma_0 + \gamma_1 v_2 + \ldots + \gamma_r v_r),$$

where $\widehat{H} = \Sigma_{h \in H} h, \gamma_k \in \mathbb{F}_{p^n}$ and $G = H \cup Hv_2 \cup \ldots \cup Hv_r$.

Let $z \in \mathbb{F}_{p^n} G$ and denote by $z(d)$ the coefficient of $d \in G$ in z. If $Hv_i = Hg^k$ for some i, then

$$\gamma_i = [g^{-1}, u](g^k) = u(g^{k+1}) - u(g^{k+1}) = 0.$$

Let $Hv_i \neq Hg^k$, for all k. In this case, i cannot be 1 and therefore $v_i \neq 1$. Then there exists $a \in G \backslash H$ such that $ag^{-1} = v_i$, so $g^{-1} u g = (a - \alpha)^{-1} u(a - \alpha)$ for some $\alpha \in \mathbb{F}_{p^n}$ and $(a - \alpha)g^{-1} \in C_{\mathbb{F}_{p^n} G}(u)$. We obtain

$$ag^{-1}u - ug^{-1}a = \alpha(g^{-1}u - ug^{-1}).$$

We remark that α cannot be 0, otherwise $v_i = ag^{-1} \in H$, i.e. $v_i = 1$, which is not the case. It yields that

$$\gamma_i = [g^{-1}, u](v_i) = \alpha^{-1}[ag^{-1}, u](v_i) = \alpha^{-1}[v_i, u](v_i) = 0.$$

We conclude that $[g^{-1}, u] = 0$, which provides the concluding contradiction. Therefore, $C_{\mathbb{F}_{p^n} G}(u)$ has the codimension $k > 1$ and by (21) p^{2n} divide the order of $|C_u|$.

We will prove that $V(\mathbb{F}_{p^n} G)$ has a conjugacy class of order p^{2n}, if G has a factor group G/H such its centre is of index p^2. Then the commutator subgroup of G has order p.

If G is a 2-group, the statement of the Theorem follows from case 1.

Assume that $P =< a, b, H > /H$ is a non-metacyclic subgroup of G/H with commutator subgroup $< cH >$ of order p. Then the centralizer of the elements aH and bH in G/H coincides with the centre of G/H and, as we have seen in case 2, $V(\mathbb{F}_{p^n} G)$ has a conjugacy class of order p^{2n}.

Therefore, we can assume that every subgroup $P =< a, b, H > /H$ of G/H is metacyclic. By the classification of metacyclic p-groups [6], a and b can be chosen in such a way that P has the following presentation:

$$P = \left\langle aH, bH \mid (aH)^{p^{r+s+u}} = H, (bH)^{p^{r+s+t}} = (aH)^{p^{r+s}}, b^{-1}abH = a^{1+p^r}H, \right\rangle$$

where s, u are non-negative integers, $r \geq 1$ and $u \leq r$. Since the element $a^{p^r} H$ is of order p, we conclude that $s + u = 1$ and the group P splits [6]. Let $Q =< b^p, H > .$ Then $P =< a, b, Q > /Q$ has the following presentation:

$$a^{p^{r+1}} \equiv 1 \pmod{Q}, \quad b^p \equiv 1 \pmod{Q}, \quad B^{-1}ab \equiv a^{1+p^r} \pmod{Q}.$$

Let $A =< a, b, Q > .$ Then $G/Q = A/Q \cdot L/Q$, where L/Q is the centre of G/Q.

Suppose that the order of some element $dQ \in L/Q$ is not less than the order of aQ. Since $< aQ, dQ >$ is abelian, it is well-known that $< aQ, dQ >=< dQ > \times < a'Q >$ and $(a'Q, bQ) = a^{p^r k}Q \in < dQ >$ for some $0 < k < p$. Thus $< a'Q, bQ >$ is not a metacyclic group, which is impossible according to our assumption.

We conclude that the element a^pQ is of maximal order in L/Q and thus L/Q is a direct product of the cyclic group $< a^pQ > /Q$ and an abelian p-group W/Q. Then W is a normal subgroup of G and we set $H_1 =< W, Q > .$ The subgroup $< aH_1, bH_1 > /H_1$ of G/H_1 is nonabelian with the commutator subgroup of order p and therefore it coincides with G/H_1. According to case 3, $V(\mathbb{F}_{p^n} G)$ has a conjugacy class of order p^{2n}.

Corollary. *If H is a normal subgroup a finite p-group G and G/H is a nonabelian group with two generators, then $V(\mathbb{F}_{p^n} G)$ has a conjugacy class of order p^{2n}.*

References

[1] Blackburn, N., *On prime-power groups with two generators*, Proc. Cambridge Phil. Soc. **54** (1958), 327-337.

[2] Bovdi, A., *The group of units of a group algebra of characteristic p*, Publ. Math. Debrecen **52** (1998), 193-244.

[3] Bovdi, A., Khripta, I., *Normal subgroups of the multiplicative group of a ring*, Math. Sbornik **16** (1972), 349-362.

[4] Bovdi, A., Patay, Z., *Units in group algebras of finite p-group*, Zapiski nauchnykh seminarov POMI **272** (2000), 14-25.

[5] Coleman, D.B., *On the modular group ring of a p-group*, Proc. Amer. Math. Soc. **5** (1964), 511-514.

[6] Newman, M.F., Mingyao Xu, *Metacyclic groups of prime-power order*, Advances in Math. **17, No.1** (1988), 106-107.

[7] Passman, D.S., *The algebraic structure off group rings*, Wiley-Interscience, New York, 1977.

[8] Rao, M., and Sandling, R., *Vanishing orbit sums in group algebras*, London Math. Soc. Lecture Note Ser. 2 (1995), 507-511.

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MODULES INJECTIVE WITH RESPECT TO MAXIMAL IDEALS

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Abstract

Let R be an associative ring with non-zero identity. An R-module D is called *m-injective* if any homomorphism from any maximal left ideal of R to D extends to R. This type of injectivity coincides with injectivity with respect to the Dickson torsion theory. We establish some properties for m-cocritical modules over a commutative ring as well as decomposition theorems for certain m-injective modules which are m-injective hulls of each of their non-zero submodules.

1 Introduction

Throughout this paper we denote by R an associative ring with non-zero identity and all modules are left unital R-modules.

Let A be an R-module. Then we denote by $Soc(A)$ the socle of A and by $E(A)$ the injective hull of A. If $0 \neq B \subseteq A$ and $0 \neq I \subseteq R$, we denote $Ann_R B = \{r \in R \mid rb = 0, \forall b \in B\}$ and $Ann_A I = \{a \in A \mid ra = 0, \forall r \in I\}$. If $0 \neq a \in A$, $Ann_R\{a\}$ is denoted by $Ann_R a$. A module A is said to be *faithful* if $Ann_R A = 0$.

A module A is said to be *semiartinian* if every non-zero homomorphic image of A contains a simple submodule [11, Chapter I, Definition 11.4.6].

A finite strictly increasing sequence $p_0 \subset p_1 \subset \ldots \subset p_n$ of prime ideals of a commutative ring R is said to be a *chain of length* n. The supremum of the lengths of all chains of prime ideals of R is called the *dimension of* R and it is denoted by $dim\,R$ [11, p.207]. If p is a prime ideal of a commutative ring R, then $dim\,R/p$ is called the dimension of p and it is denoted by $dim\,p$ [7, p.227].

2   A short survey on the topic

J.L. Johnson studied in [9] modules injective with respect to primes in a commutative ring, i.e. modules satisfying a Baer-type criterion only for prime ideals. We have considered in [3] and [5] the following generalization of injectivity for modules over a not necessarily commutative ring: in the well known Baer criterion for injective modules take only the maximal left ideals. Thus, an R-module D is called *m-injective* if any homomorphism from any maximal left ideal of R to D extends to R [3, Definition 1].

The following theorem gives a characterization of m-injective modules in terms of injectivity with respect to certain exact sequences of modules.

Theorem 2.1. [3, Theorem 3] *The following statements are equivalent for a module D:*

(i) D is m-injective.

(ii) D is injective with respect to every short exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

where C is a semiartinian module.

In order to justify the study of this type of injectivity we make a connection with torsion theories. Consider the following torsion theory:

Let $\mathcal{T}$ be the class of all semiartinian R-modules and let $\mathcal{F}$ be the class of all R-modules with zero socle. Then $\tau_D = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory, called the Dickson torsion theory [6]. The corresponding Gabriel filter F consists of all τ_D-dense left ideals I of R, i.e. all left ideals of R for which R/I is a left semiartinian R-module.

An R-module D is τ_D-*injective* if any homomorphism from any left ideal $I \in F$ to D extends to R or equivalently if D is injective with respect to every short exact sequence of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where C is τ_D-torsion.

Theorem 2.1 proves that when checking τ_D-injectivity it is enough to consider only a subset of ideals of the filter F, namely the maximal left ideals. By Theorem 2.1, a module is m-injective if and only if it is τ_D-injective. Therefore the goal is the study of modules injective with respect to the Dickson torsion theory.

We will continue by presenting some cases when the notions of injectivity and m-injectivity coincide or not. A characterization is given for commutative noetherian domains.

Proposition 2.2. [3, Corollary 13] *Let R be a commutative noetherian domain. Then every m-injective R-module is injective if and only if $\dim R \leq 1$.*

Therefore, there exist non-injective m-injective modules. An example is the following one: if R is a unique factorization domain whose maximal ideals are not principal, then R is a non-injective m-injective R-module [5, Theorem 15]. For instance, that hypothesis is verified by the ring $K[[X_1, \ldots, X_n]]$ of formal power series in n indeterminates over a field K or by the ring $K[X_1, \ldots, X_n]$ of polynomials in n indeterminates over an algebraic closed field K, where $n \geq 2$ in both cases.

Some classes of rings that have the property that every m-injective module is injective are the following ones: Dedekind domains, left semiartinian rings and left τ_D-cocritical rings.

From the general context of torsion theories, every module A has an m-injective hull, denoted by $E_m(A)$, contained in $E(A)$, unique up to an isomorphism. The structure of the m-injective hull of R/p, where p is a prime ideal of a commutative ring R, is established if $dim\, p = 1$ and $E_m(R/p)$ is a minimal m-injective module.

Theorem 2.3. [5, Theorem 4.7] *Let p be a prime ideal of a commutative ring R with $dim\, p = 1$. Suppose that $E_m(R/p)$ is a minimal m-injective submodule of $E(R/p)$. Then $E_m(R/p) = Ann_{E(R/p)}p$, there exists an R/p-isomorphism between $E_m(R/p)$ and the field of fractions of R/p and $R/p \neq E_m(R/p)$.*

A non-zero m-injective module D is said to be minimal m-injective if D is an m-injective hull of each of its non-zero submodules [5, p.147]. For a commutative noetherian ring we are able to give the structure of minimal m-injective modules.

Theorem 2.4. [5, Corollary 4.6] *Let R be a commutative noetherian ring. Then the following statements are equivalent:*

(i) D is a minimal m-injective R-module.

(ii) $D \cong E_m(R/p)$, where p is a prime ideal of R with $dim\, p \leq 1$.

Note that every minimal m-injective module is uniform. Minimal m-injective modules play an important part in the decomposition of m-injective modules. Some decomposition properties could be easily obtained from general results for torsion theories (see [1], [10]).

For all notions and results concerning torsion theories we refer to [8].

3   Some further results

We begin this section with some properties of certain minimal m-injective modules. By an *m-cocritical* module we will understand a τ_D-cocritical module, where τ_D is the Dickson torsion theory. Thus, a non-zero module A is said to be *m-cocritical* if $Soc(A) = 0$ and $Soc(A/B) \neq 0$ for every non-zero proper

submodule B of A. If the R-module R is m-cocritical, it is said that the ring R is m-cocritical. It is worth of being mentioned that minimal m-injective modules with zero socle (i.e. torsionfree with respect to the Dickson torsion theory) are exactly m-injective m-cocritical modules.

Theorem 3.1. *Let A be an m-cocritical module over a commutative ring R. Then:*

(i) $Ann_R a = Ann_R A$, for every non-zero element $a \in A$.

(ii) $Ann_R A$ is a prime ideal of R.

(iii) A is a torsion-free $R/Ann_R A$-module.

(iv) A is isomorphic to a submodule of $Ann_{E(R/Ann_R A)}(Ann_R A)$.

(v) $R/Ann_R A$ is m-cocritical.

Proof. m-cocritical modules have the property that every endomorphism of such a module is a monomorphism. Then the statements $(i) - (iv)$ follow [4, Theorem 3].

(v). Let a be a non-zero element of A. Then $R/Ann_R A = R/Ann_R a \cong Ra$. But the class of m-cocritical modules is closed under taking submodules. Since A is m-cocritical, it follows that $R/Ann_R A$ is m-cocritical. $\qquad\square$

Corollary 3.2. *Let A be an m-cocritical module over a commutative ring R. Denote $p = Ann_R A$ and let B be a non-zero submodule of $Ann_{E(A)}p$. Then B is m-cocritical.*

Proof. Since $Soc(A) = 0$, it follows $Soc(E(A)) = 0$, hence $Soc(B) = 0$. Since A is m-cocritcal, A is uniform. Now let D be a non-zero proper submodule of B. Then D is essential in B. Let $b \in B \setminus D$. Since $Ann_R b = p$, we have $Rb \cong R/p$, hence by Theorem 3.1, Rb is m-cocritical. We have $Rb/(Rb \cap D) \cong (Rb+D)/D$. Since $Rb \cap D \neq 0$ and Rb is m-cocritical, it follows that $Rb/(Rb \cap D)$ is semiartinian. Then $Soc(Rb/(Rb \cap D)) \neq 0$, hence $Soc(B/D) \neq 0$. Therefore B is m-cocritical. $\qquad\square$

Corollary 3.3. *Let A be a faithful m-cocritical module over a commutative ring R. Then:*

(i) R is a domain and $E(A) \cong E(R)$.

(ii) Every non-zero submodule of $E(A)$ is m-cocritical.

(iii) Every non-zero prime ideal of R is a maximal ideal.

Proof. (i) By Theorem 3.1, $Ann_R A = 0$ is a prime ideal of R. Hence R is a domain. Let a be a non-zero element of A. By Theorem 3.1, $Ann_R a = 0$, hence $Ra \cong R$ is m-cocritical. Since A is uniform, we have $E(A) = E(Ra) \cong E(R)$.

(ii) and (iii) They follow by Corollary 3.2 and [2, Theorem 2.6] respectively. $\square$

We continue with several results that generalize the corresponding ones for indecomposable injective modules. For recall that minimal m-injective modules have local endomorphism rings ([3], Theorem 3.2). Therefore, we may state a Krull-Schmidt-Remak-Azumaya-type theorem:

Theorem 3.4. *Let A be a module that is a direct sum of minimal m-injective modules. Then any two direct sum decompositions into indecomposable direct summands are isomorphic.*

Theorem 3.5. *Let A be a module and let $B = B_1 \cap \ldots \cap B_n$ be an irredundant intersection of submodules of A such that $E_m(A/B_i)$ is a minimal m-injective module for every $i \in \{1, \ldots, n\}$ Then*

$$E_m(A/B) \cong \oplus_{i=1}^n E_m(A/B_i)$$

and any two such direct sum decompositions are isomorphic.

Proof. Let $f : A \to \oplus_{i=1}^n E_m(A/B_i)$ be defined by

$$f(a) = (a + B_1, \ldots, a + B_n).$$

Then f is a homomorphism with $Ker\, f = B$. Hence f induces a monomorphism $g : A/B \to \oplus_{i=1}^n E_m(A/B_i)$. For each $i \in \{1, \ldots, n\}$, let $q_i : E_m(A/B_i) \to \oplus_{i=1}^n E_m(A/B_i)$ denote the canonical injection. Since the intersection $B = B_1 \cap \ldots \cap B_n$ is irredundant, for every i there exists $b_i \in B_1 \cap \ldots \cap B_{i-1} \cap B_{i+1} \cap \ldots \cap B_n$ and $b_i \notin B_i$. Then $g(b_i + B) = q_i(b_i + B_i)$ is a non-zero element of $g(A/B) \cap q_i(A/B_i)$. But $E_m(A/B_i)$ is minimal m-injective, hence $q_i(E_m(A/B_i))$ has the same property. Then $q_i(E_m(A/B_i))$ is an m-injective envelope of $g(A/B) \cap q_i(A/B_i)$. Hence

$$\oplus_{i=1}^n E_m(A/B_i) = \oplus_{i=1}^n q_i(E_m(A/B_i)) =$$
$$= E_m(\oplus_{i=1}^n (g(A/B) \cap q_i(A/B_i))) = E_m(g(A/B)).$$

But $E_m(g(A/B) \cong E_m(A/B)$ [5, Lemma 2.2]. It follows that $E_m(A/B) \cong \oplus_{i=1}^n E_m(A/B_i)$.

Now let $B = C_1 \cap \ldots \cap C_m$ be another irredundant intersection of submodules of A such that $E_m(A/C_j)$ is a minimal m-injective module for every $j \in \{1, \ldots, m\}$. We have the isomorphisms

$$E_m(A/B) \cong \oplus_{i=1}^n E_m(A/B_i) \cong \oplus_{i=1}^n E_m(A/C_j).$$

Now by Theorem 3.4, it follows that $m = n$ and there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $E_m(A/B_i) \cong E_m(A/C_{\sigma(i)})$ for every $i \in \{1 \ldots, n\}$. $\square$

Remark. In the context of the previous theorem, since $E(A/B_i)$ is indecomposable, B_i is irreducible for every $i \in \{1, \ldots, n\}$.

As a consequence of Theorem 3.5, we obtain the following result previously established in [5, Theorem 4.15].

Corollary 3.6. *Let R be a commutative ring with $\dim R \geq 1$ and let $p = p_1 \cap \ldots \cap p_n$ be an irredundant intersection of prime ideals of R such that $E_m(R/p_i)$ is a minimal m-injective R-module for every $i \in \{1, \ldots, n\}$. Then:*

(i) $E_m(R/p) \cong \bigoplus_{i=1}^n E_m(R/p_i)$.

(ii) n and $p_1, \ldots, p_n$ are uniquely determined by p.

Theorem 3.7. *Let n be a positive integer, A a module and B a submodule of A. Then the following statements are equivalent:*

(i) $E_m(A/B) = \bigoplus_{i=1}^n E_i$, *where E_i is a minimal m-injective module for every $i \in \{1, \ldots, n\}$.*

(ii) There exists an irredundant intersection $B = B_1 \cap \ldots \cap B_n$ of submodules of A such that $E_m(A/B_i)$ is a minimal m-injective module for every $i \in \{1, \ldots, n\}$.

Proof. $(ii) \Rightarrow (i)$ This is Theorem 3.5.

$(i) \Rightarrow (ii)$ Let $p : A \to A/B$ and $k : A/B \to E_m(A/B)$ be the natural epimorphism and the inclusion homomorphism respectively. For every $i \in \{1, \ldots, n\}$, let $q_i : E_m(A/B) \to E_i$ be the canonical projection, let $g_i : A \to E_i$ be the combined homomorphism $g_i = q_i k p$ and put $B_i = Ker\, g_i$. Then $B = B_1 \cap \ldots \cap B_n$. Now let $i \in I$. Since $E_i \cap (A/B) \neq 0$, it follows that $B_i \neq A$. We have $A/B_i \cong g_i(A) \subseteq E_i$, hence $E_m(A/B_i) \cong E_i$, because E_i is a minimal m-injective module. Suppose that the intersection $B = B_1 \cap \ldots \cap B_n$ is not irredundant. Then we can refine from it an irredundant intersection with fewer terms by omission. By Theorem 3.5, $E_m(A/B)$ is isomorphic to a direct sum of less than n minimal m-injective modules, which is a contradiction. $\square$

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References

[1] J.L. Bueso, P. Jara and B. Torrecillas, *Decomposition of injective modules relative to a torsion theory*, Israel J.Math. **52** (1985), 266-272.

[2] I. Crivei, *On a class of modules closed to non-zero submodules*, Automat. Comput. Appl. Math. **1**(2) (1992), 94-98.

[3] S. Crivei, *m-injective modules*, Mathematica (Cluj) **40** (63), No.1 (1998), 71-78.

[4] S. Crivei, *On a class of modules whose non-zero endomorphisms are monomorphisms*, Studia Univ. "Babeş-Bolyai", Mathematica **43**, No.1 (1998), 23-28.

[5] S. Crivei, *m-injective envelopes of modules*, Mathematica (Cluj) **41** (64), No.2 (1999), 149-159.

[6] S.E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. **121** (1966), 223-235.

[7] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Springer-Verlag, 1995.

[8] J.S. Golan, Torsion theories, Longman Scientific and Technical, New York, 1986.

[9] J.L. Johnson, *Modules injective with respect to primes*, Comm.Alg. **7**(3) (1979), 327-332.

[10] S. Mohamed and S. Singh, *Decomposition of σ-injective modules*, Comm. Alg. **9**(6) (1981), 601-611.

[11] C. Năstăsescu, Inele. Module. Categorii, Ed. Academiei, Bucureşti, 1976.

[12] D.W. Sharpe, P. Vámos, Injective modules, Cambridge Univ. Press, 1972.

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$\mathcal{G}_1$ we have :

$$f([u_i, u_j], u_k) - f([u_i, u_k], u_j) + f([u_j, u_k], u_i) - [u_i, f(u_j, u_k)] +$$
$$[u_j, f(u_i, u_k)] - [u_k, f(u_i, u_j)] = 0$$
$$f([u_i, u_j], v_t) + f(u_j, [u_i, v_t]) - f(u_i, [u_j, v_t]) - [u_i, f(u_j, v_t)] +$$
$$[u_j, f(u_i, v_t)] - [v_t, f(u_i, u_j)] = 0$$
$$f([u_i, v_t], v_r) + f([u_i, v_r], v_t) + f([v_t, v_r], u_i) - [u_i, f(v_t, v_r)] +$$
$$[v_t, f(u_i, v_r)] + [v_r, f(u_i, v_t)] = 0$$
$$f([v_t, v_r], v_s) + f([v_t, v_s], v_r) + f([v_r, v_s], v_t) - [v_t, f(v_r, v_s)] -$$
$$[v_r, f(v_t, v_s)] - [v_s, f(v_t, v_r)] = 0.$$

We denote by $Z_0^2(\mathcal{G}, \mathcal{G})$ the linear space of these cocycles.

1.3 On the variety of Lie superalgebras

Let $C_{i,j}^k$, $D_{i,j}^k$ and $E_{i,j}^k$ be elements of an algebraic closed field $\mathbb{K}$ such that $C_{j,i}^k = -C_{i,j}^k$ and $E_{j,i}^k = E_{i,j}^k$. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a superalgebra with $\{X_1, X_2, \ldots X_n, Y_1, Y_2, \ldots, Y_m\}$ as a basis with $X_i \in \mathcal{G}_0$ and $Y_i \in \mathcal{G}_1$. Assume that : $[X_i, X_j] = \sum_{k=1}^n C_{i,j}^k X_k$, $[X_i, Y_j] = \sum_{k=1}^m D_{i,j}^k Y_k$ and $[Y_i, Y_j] = \sum_{k=1}^n E_{i,j}^k X_k$. Then the Jacobi's super-relations can be seen as polynomials relations, see [5], on the constants of structure $C_{i,j}^k$, $D_{i,j}^k$ and $E_{i,j}^k$. Therefore the set of Lie superalgebras is an algebraic variety, we will denote it by $\mathcal{L}_{n,m}$.

Definition 1.3. Let ν_0 be a Lie superalgebra of $\mathcal{L}_{n,m}$. A *deformation* ν_t in $\mathcal{L}_{n,m}$ of ν_0 is a formal power series in one parameter t such that $\nu_t = \nu_0 + t\nu_1 + t^2\nu_2 + \ldots$ where ν_i are bilinear maps such that $\nu_i(g_\alpha, g_\beta) = (-1)^{\alpha \cdot \beta} \nu_i(g_\beta, g_\alpha)$, for all $i \in \mathbb{N}$, $g_\alpha \in \mathcal{G}_\alpha$ and $g_\beta \in \mathcal{G}_\beta$ and satisfying Jacobi's formal-relation :

$$(-1)^{\gamma \cdot \alpha} \nu_t(A, \nu_t(B, C)) + (-1)^{\alpha \cdot \beta} \nu_t(B, \nu_t(C, A)) + (-1)^{\beta \cdot \gamma} \nu_t(C, \nu_t(A, B)) = 0,$$

for all $A \in \mathcal{G}_\alpha$, $B \in \mathcal{G}_\beta$ et $C \in \mathcal{G}_\gamma$.

The Jacobi's formal-relations implies the following proposition :

Proposition 1.1. *Let μ_0 be a Lie superalgebra and ν_t a deformation of ν_0 such that $\nu_t = \nu_0 + t\nu_1 + t^2\nu_2 + \ldots$. Then $\nu_1 \in Z_0^2(\nu_0, \nu_0)$.*

2   Nilpotent Lie superalgebras

2.1   Definitions

For a Lie superalgebra $\mathcal{G}$ over an algebraic closed field of characteristic 0 we define the lower central series $C^0(\mathcal{G}) = \mathcal{G}$, $C^{i+1}(\mathcal{G}) = [\mathcal{G}, C^i(\mathcal{G})]$.

Definition 2.1. A Lie superalgebra $\mathcal{G}$ is *nilpotent* if there exists an integer n such that $C^n(\mathcal{G}) = \{0\}$.

This definition is not easy to use. We define for a Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ two sequences :

$$C^0(\mathcal{G}_0) = \mathcal{G}_0, \qquad C^{i+1}(\mathcal{G}_0) = [\mathcal{G}_0, C^i(\mathcal{G}_0)]$$

and

$$C^0(\mathcal{G}_1) = \mathcal{G}_1, \qquad C^{i+1}(\mathcal{G}_1) = [\mathcal{G}_0, C^i(\mathcal{G}_1)].$$

It is easy to see that if $\mathcal{G}$ is nilpotent, there exists (p, q) such that : $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.

Theorem 2.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebras. Then $\mathcal{G}$ is nilpotent if and only if there exists $(p, q) \in \mathbb{N}^\star \times \mathbb{N}^\star$ such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.*

The proof is based on the classical Engel's theorem.

Definition 2.2. Let $\mathcal{G}$ be a nilpotent Lie superalgebra, the *super-nilindex* of $\mathcal{G}$ is the pair $(p, q) \in \mathbb{N}^\star \times \mathbb{N}^\star$ such that : $C^p(\mathcal{G}_0) = \{0\}$, $C^{p-1}(\mathcal{G}_0) \neq \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, $C^{q-1}(\mathcal{G}_1) \neq \{0\}$. It is and invariant up to isomorphism.

3   Filiform Lie superalgebras

3.1   Adapted basis

Definition 3.1. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a nilpotent Lie superalgebra with $\dim \mathcal{G}_0 = n + 1$ and $\dim \mathcal{G}_1 = m$. $\mathcal{G}$ is called *filiform* if its super-nilindex is (n, m).

Remark. We can view the set of filiform Lie superalgebras as the complementary of the closed set, for the Zariski topology, of the nilpotent superalgebas with super-nilindex (k, p) such that $k \leq n-1$ and $p \leq m-1$. Hence the set of filiform Lie superalgebras is an open set of the variety of nilpotent Lie superalgebras.

Like for the filiform Lie algebras [7], there exists an adapted basis of a filiform Lie superalgebra :

Theorem 3.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra with $\dim \mathcal{G}_0 = n+1$ and $\dim \mathcal{G}_1 = m$. Then there exists a basis $\{X_0, X_1, \ldots X_n, Y_1, Y_2, \ldots Y_m\}$ of $\mathcal{G}$ with $X_i \in \mathcal{G}_0$ and $Y_j \in \mathcal{G}_1$ such that :*

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1, \quad [X_0, X_n] = 0; \\ [X_1, X_2] \in \mathbb{C}.X_4 + \mathbb{C}.X_5 + \cdots + \mathbb{C}.X_n; \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1, \quad [X_0, Y_m] = 0. \end{cases}$$

Definition 3.2. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a nilpotent Lie superalgebra. Let $gz_0(X)$ and $gz_1(X)$ be the ordered sequences of Jordan block's dimensions of the nilpotent operator $ad(X)$ restricted to $\mathcal{G}_0$ and $\mathcal{G}_1$, where $X \in \mathcal{G}_0$. In the set of sequences, we consider the lexicographical order. Then the pair :

$$gz(\mathcal{G}) = \left(\max_{X \in \mathcal{G}_0 \setminus [\mathcal{G}_0, \mathcal{G}_0]} gz_0(X) \,\middle|\, \max_{\widetilde{X} \in \mathcal{G}_0 \setminus [\mathcal{G}_0, \mathcal{G}_0]} gz_1(\widetilde{X}) \right)$$

is an invariant up to an isomorphism.

This extends the Goze's invariant for nilpotent Lie algebras. The following proposition give a characterization of filiform Lie superalgebras with this invariant. The proof of this proposition is based on the Theorem 3.1.

Proposition 3.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra with $\dim \mathcal{G}_0 = n + 1$ and $\dim \mathcal{G}_1 = m$. Then $gz(\mathcal{G}) = (gz_0(X_0)|gz_1(X_0)) = (n|m)$, where $X_0 \in \mathcal{G}_0 \setminus [\mathcal{G}_0, \mathcal{G}_0]$.*

Let's define the superalgebra $L_{n,m}$ by

$$\begin{aligned} [X_0, X_i] = \mu_0(X_0, X_i) = X_{i+1} & \quad 1 \leq i \leq n-1, \\ [X_0, Y_i] = \rho_0(X_0, Y_i) = Y_{i+1} & \quad 1 \leq i \leq m-1, \end{aligned}$$

the other brackets vanished.

Theorem 3.2. *Every filiform Lie superalgebra is isomorphic to a linear deformation of $L_{n,m}$.*

Proof. Let $\mathcal{G}$ be a filiform Lie superalgebra. The Theorem 3.1 shows that the product of $\mathcal{G}$ can be written as $\mu_0 + \rho_0 + \Phi$ where $\Phi(X_0, \bullet) = 0$. As this product satisfies the Jacobi's super-relation, we have a linear deformation of $L_{n,m}$. $\qquad \square$

Using Theorem 3.2 and Proposition 1.1, the study of the set of filiform Lie superalgebras $\mu_0 + \rho_0 + \Phi$ can be reduced to the study of the even 2-cocycles $\Phi \in Z_0^2(L_{n,m}, L_{n,m})$.

3.2 Description of $Z_0^2(L_{n,m}, L_{n,m})$

Proposition 3.2. *Let the Lie superalgebra $L_{n,m} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be filiform. Any even 2-cocycle $\Phi \in Z_0^2(L_{n,m}, L_{n,m})$ is a sum of three cocycles $\Psi, \rho, b \in Z_0^2(L_{n,m}, L_{n,m})$, $\Phi = \Psi + \rho + b$ such that $\psi \in Hom(\mathcal{G}_0 \wedge \mathcal{G}_0, \mathcal{G}_0)$, $\rho \in Hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1)$ and $b \in Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0)$.*

This helps us to compute a basis for these cocycles. Hence, the structure of filiform Lie superalgebras is well known. We can precise the expression of the cocycle b. For that we introduce the symmetric mapping $f_{p,s}$, for $1 \leq s \leq n$ and $1 \leq p \leq m-1$, by putting :

$$f_{p,s}(Y_i, Y_i) = X_s, \text{ if } i = p, \text{ otherwise } 0.$$

Assume that, for $1 \leq i, j \leq m-1$:

$$[X_0, f_{p,s}(Y_i, Y_j)] = f_{p,s}(Y_{i+1}, Y_j) + f_{p,s}(Y_i, Y_{j+1}).$$

Then we have, if $1 \leq i \leq p < j \leq m$,

$$f_{p,s}(Y_i, Y_j) = \frac{(-1)^{p-i}}{2} \left(C_{j-p}^{p-i} + C_{j-p-1}^{p-i-1} \right) X_{s-2p+i+j}.$$

To simplify the notations, we define the mappings : $f_{n,m}(Y_m, Y_m) = X_n$ and $f_{n,m}(Y_i, Y_j) = 0$ if $i \neq m$ or $j \neq m$, and $f_{i,m} = 0$, for $1 \leq i < n$.

Theorem 3.3. *For a 2-cocycle $b \in Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0)$ in $Z_0^2(L_{n,m}, L_{n,m})$, we have*

$$b = \sum_{p=1}^{m} \sum_{s=1}^{n} a_{p,s} \, f_{p,s},$$

where the coefficients $a_{p,s}$ satisfy the linear relations given by :

$$[X_0, b(Y_i, Y_m)] = b(Y_{i+1}, Y_m) \text{ for } 1 \leq i \leq m-1.$$

Remark. Here b is not decomposed in cocycles. But this result permits to have informations on the dimension of $Z_0^2(L_{n,m}, L_{n,m})$ and to get classification in lower dimensions.

4   Classifications of filiforms over $\mathbb{C}$ in lower dimensions

Using the basis of the cocycles and adapted changes of basis, like it was done for Lie algebras in [4], we have classifications of filiform Lie superalgebras.

The next table gives us for the dimensions of $\mathcal{G}_0$ and $\mathcal{G}_1$ the number of nonisomorphic filiform Lie algebras and Lie superalgebras. The descriptions of these superalgebras can be found in [2].

$\dim \mathcal{G}_0$	$\dim \mathcal{G}_1$	Lie algebras	Lie superalgebras
2	2	1	3
2	3	1	6
2	4	1	9
3	2	1	5
3	3	1	16
4	2	1	6
5	2	2	11
6	2	5	35

References

[1] D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*, Plenum publishing Copr. ISBN 0-306-10990-5.

[2] M. Gilg, *Super-algèbres de Lie nilpotentes*, Thèse, Université de Haute-Alsace, 2000.

[3] M. Gilg, *Low-dimensional filiform Lie superalgebras*, Revista Mathemática Complutense, (To be published).

[4] J.R. Gómez, A. Jimenéz-Merchán, Y. Khakimdjanov, *Low-dimensional filiform Lie algebras*, Journal of pure and applied algebras, **130**(1998), 133-158.

[5] M. Goze, *Perturbations des super-algèbres de Lie*, JPG, **6**, 4(1889).

[6] M. Scheunert, *The Theory of Lie Superalgebras*, Lecture Notes in Math., 716, Springer, 1979.

[7] M. Vergne, *Cohomologie des algèbres de Lie nilpotentes. Apllication à l'etude de la variété des algèbres de Lie nilpotentes*, Bull. Soc. math. France, **98**(1970), 81-116.

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ON THE NUMBER OF NONEQUIVALENT INDECOMPOSABLE MATRIX REPRESENTATIONS OF THE GIVEN DEGREE OF A FINITE p-GROUP OVER COMMUTATIVE LOCAL RING OF CHARACTERISTIC p^s

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Abstract

It is settled that the number of nonequivalent indecomposable matrix representations of an arbitrary degree of a finite p-group G of order $|G| > 2$ over a commutative local ring K of characteristic p^s ($s > 0$) is infinite, if $\mathrm{Rad}K \neq 0$ and $K/\mathrm{Rad}K$ is an infinite field ($\mathrm{Rad}K$ is the Jacobson radical of the ring K).

Higman [1] has cleared up when the number of nonequivalent indecomposable matrix representations of a finite group over a field of characteristic $p > 0$ is finite. The analogous problem for matrix representations of finite groups over residue class ring modulo p^s ($s > 0$) was solved in [2]. The known problem on finiteness of the set of degrees of all indecomposable matrix representations of a finite p-group over an arbitrary commutative local ring of characteristic p^s has been solved in [3-4]. Roggenkamp [5] has proved that if a noetherian domain R of characteristic $p > 0$ is not a field and p divides the order $|G|$ of a finite group G then there exists infinite number of nonisomorphic indecomposable RG-lattices of a finite R-rank less then $|G| + 1$ (RG is the group ring of a group G over the ring R). It is shown in [6] that the number of nonequivalent indecomposable matrix representations of an arbitrary degree $n > 1$ of a finite p-group ($p \neq 2$) over an infinite field of characteristic p is infinite. The similar statement have been proved in [7] for matrix representations of a finite noncyclic p-group ($p \neq 2$) over a commutative local ring K of characteristic p^s ($s > 0$) when K is an infinite ring of characteristic p or $K/\mathrm{Rad}K$ is an infinite field ($\mathrm{Rad}K$ is the Jacobson radical of the ring K).

In this paper we are investigating when a finite p-group has infinite number of nonequivalent indecomposable matrix representations of an arbitrary degree $n > 1$ over commutative local rings of characteristic p^s.

Lemma 1 [8]. *Let K be a commutative local ring, and G be a finite group, and $\Gamma : g \to \Gamma_g$ be a matrix K-representation of degree n of the group G ($g \in G$, $\Gamma_g \in GL(n, K)$), and $W(\Gamma) = \{C \in M(n, K) \mid C\Gamma_g = \Gamma_g C, \quad g \in G\}$, where $M(n, K)$ is the set of all $n \times n$-matrices over the ring K. If $W(\Gamma)$ is a local ring, then Γ is an indecomposable matrix K-representation of the group G.*

Lemma 2. *Let $H = \langle a \rangle$ be a cyclic p-group of order p, and K be a commutative local ring of characteristic p^s ($s > 0$) which contains a nonzero nilpotent element. If the factor ring $K/\text{Rad}K$ is an infinite field, then the number of nonequivalent indecomposable matrix K-representations of an arbitrary degree $n > 1$ of the group H is infinite.*

Proof. Let E_n be the identity $n \times n$-matrix, $I_n(\lambda)$ be the Jordan $n \times n$-matrix with element λ at the principal diagonal ($\lambda \in K$), and t be a nonzero nilpotent element of the ring K ($t^r = 0$, $t^{r-1} \neq 0$). Suppose that $t = p$ if $s > 1$. Using the Lemma 1, it is easy to show that the map

$$\Gamma(\lambda) : \ a \to E_n + t^{r-1} I_n(\lambda)$$

is an indecomposable matrix representation of the group $H = \langle a \rangle$ over the ring K. Obviously the representations $\Gamma(\lambda_1)$ and $\Gamma(\lambda_2)$ are nonequivalent over the ring K for $\lambda_1 \not\equiv \lambda_2 (\text{mod Rad}K)$ ($\lambda_i \in K$: $i = 1, 2$). The lemma is proved.

Lemma 3. *Let $H = \langle a \rangle$ be a cyclic p-group of order $|H| > 2$, and K be a commutative local ring of characteristic p which is not integral domain, and $K/\text{Rad}K$ be an infinite field. Then the number of nonequivalent indecomposable matrix K-representations of an arbitrary degree $n > 1$ of the group H is infinite.*

Proof. Using the Lemma 2, it is enough to consider the case if the ring K has not nonzero nilpotent elements. Then in the ring K there exist elements u and v such that $uv = 0$, $u \notin Kv$, $v \notin Ku$.

Let n be even, that is, $n = 2m$. It is easy to see that the map

$$\Gamma_1(\lambda) : a \to \begin{pmatrix} E_m & uE_m + vI_m(\lambda) \\ 0 & E_m \end{pmatrix} = A(\lambda) \quad (\lambda \in K) \tag{1}$$

is a matrix K-representation of the group $H = \langle a \rangle$. Denote $D_m(\lambda) = uE_m + vI_m(\lambda)$. Let us show that the representations $\Gamma_1(\lambda_1)$ and $\Gamma_1(\lambda_2)$ are

nonequivalent over the ring K for $\lambda_1 \not\equiv \lambda_2$ (mod Rad$K$). Assume the representations $\Gamma_1(\lambda_1)$ and $\Gamma_1(\lambda_2)$ are K-equivalent. Then there exists an invertible matrix C over the ring K such that

$$A(\lambda_1)C = CA(\lambda_2). \tag{2}$$

We shall represent the matrix C as

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \tag{3}$$

where C_i is an $m \times m$-matrix $(i = 1, 2, 3, 4)$. Then by (1)-(3), we have obtained that

$$D_m(\lambda_1)C_3 = 0, \ D_m(\lambda_1)C_4 = C_1 D_m(\lambda_2).$$

From here it follows that

$$C_3 \equiv 0 \ (\text{mod Rad}K), \ C_1 \equiv C_4 \ (\text{mod Rad}K),$$

$$I_m(\lambda_1)C_1 \equiv C_1 I_m(\lambda_2) \ (\text{mod Rad}K). \tag{4}$$

Therefore the matrix C is a noninvertible matrix over the ring K. This contradiction proves the nonequivalence of the K-representations $\Gamma_1(\lambda_1)$ and $\Gamma_1(\lambda_2)$ for $\lambda_1 \not\equiv \lambda_2$(mod Rad$K$).

Next we shall establish that $\Gamma_1(\lambda)$ is an indecomposable K-representation of the group H. If we put in (2) $\lambda_1 = \lambda_2 = \lambda$, then, using (1)-(4), we receive

$$C \equiv \begin{pmatrix} \alpha & & * \\ & \ddots & \\ 0 & & \alpha \end{pmatrix} \ (\text{mod Rad}K), \tag{5}$$

where $\alpha \in K$. That is, C or $E_{2m} - C$ is an invertible matrix over the ring K. Then by the Lemma 1, we get that $\Gamma_1(\lambda)$ is indecomposable representation of the group H over the ring K. The group $H = \langle a \rangle$ has an infinite number of nonequivalent indecomposable K-representations of the degree $n = 2m$.

Next we consider the case when $n = 2m + 1$ $(m \geq 1)$. We denote by $\langle 1 \rangle$ the $m \times 1$-matrix:

$$\langle 1 \rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Obviously the map

$$\Gamma_2(\lambda): \ a \rightarrow \begin{pmatrix} E_m & D_m(\lambda) & 0 \\ 0 & E_m & \langle 1 \rangle \\ 0 & 0 & 1 \end{pmatrix} = B(\lambda) \tag{6}$$

is a matrix K-representation of the group H of order $|H| > 2$. Let us see that the representations $\Gamma_2(\lambda_1)$ and $\Gamma_2(\lambda_2)$ are nonequivalent over the ring K for $\lambda_1 \not\equiv \lambda_2 \pmod{\operatorname{Rad}K}$ ($\lambda_i \in K$; $i = 1, 2$). Suppose that the representations $\Gamma_2(\lambda_1)$ and $\Gamma_2(\lambda_2)$ are K-equivalent. Then there exists an invertible $n \times n-$matrix C' over the ring K such that

$$B(\lambda_1)C' = C'B(\lambda_2). \tag{7}$$

Denote

$$C' = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix},$$

where C_{11} and C_{22} are $m \times m-$matrices. Then from (6)-(7) we obtain that

$$\langle 1 \rangle C_{31} = 0, \ \langle 1 \rangle C_{32} = C_{21}D_m(\lambda_2), \ D_m(\lambda_1)C_{21} = 0,$$

$$D_m(\lambda_1)C_{22} = C_{11}D_m(\lambda_2), \ \langle 1 \rangle C_{33} = C_{22}\langle 1 \rangle.$$

From here it follows that

$$C_{31} \equiv C_{32} \equiv 0 \ (\operatorname{mod} \operatorname{Rad}K), \ C_{21} \equiv 0 \ (\operatorname{mod} \operatorname{Rad}K),$$

$$C_{11} \equiv C_{22} \ (\operatorname{mod} \operatorname{Rad}K), \ C_{11}I_m(\lambda_2) \equiv I_m(\lambda_1)C_{11} \ (\operatorname{mod} \operatorname{Rad}K). \tag{8}$$

That is, C' is a noninvertible matrix over the ring K. This contradiction proves that the representations $\Gamma_2(\lambda_1)$ and $\Gamma_2(\lambda_2)$ are nonequivalent for $\lambda_1 \not\equiv \lambda_2 \pmod{\operatorname{Rad}K}$. Next we prove that $\Gamma_2(\lambda)$ is an indecomposable K-representation of the group H. If we put in (7) $\lambda_1 = \lambda_2 = \lambda$, then, using (7)-(8), we obtain that the matrix C' looks like (5). From here and from the Lemma 1, it follows that $\Gamma_2(\lambda)$ is indecomposable K-representation of the group H. The lemma is proved.

Lemma 4. Let $H = \langle a \rangle$ be a cyclic p-group of order $|H| > 2$, and K be a local integral domain of characteristic p, $\operatorname{Rad}K \neq 0$, and $K/\operatorname{Rad}K$ be an infinite field. Then the number of nonequivalent indecomposable matrix K-representations of an arbitrary degree $n > 1$ of the group H is infinite.

Proof. We consider four cases.
1) Let $n = 2$. It is easy to see that the K-representation

$$\Delta(\lambda): \ a \to \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad (\lambda \in K)$$

of the group H for $\lambda \neq 0$ is indecomposable. It is obvious that the representations $\Delta(\lambda_1)$ and $\Delta(\lambda_2)$ are not equivalent over the ring K for $\lambda_1 \not\equiv \lambda_2 \pmod{\operatorname{Rad}K}$.

2) Let $n = 3m$. It is easy to check that the map

$$\Gamma(\lambda): \quad a \to \begin{pmatrix} E_m & tE_m & I_m(\lambda) \\ 0 & E_m & tE_m \\ 0 & 0 & E_m \end{pmatrix} = D(\lambda) \ (t \in \mathrm{Rad}K, \ t \neq 0, \ \lambda \in K) \quad (9)$$

is a matrix representation of the group $H = \langle a \rangle$ over the ring K. We show that, for $\lambda_1 \not\equiv \lambda_2 \ (\mathrm{mod} \ \mathrm{Rad}K)$, the representations $\Gamma(\lambda_1)$ and $\Gamma(\lambda_2)$ are nonequivalent over the ring K. Let us assume that the representations $\Gamma(\lambda_1)$ and $\Gamma(\lambda_2)$ are K-equivalent. Then there exists an invertible matrix over the ring K such that

$$D(\lambda_1)C = CD(\lambda_2). \quad (10)$$

We shall represent the matrix C as

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix},$$

where C_{ii} $(i = 1, 2, 3)$ are $m \times m$−matrices. Then, by (9)-(10), we obtain

$$C_{31} = 0, \ C_{32} = 0, \ C_{21} = 0, \ C_{11} = C_{22} = C_{33},$$

$$I_m(\lambda_1)C_{11} \equiv C_{11}I_m(\lambda_2) \ (\mathrm{mod} \ \mathrm{Rad}K). \quad (11)$$

From here it follows that C is a noninvertible matrix over the ring K. So we obtain a contradiction, that is, the representations $\Gamma(\lambda_1)$ and $\Gamma(\lambda_2)$ are nonequivalent over the ring K for $\lambda_1 \not\equiv \lambda_2 \ (\mathrm{mod} \ \mathrm{Rad}K)$.

Clearly if we shall put in (10) $\lambda_1 = \lambda_2 = \lambda$, then using (11), the matrix C will look like (5). From here and from the Lemma 1 we obtain that $\Gamma(\lambda)$ is an indecomposable representation of the group H over the ring K. Hence, the group H of order $|H| > 2$ has an infinite number of nonequivalent indecomposable matrix K-representations of degree $n = 3m$.

3) Let $n = 3m + 1$ $(m \geq 1)$. It is easy to check that the map

$$\Gamma_1(\lambda): \quad a \to \begin{pmatrix} E_m & t^2E_m & I_m(\lambda) & t\langle 1 \rangle \\ 0 & E_m & tE_m & 0 \\ 0 & 0 & E_m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S(\lambda) \quad (12)$$

is a matrix K-representation of the group $H = \langle a \rangle$ $(t \in RadK, \ t \neq 0, \ \lambda \in K)$.

We shall prove that, for $\lambda_1 \not\equiv \lambda_2 \ (\mathrm{mod} \ \mathrm{Rad}K)$, the representations $\Gamma_1(\lambda_1)$ and $\Gamma_1(\lambda_2)$ are nonequivalent over the ring K. Suppose that the representations

$\Gamma_1(\lambda_1)$ and $\Gamma_1(\lambda_2)$ are K-equivalent. Then there exists an invertible $n \times n$-matrix C over the ring K such that

$$S(\lambda_1)C = CS(\lambda_2). \tag{13}$$

Obviously

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix},$$

where C_{ii} $(i = 1, 2, 3)$ are $m \times m$-matrices. From (12) and (13) it follows that

$$C_{21} = 0, \; C_{31} = 0, C_{32} = 0, \; C_{41} = 0, \; C_{42} = 0, \; C_{34} = 0, \tag{14}$$

$$C_{11} = C_{22} = C_{33}, \; C_{11}\langle 1 \rangle \equiv \langle 1 \rangle C_{44} \pmod{\mathrm{Rad}K}, \tag{15}$$

$$C_{11}I_m(\lambda_1) \equiv I_m(\lambda_2)C_{11} \pmod{\mathrm{Rad}K}. \tag{16}$$

Hence, C is a noninvertible matrix over the ring K. We have obtained a contradiction, that is, the representations $\Gamma_1(\lambda_1)$ and $\Gamma_1(\lambda_2)$ are nonequivalent over the ring K for $\lambda_1 \not\equiv \not\equiv \lambda_2 \pmod{\mathrm{Rad}K}$. If we shall put in (13) $\lambda_1 = \lambda_2 = \lambda$, then, using (14)-(16), we shall receive that $C_{44} \equiv \alpha \pmod{\mathrm{Rad}K}$ and the matrix C_{11} will looks like (5) $(\alpha \in K)$. From here and from the Lemma 1 it follows that $\Gamma_1(\lambda)$ is an indecomposable K-representation of the group H. Hence, the group H has an infinite number of nonequivalent indecomposable K-representations of degree $n = 3m + 1$.

4) Let $n = 3m + 2$ $(m \geq 1)$. It is easy to show that the map

$$\Gamma_2(\lambda): \quad a \to \begin{pmatrix} E_m & tE_m & I_m(\lambda) & t^2\langle 1 \rangle & t\langle 1 \rangle \\ 0 & E_m & t^2 E_m & 0 & t^2\langle 1 \rangle \\ 0 & 0 & E_m & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = T(\lambda) \tag{17}$$

is a matrix K-representation of the group $H = \langle a \rangle$ $(t \in \mathrm{Rad}K, \; t \neq 0, \; \lambda \in K)$. We shall prove that, for $\lambda_1 \not\equiv \lambda_2 \pmod{\mathrm{Rad}K}$, the representations $\Gamma_2(\lambda_1)$ and $\Gamma_2(\lambda_2)$ are nonequivalent over the ring K. Let us assume that the representations $\Gamma_2(\lambda_1)$ and $\Gamma_2(\lambda_2)$ are K-equivalent, hence there exists an invertible $n \times n$-matrix C over the ring K such that

$$T(\lambda_1)C = CT(\lambda_2). \tag{18}$$

Clearly

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{pmatrix},$$

where C_{ii} (i=1,2,3) are $m \times m-$matrices and C_{jj} ($j = 4, 5$) are $1 \times 1-$matrices. By (17)-(18), we obtain

$$C_{21} = 0, \ C_{31} = 0, \ C_{41} = 0, \ C_{51} = 0, \ C_{32} = 0, \ C_{52} = 0, C_{54} = 0, \ C_{34} = 0,$$
$$(19)$$
$$C_{24} \equiv 0 \ (\text{mod Rad}K), \ C_{53} \equiv 0 \ (\text{mod Rad}K), \ C_{11} \equiv C_{22} \equiv C_{33} \ (\text{mod Rad}K),$$
$$(20)$$
$$C_{44} \equiv C_{55} \ (\text{mod Rad}K), \ C_{11}\langle 1 \rangle \equiv \langle 1 \rangle C_{44} \ (\text{mod Rad}K),$$
$$(21)$$
$$C_{11} I_m(\lambda_1) \equiv I_m(\lambda_2) C_{11} \ (\text{mod Rad}K).$$

Hence, C is a noninvertible matrix over K. We have obtained a contradiction, therefore the representations $\Gamma_2(\lambda_1)$, $\Gamma_2(\lambda_2)$ are nonequivalent over the ring K for $\lambda_1 \not\equiv \lambda_2$ (mod Rad$K$). Putting $\lambda_1 = \lambda_2 = \lambda$ in (18), we receive, from (18)-(21), that $C_{44} \equiv \alpha$ (mod Rad$K$) ($\alpha \in K$) and the matrix C_{11} looks like (5). That is, the matrix C or $E_n - C$ is invertible over the ring K. Then, from the Lemma 1, we obtain that $\Gamma_2(\lambda)$ is an indecomposable K-representation of the group H. Hence, the group H of order $|H| > 2$ has an infinite number nonequivalent indecomposable K-representations of degree $n = 3m + 2$. The lemma is proved.

Lemma 5. *Let H be an Abelian group of the type $(2, 2)$, and K be a commutative local ring of characteristic 2 without nonzero nilpotent elements which is not a field. Then the number of nonequivalent indecomposable K-representations of an arbitrary degree $n > 1$ of the group H is infinite.*

Proof. H is an Abelian group of the type $(2, 2)$, that is,

$$H = < a > \times < b > \quad (a^2 = b^2 = 1, \quad ab = ba).$$

We consider n even, that is, $n = 2m$. It is easy to check that the map

$$\Gamma_i : \quad a \to \begin{pmatrix} E_m & t^i E_m \\ 0 & E_m \end{pmatrix} = A_i, \ b \to \begin{pmatrix} E_m & I_m(1) \\ 0 & E_m \end{pmatrix} = B \ (i \in \mathbb{N}) \quad (22)$$

is a matrix representation of the group H over the ring K ($t \in$ Rad$K$, $t \neq 0$). Now we shall show that, for $i \neq j$, the representations Γ_i and Γ_j are

nonequivalent over the ring K. Suppose that the representation Γ_i and Γ_j are K-equivalent. Then there exists an invertible matrix C over the ring K such that

$$A_i C = C A_j, \tag{23}$$

$$BC = CB. \tag{24}$$

Obviously,

$$C = \left(\begin{array}{cc} C_1 & C_2 \\ C_3 & C_4 \end{array} \right),$$

where C_i is an $m \times m$−matrix $(i = 1, 2, 3, 4)$. Then, by (22)-(24), we obtain that

$$C_3 \equiv 0 \ (\text{mod Rad}K), \ C_1 t^j = t^i C_4. \tag{25}$$

From here it follows that C is a noninvertible matrix over the ring K. This contradiction proves that the representations Γ_i and Γ_j are nonequivalent over the ring K, for $i \neq j$. We have proved that Γ_i is an indecomposable K-representation of the group H. If we shall put $i = j$ in (22), then, by (24)-(25), we obtain

$$I_m(1)C_1 \equiv C_1 I_m(1) \ (\text{mod Rad}K).$$

Hence, the matrix C_1 will looks like (5). That is, the matrix C or $E_{2m} - C$ is invertible over the ring K. Then, from the Lemma 1, we have obtained that Γ_i is an indecomposable representation of the group H over the ring K. Therefore the group H has an infinite number of nonequivalent indecomposable K-representations of degree $n = 2m$.

Now let $n = 2m+1 \ (m \geq 1)$. Let us consider the following K-representation of H:

$$\Gamma'_i : a \rightarrow \left(\begin{array}{ccc} E_m & t^i E_m & 0 \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{array} \right) = A'_i, \ b \rightarrow \left(\begin{array}{ccc} E_m & I_m(1) & t^i\langle 1\rangle \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{array} \right) = B'_i \ (i \in \mathbb{N}). \tag{26}$$

We shall check that for $i \neq j$ the representations Γ'_i and Γ'_j are nonequivalent over the ring K. Let us assume that the representations Γ'_i and Γ'_j are K-equivalent. Then there exists an invertible matrix C' over the ring K such that

$$A'_i C' = C' A'_j, \tag{27}$$

$$B'_i C' = C' B'_j. \tag{28}$$

We represent matrix C' as

$$C' = \left(\begin{array}{ccc} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{array} \right),$$

where C_{11} and C_{22} are $m \times m$ matrices. By (26) and (27), we obtain

$$C_{21} \equiv 0 \ (\text{mod RadK}), \ C_{31} \equiv 0 \ (\text{mod RadK}), \tag{29}$$

$$C_{23} \equiv 0 \ (\text{mod RadK}), \tag{30}$$

$$t^i C_{22} = t^j C_{11}. \tag{31}$$

From here it follows that, if $i \neq j$, then C' is noninvertible matrix over the ring K. The given contradiction proves that for $i \neq j$ the representations Γ_i' and Γ_j' are nonequivalent over the ring K.

Finally, we shall prove that Γ_i' is an indecomposable K-representation of the group H. If in (28) we shall put $i = j$, then we obtain

$$C_{11} I_m(1) = I_m(1) C_{22} + t^i \langle 1 \rangle C_{32}, \tag{32}$$

$$t^i C_{11} \langle 1 \rangle = I_m(1) C_{23} + t^i \langle 1 \rangle C_{33}. \tag{33}$$

For $i = j$, from the equations (27), (29)-(31), we get that

$$C' \equiv \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ 0 & C_{11} & 0 \\ 0 & C_{32} & C_{33} \end{pmatrix} \ (\text{mod RadK}). \tag{34}$$

Then from (33), it follows that

$$C_{11} \langle 1 \rangle = \langle 1 \rangle C_{33} \ (\text{mod RadK}). \tag{35}$$

Using (32)-(35), we obtain that $C_{33} \equiv \alpha \ (\text{mod RadK})$ and the matrix C_{11} looks like (5). Hence, C' or $E_{2m+1} - C'$ is an invertible matrix over the ring K. That is, by the Lemma 1, Γ_i' is an indecomposable representation of the group H over the ring K.

Thus, we have shown that there exists an infinite number of nonequivalent indecomposable matrix K-representations of degree $n = 2m + 1$ of the group H. This completes the proof of the lemma.

Theorem 1. *Let G be a finite p-group of order $|G| > 2$, and K be a commutative local ring of characteristic p^s ($s \geq 1$), $\text{RadK} \neq 0$ and K/RadK be an infinite field. Then the number of nonequivalent indecomposable matrix K-representations of an arbitrary degree $n > 1$ of the group G is infinite.*

The proof of the theorem follows from the Lemmas 2-5.

References

[1] D. G. HIGMAN. Indecomposable representations at characteristic p, *Duke Math. J.* **21** (1954), 377-381.

[2] P. M. GUDIVOK, V. S. DROBOTENKO, A. I. LICHTMAN. On representations of finite groups over residue class ring modulo m, *Ukr. Math. J.* **16** № 1 (1964), 82-89.

[3] P. M. GUDIVOK, V. I. POGORILJAK. On indecomposable representations of finite p-groups over commutative local rings, *Dopovidi NAN Ukraini* **5** (1996), 7-11.

[4] P. M. GUDIVOK, V. I. POGORILJAK. On indecomposable matrix representations of finite p-groups over commutative local rings of characteristic p^s, *Nauk. Visnik Uzhgorod. Univ., ser. Matem.* **4** (1999), 43-46.

[5] K. W. ROGGENKAMP. Gruppenringe von unendlichem Darstellungstyp, *Math.Z.* **96** (1967), 393-398.

[6] P. M. GUDIVOK. On modular representations of finite groups, *Dokl. i Soob. Uzhgorod. Univ, ser. Phys.-Matem.Nauk.* **4** (1961), 86–87.

[7] P. M. GUDIVOK, I. P. SIGETIJ, I. B. CHUKHRAJ. On the number matrix representations of the given degree of the finite p-group over some commutative local rings of characteristic p^s, *Nauk. Visnik Uzhgorod. Univ., ser. Matem.* **4** (1999), 47-53.

[8] F. KASH. Modules and rings, *Mir, Moscow*, 1981.

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ON THE TRADITIONAL WAY OF DESCRIPTION OF THE PRIME VARIETIES IN CHARACTERISTIC P

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Abstract

In [2] the multilinear components of the prime varieties of the matrix type 2 were classified in terms of the multilinear generators, but the generators were not calculated. In this paper we calculate with the help of PC the generators of one of these multilinear components. The result shows that the problem of description of the prime varieties in terms of generators is quite wild.

1. Prime Varieties.

In this section we recall some results about the prime varieties in the case of characteristic p.

Let X be a countable set, $R\langle X \rangle, \widetilde{R}\langle X \rangle$ are the free associative algebra and the free associative algebra with trace, respectively, generated by the set X over an associative and commutative ring R. We assume that

$$X \subseteq R\langle X \rangle \subseteq \widetilde{R}\langle X \rangle.$$

We recall that an arbitrary ideal Γ of the algebra $R\langle X \rangle$ is called a T- *ideal* if the ideal Γ is an ideal of identities of some R-algebra. Similarly an ideal $\widetilde{\Gamma}$ of the algebra $\widetilde{R}\langle X \rangle$ is called a $\widetilde{T}$-*ideal* if the ideal $\widetilde{\Gamma}$ is an ideal of trace identities of some algebra with trace.

A T-ideal Γ of the algebra $R\langle X \rangle$ is called *verbally prime* if for every T-ideals Γ_1, Γ_2, the inclusion $\Gamma_1 \Gamma_2 \subseteq \Gamma$ implies one of the inclusions $\Gamma_1 \subseteq \Gamma$ or $\Gamma_2 \subseteq \Gamma$. A T-ideal Γ is called *verbally semiprime* if there are no non-trivial nilpotent modulo Γ T-ideals. A variety of algebras is called prime (semiprime) if the ideal of identities of this variety is verbally *prime (semiprime)*. In the same manner we define the prime and semiprime varieties of the algebras with trace.

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In the case of the algebras over a field of characteristic zero the prime varieties were described in [1]. The following structure theorem is valid.

Theorem. [1].*Let R be a field of characteristic zero.*
1. For every non-zero T-ideal Γ of the algebra $F\langle X \rangle$ there exists a nilpotent modulo Γ verbally semiprime T-ideal containing Γ.
2. A T-ideal Γ is verbally semiprime if Γ is the intersection of a finite number of verbally prime T-ideals.
3. A variety is prime if it is generated by the matrix superalgebra $M_{n,k}$ or by the algebra $M_n(G)$, where G is the Grassmann algebra of infinite rank.

The problem of classification of the prime varieties in the case of the algebras over a field of characteristic p is open. At first we remark that this problem scarcely can be solved in full sense because the Finite Bases Problem has a negative solution. Perhaps it can be solved at the multilinear level.

In this section we talk about some results concerning the multilinear components of the prime varieties.

In [3] the prime varieties were divided into two classes: classical and non-classical. We recall the definitions.

Let $\widetilde{P}_n$ be the set of all multilinear polynomials with trace of degree n depending on the variables $x_1, \cdots, x_n$. It follows from the definition of the free algebra with trace that any polynomial $f \in \widetilde{P}_n$ can be written in a unique way as an R-linear combination of the monomials

$$u_0(Tr(1))^l Tr(u_1) \cdots Tr(u_n), \quad u_i \in \langle X \rangle, \quad n, l \geq 0,$$

which belong to $\widetilde{P}_n$ and satisfy the properties:
1. $u_i \neq 1$ for every $i > 0$;
2. For all $i > 0$ the least number j, such that x_j occurs in u_{i+1}, is greater than the least number k, such that x_k occurs in u_i.

Denote by K the R-subalgebra with unity of the algebra $\widetilde{R}\langle X \rangle$ generated by the element $Tr(1)$. Let KS_{n+1} be the group algebra (over K) of the symmetric group of permutations S_{n+1} acting on the set $\{0, 1, \cdots, n\}$. We define a K-linear mapping $\lambda_n : \widetilde{P}_n \to KS_{n+1}$, by putting

$$\lambda_n(x_{i_1} \cdots x_{i_s} Tr(x_{j_1} \cdots x_{j_t}) Tr(x_{k_1} \cdots x_{k_l}) \cdots) = \sigma \in S_{n+1},$$

where σ is the permutation whose decomposition into the cycles is the following

$$\sigma = (0, i_1, \cdots, i_s)(j_1, \cdots, j_t)(k_1, \cdots, k_l) \cdots.$$

The symbol 0 plays a role of label, which indicates the non-trace part of the monomial. It follows from the definition of the free algebra with trace that

the mapping λ_n is an isomorphism of K-modules. Put

$$X_n = \lambda_n^{-1} \sum_{\sigma \in S_{n+1}} (-1)^\sigma \sigma.$$

This polynomial is said to be the *Cayley-Hamilton polynomial of degree n*. Put also

$$X_n^+ = \lambda_n^{-1} \sum_{\sigma \in S_{n+1}} \sigma.$$

Take an arbitrary $\gamma \in R$. We call a $\widetilde{T}$-ideal $\widetilde{\Gamma}$ γ-*classical* if

1. $Tr(1) - \gamma \in \widetilde{\Gamma}$;

2. For every n, the set $\lambda_n(\widetilde{\Gamma} \cap \widetilde{P}_n)$ is a two-sided ideal of the group algebra KS_{n+1}.

A variety $\widetilde{V}$ of the algebras with trace is called γ-*classical* if the ideal of trace identities of $\widetilde{V}$ is γ-classical. A variety of ordinary algebras is called γ-*classical* if it is generated by some algebra with trace which generates a γ-classical variety of the algebras with trace.

In the case of the algebras over a field of characteristic zero the variety generated by the superalgebra $M_{n,k}$ is $(n - k)$-classical [7]. The variety generated by the matrix algebra over the Grassmann algebra is non-classical. We mention two conjectures about the non-classical prime varieties.

Yu. P. Razmyslov [7] has formulated the conjecture about the identities of the algebras $M_n(G)$.

Conjecture 1. *If R is a field of characteristic zero then*

$$T[M_n(G)] = (\widetilde{T}[M_{n,n}] + \{Tr(x)\}^{\widetilde{T}}) \cap F\langle X \rangle,$$

where $\{g\}^{\widetilde{T}}$ is the $\widetilde{T}$-ideal generated by g, in $\widetilde{R}\langle X \rangle$.

The following conjecture about the multilinear components of the non-classical prime varieties in arbitrary characteristic generalizes the Conjecture 1 and also looks quite probable.

Conjecture 2. *If V is a non-classical prime variety then*

$$P \cap T[V] = P \cap (\widetilde{T}[\widetilde{V}] + \{Tr(x)\}^{\widetilde{T}}),$$

for some 0-classical prime variety $\widetilde{V}$.

The traditional natural way of description of the multilinear components of the γ-classical varieties is the following. Let $\widetilde{\Gamma}$ be a γ-classical $\widetilde{T}$-ideal. At first we choose some non-zero polynomial $f_1 \in \widetilde{\Gamma}$ of minimal degree and generate by this polynomial a γ-classical $\widetilde{T}$-ideal $\widetilde{\Gamma}_1$. Then (if it is possible) we choose a polynomial of minimal degree f_2 belonging to the set

$$\widetilde{\Gamma} \setminus \widetilde{\Gamma}_1$$

and generate by f_1 and f_2 a γ-classical $\widetilde{T}$-ideal $\widetilde{\Gamma}_2$ etc. Obviously we can assume that the polynomials f_i satisfy the additional properties: 1. $f_i \in \widetilde{P}_{n_i}$ for some n_i; 2. The left and right RS_{n_i+1}-modules generated by the polynomial f_i are irreducible modulo RS_{n_i+1}-bimodule $N_i = \widetilde{P}_{n_i} \cap \widetilde{\Gamma}_{i-1}$ ($\widetilde{\Gamma}_0 = 0$). Put

$$M_i = (FS_{n_i+1}f_i + N_i)/N_i.$$

The modules M_i satisfy the following property

Theorem. [3]. *The RS_{n_i+1}-module M_i is irreducible as a RS_{n_i}-module.*

It is well-known that if R is a field of characteristic zero then an RS_{n+1}-module M is irreducible as an RS_n-module if and only if M corresponds to the rectangular Young diagram. In the case of characteristic p the description of such modules was obtained by A. Kleshchev [5].

Theorem. [5]. *Let R be a field of characteristic p. An RS_{n+1} - module M is irreducible as RS_n - module if and only if M corresponds to a p-regular partition $\lambda = (l_1^{(a_1)}, \ldots, l_k^{(a_k)})$ satisfying the property: For every $j \leq k$ and for every $i < j$, the numbers*

$$B(i,j) = l_i - l_j + \sum_{k=i}^{j} a_k$$

are divisible by p.

We see that the multilinear generators of the multilinear components of the classical varieties satisfy a nice module property. Of course this information is not enough for describing them. Moreover further we'll see that the problem of the calculation of these generators looks wild. At least this problem is very difficult.

2. Prime subvarieties of the variety generated by the matrix algebra of order 2.

In [4] one of the authors of this paper, following the traditional natural way described above, has classified the multilinear components of the prime subvarieties of the variety $\mathrm{Var}\, M_2(F)$ generated by the matrix algebra of order 2 over a field F of characteristic $p > 0$. To formulate the classification theorem we need the following lemma.

Lemma 1. *If $p \neq 2$ or $k \neq 1$ then there exists a uniquely defined modulo the ideal of trace identities of the algebra $M_2(F)$ polynomial $f_k \in \widetilde{P}_n$, where $n = p^k - 2$, such that:*
1. The algebra $M_2(F)$ does not satisfy trace identity $f_k = 0$, but satisfies the identities $\sigma f_k = f_k \sigma = f_k$ for every $\sigma \in S_{n+1}$.
2. The sum of the coefficients of f_k is equal to 1.

Denote by $\widetilde{\Gamma}_k$ the $\widetilde{T}$-ideal of the algebra $\widetilde{F}\langle X \rangle$ generated by the polynomial f_k, the Cayley - Hamilton polynomial X_2 and $Tr(1) - 2$. Let V_k be the variety (of ordinary algebras) corresponding to the T-ideal

$$\Gamma_k = F\langle X \rangle \cap \widetilde{\Gamma}_k.$$

We call a subvariety V of $\mathrm{Var}\,M_2(F)$ *trivial*, if either V is a subvariety of the variety of commutative algebras or the multilinear components of the varieties V and $\mathrm{Var}\,M_2(F)$ are equal. Now we can state the theorem which classifies the multilinear components of the prime subvarieties of the variety $\mathrm{Var}\,M_2(F)$.

Theorem. [4]. *1. If V is a non-trivial prime subvariety of $\mathrm{Var}M_2(F)$ then for some k the multilinear component of the varieties V and V_k are equal.*
2. For every k, there exists a prime subvariety V whose multilinear component equals the multilinear component of V_k.
3. If $k < s$, then the multilinear component of V_s is a proper subset of the multilinear component of V_k.

It is easy to calculate the polynomial f_1 in every characteristic:

$$f_1 = -X_{p-2}^+.$$

The prime variety V_1 is well-known and was found by Yu. P. Razmyslov [6]. The relatively free algebra of countable rank of this variety is very interesting. This algebra satisfies Engel identity of degree $p - 1$ but is not Lie nilpotent.

It is quite easy to calculate the polynomial f_2 for $p = 2$.

$$f_2 = x_1 \circ x_2 + Tr(x_1)Tr(x_2).$$

The relatively free algebra of countable rank satisfying identities $f_2 = 0$ and $Tr(1) = 0$ is also interesting and can be considered as the Grassmann algebra in characteristic 2.

3 Calculation of the polynomials f_k.

In this section we give the algorithm for the calculation of the polynomials f_k.

Let $\widetilde{Z}\langle X \rangle$, $\widetilde{Q}\langle X \rangle$ be the free algebras with trace over the ring of integers Z and over the field of the rational numbers Q, respectively and let F be a field of characteristic p. Denote by ϕ the natural homomorphism

$$\phi : \widetilde{Z}\langle X \rangle \to \widetilde{F}\langle X \rangle.$$

At first we prove

Lemma 2. *Let* $n = p^k - 2$, *and* l *be a maximal number with property:* p^l *divides* $(n + 1)!$. *Then:*
(i) There exists a uniquely defined modulo the ideal of trace identities of the algebra $M_2(Q)$ *polynomial* $u_k \in \widetilde{Z}\langle X \rangle$ *such that the algebra* $M_2(Q)$ *satisfies the trace identity* $X_n^+ = p^l u_k$.
(ii) $f_k = \phi(u_k)$.

Proof. Indeed let s be the maximal non-negative integer with property: the algebra $M_2(Q)$ satisfies the trace identity

$$X_n^+ = p^s v, \tag{1}$$

for some multilinear polynomial $v \in \widetilde{Z}\langle X \rangle$. Prove that $s = l$. Let

$$\lambda_n(v) = \sum_{\sigma \in S_{n+1}} \alpha_\sigma \sigma.$$

Take an arbitrary partition $\Lambda = (\Lambda_1, \Lambda_2)$ of the set $\{0, 1, \ldots, n\}, |\Lambda_2| = m$, and $\tau \in S_{n+1}$. Let S_Λ be the Young subgroup corresponding to the partition Λ. By the Lemma from [2], we have the equalities in Q

$$\sum_{\sigma \in \tau S_\Lambda} \alpha_\sigma = p^{-s} m!(n + 1 - m)! = \frac{(n + 1)!}{p^s C_{n+1}^m}.$$

Hence, if $s < l$, then

$$\sum_{\sigma \in \tau S_\Lambda} \alpha_\sigma = 0 \text{ modulo } p$$

for every Λ and τ, since $C_{n+1}^m = (-1)^m$ modulo p. It means, by Lemma in [2], that the algebra $M_2(F)$ satisfies the trace identity $\phi(v) = 0$. Therefore, by the Theorem from the paper [2], the polynomial $\phi(v)$ can be written in the algebra $\widetilde{F}\langle X \rangle$ as a linear combination (with the coefficients from the simple subfield) of the polynomials of the form $(Tr(1) - 2)w$, $w_1 X_2(w_2, w_3) w_4$ and $w_1 Tr(X_2(w_2, w_3) w_4)$, where the w's are the trace monomials. It follows from this that $v = pv_1 + g$, where v_1 is a multilinear polynomial with

integer coefficients and the polynomial g belongs to the ideal of trace identities of the algebra $M_2(Q)$. Using the last equality, the identity (1) can be written in the form $X_n^+ = p^{s+1}v_1$. This contradicts the maximality of s.

The identity $(1)(s = l, v = u_k)$ implies that the sum of the coefficients of the polynomial u_k is equal to 1 modulo p. Hence, in particular, the algebra $M_2(F)$ does not satisfy the identity $\phi(u_k) = 0$. Let $\sigma \in S_{n+1}$. Since the ideal of trace identities of the algebra $M_2(Q)$ is 2-classical and $\sigma X_n^+ = X_n^+ \sigma = X_n^+$, then, by (1), the algebra $M_2(F)$ satisfies the identities $\sigma\phi(u_k) = \phi(u_k)\sigma = \phi(u_k)$. It means $\phi(u_k) = f_k$ and the Lemma 2 is proved.

By Lemma 2, it is sufficient to calculate the polynomials u_k. We consider a more general problem. Let $h = h(x_1, \ldots x_n) \in \widetilde{Z}\langle X \rangle$ be a multilinear trace polynomial such that the algebra $M_m(Q)$ does not satisfy the trace identity $h = 0$. Since the algebra $M_m(F)$ is finitely dimensional then there exist the non-negative integer l and the uniquely defined, modulo the ideal of the trace identities of the algebra $M_m(Q)$, polynomial $u \in \widetilde{Z}\langle X \rangle$ such that the algebra $M_m(Q)$ satisfies the identity

$$h = p^l u, \qquad (2)$$

but the algebra $M_m(F)$ does not satisfy the identity $\phi(u) = 0$. We give the algorithm for the calculation of the polynomial u.

This algorithm can be extracted from the proof of the Theorem 1 [2]. Let $g \in \widetilde{P}_n$

$$\lambda_n(g) = \sum_{\sigma \in S_{n+1}} \gamma_\sigma \sigma,$$

where $\gamma_\sigma \in Z$.

We order the set S_{n+1}, putting $\sigma > \tau$ if and only if there exists a number $t \geq 0$ such that $\sigma(i) = \tau(i)$ for $i < t$ and $\sigma(t) > \tau(t)$. The permutation $\sigma \in S_{n+1}$ is called m-decomposable if the series $\sigma(0), \cdots, \sigma(n)$ contains the descending subseries of length $m + 1$.

Assume that $\gamma_\sigma \neq 0$ for some m-decomposable permutation σ. Let $\tau \in S_{n+1}$ be a maximal m-decomposable permutation such that $\gamma_\tau \neq 0$. Then $\sigma(i_1) > \ldots > \sigma(i_{m+1})$ for some $i_1 < \ldots < i_{m+1}$. Denote by S the subgroup of the group S_{n+1} consisting of all permutations σ such that $\sigma(j) = j$ for all

$$j \notin \{i_1, \ldots i_{m+1}\}.$$

Consider the element

$$d = (\sum_{\sigma \in S} (-1)^\sigma \sigma\tau).$$

By what is proved in [6], the variety $\mathrm{Var} M_m(Q)$ is m-classical and every trace identity of the algebra $M_m(Q)$ follows from the Cayley - Hamilton identity

$X_m = 0$. It follows from this that $\lambda_n^{-1}(d) = 0$ is an identity of the algebra $M_m(Q)$ since the group S is isomorphic to the group S_{m+1}. Modulo the two-sided ideal generated by d, the permutation τ can be written as a Z-linear combination of the permutations which are less than τ.

Applying the described procedure to the polynomial h, we can find the polynomial w such that the algebra $M_m(Q)$ satisfies the identity $h = w$ and the element $\lambda_n(w)$ is a Z-linear combination of m-indecomposable permutations.

Lemma 3. *The algebra $M_m(Q)$ satisfies the identity $w = p^l u$.*

Proof. Applying the described algorithm to the polynomial u, we can assume that the element $\lambda_n(u)$ is a Z-linear combination of m-indecomposable permutations. Then, by [2], $w = p^l u$.

4. The polynomial f_3 for $p = 2$.

Applying the algorithm described in the previous section, we have calculated, with a help of PC, the polynomial f_3 for $p = 2$:

$$f_3(x,y,z,t,u,v) = xytvzu + xyuvtz + xyuzvt + xyvutz + xyztuv +$$

$$+xyztvu + xyzvut + xtyuvz + xtyvuz + xtuvzy + xtvzuy + xuvtzy +$$

$$+xvutzy + xzvuyt + ytuvxz + ytvxzu + yuxtvz + yuxzvt + yuvtxz +$$

$$+yuzvtx + yvtxuz + yvutxz + yztuvx + yztvux + yzvutx + txuyvz +$$

$$+tyuvzx + tuvzyx + tvzxuy + tvzuyx + uvtzyx + uzxvty + uzvtyx +$$

$$+vutzyx + zxtuvy + zxtvuy + zxvuty + ztuvyx + ztvuyx + zuxtvy +$$

$$+zuvytx + zvyuxt + zvtxuy + zvuytx + xzutyTrv + xtyuzTrv +$$

$$+ytuzxTrv + zuytxTrv + xytzTruv + xyuzTrtv + xyzuTrtv +$$

$$+xuzyTrtv + xzvtTryu + yuxtTrzv + yuxzTrtv + yvuzTrxt +$$

$$+yzuxTrtv + txuyTrzv + tvzxTryu + uzyxTrtv + zxuyTrtv +$$

$$+ztyxTruv + zuyxTrtv + zuvyTrxt + xyztTruTrv +$$

$$+xztyTruTrv + xtzyTruTrv + yztxTruTrv + ytzxTruTrv +$$

$$+tzyxTruTrv + xytTrzuv + xytTrzvu + xyzTrtuv +$$

$$+xtyTrzvu + xuzTryvt + xzuTrytv + ytxTrzuv + tyxTrzuv +$$

$$+tyxTrzvu + uzxTryvt + zyxTrtvu + zuxTrytv + xytTrzuTrv+$$

$$+xyzTrtuTrv + xyzTruvTrt + xtyTrzuTrv + ytxTrzuTrv+$$

$$+yuzTrxtTrv + tyxTrzuTrv + zyxTrtuTrv + zyxTruvTrt+$$

$$+zuyTrxtTrv + xyzTrtTruTrv + zyxTrtTruTrv+$$

$$+xyTrztuv + xzTrytuv + xzTrytvu + xzTryvut + yxTrzvut+$$

$$+ytTrxzuv + ytTrxzvu + tyTrxuvz + tyTrxvuz + zxTrytuv+$$

$$+zxTryuvt + zxTryvut + xyTrtvuTrz + xzTrytuTrv+$$

$$+yxTrtuvTrz + ytTrxzuTrv + zxTryutTrv + tyTrxuzTrv+$$

$$+xyTrztTruv + xtTryuTrzv + yxTrztTruv + txTryuTrzv+$$

$$+xyTruvTrzTrt + yxTruvTrzTrt + xyTrzTrtTruTrv+$$

$$+yxTrzTrtTruTrv + xTrytuvz + xTryzvut + yTrxtuvz+$$

$$+yTrxtvzu + yTrxuvtz + yTrxuzvt + yTrxvutz + yTrxztuv+$$

$$+yTrxztvu + yTrxzvut + zTrxtvyu + zTrxuyvt + xTrztuvTry+$$

$$+xTrzvutTry + xTryuzTrtv + xTryzuTrtv + xTrtuvTryz+$$

$$xTrtvuTryz + yTrxtzTruv + yTrxztTruv + zTryuvTrxt+$$

$$zTryvuTrxt + xTrtuvTryTrz + xTrtvuTryTrz+$$

$$+Trxytvuz + Trxtyuvz + Trxzuvty + Trxzvuyt+$$

$$+TrytvuzTrx + TryuvtzTrx + TryvutzTrx + TryztuvTrx+$$

$$+TryztvuTrx + TryzuvtTrx + TrxtvzTryu + TrxzvtTryu+$$

$$+TrytuvTrxz + TrytvuTrxz + TryuvtTrxz + TryvutTrxz+$$

$$+TrztuvTrxy + TrzvutTrxy + TrxytzTruTrv + TrxztyTruTrv$$

$$+TrytuzTrxTrv + TryzutTrxTrv + TrztuvTrxTry+$$

$$+TrzvutTrxTry + TrxytTrzuv + TrxtyTrzvu + TrxuzTryvt+$$

$$+TrxzuTrytv + TrxytTrzuTrv + TrxyzTrtuTrv+$$

$$+TrxtyTrzuTrv + TrxzyTrtuTrv + TrytuTrxzTrv+$$

$$+TryutTrxzTrv + TrztuTrxyTrv + TrzutTrxyTrv+$$

$$+TrxyzTrtTruTrv + TrxzyTrtTruTrv+$$

$$+TryztTrxTruTrv + TrytzTrxTruTrv+$$

$$+TrtuvTrxTryTrz + TrtvuTrxTryTrz+$$
$$+TrztuTrxTryTrv + TrzutTrxTryTrv+$$
$$+TrxTryTrzTrtTruTrv.$$

We have also calculated the polynomial f_2 for $p = 3$, but this polynomial cannot be written down since it contains more then 1000 summands. Finally, we are sure that the other polynomials f_k cannot be calculated by any computer.

References

[1]   Kemer A.R. (1991) Ideals of identities of associative algebras, *Amer.Math. Soc. Translations of Math. Monographs*, Vol **87, Providence, R.I.**

[2]   Kemer A. (1995) Multilinear identities of the algebras over a field of characteristic p, *Intern. J. of Algebra and Computations*, no **2**, pp 1-9.

[3]   Kemer A. (1998) Multilinear components of the regular prime varieties, *Lecture notes in pure and applied mathematics*, Vol **198**, pp 171-183.

[4]   Kemer A. (2001) Multilinear components of the prime subvarieties of the variety Var($M_2(F)$), *Algebras and Representations (submitted)*.

[5]   Kleshchev A (1994) On restrictions of irreducible modular representations of semisimple algebraic groups and symmetric groups to some natural subgroups, I, *Proc. Lond. Math. Soc.*, Vol **69, no 3**, pp 515-540.

[6]   Razmyslov Yu. P. (1974) Trace identities of the matrix algebras over field of characteristic zero, *Izv. AN SSSR (Russian)*, Vol. **38, no 4**, pp 723-756.

[7]   Razmyslov Yu. P. (1985) Trace identities and central polynomials of the matrix superalgebras $M_{n,k}$, *Mat. sbornik (Russian)*, Vol. **128, no 2**, pp 194-215.

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MODULES WHICH ARE SELF-PROJECTIVE RELATIVE TO COCLOSED SUBMODULES

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Abstract

Let M_1 and M_2 be modules. The module M_2 is called M_1-*cc-projective* if every homomorphism $\alpha : M_2 \longrightarrow M_1/K$, where K is a coclosed submodule of M_1, can be lifted to a homomorphism $\beta : M_2 \longrightarrow M_1$. Let M be an amply supplemented module. Then M is lifting if and only if every module is M-cc-projective.

Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of relatively projective modules M_i. Assume M is amply supplemented. Then M is self-cc-projective if and only if each M_i is self-cc-projective.

1   Introduction and Preliminaries

Throughout this paper all rings will have an identity and all modules will be unital right modules. Let M be a module. If N is a submodule of M, we write $N \leq M$ and if N is small in M, we write $N \ll M$.

Let M be a module and $A \leq B \leq M$. If $B/A \ll M/A$, then A is called a *coessential submodule* of B in M. A submodule K of M is called *coclosed* (denoted by $K \leq_{cc} M$) if K has no proper coessential submodule in M. Given a submodule N of M, a submodule K of M is called a *supplement* of N in M if K is minimal in the collection of submodules L of M such that $M = L + N$, equivalently, $M = N + K$ and $N \cap K \ll K$. A submodule K of M is called a *supplement* in M if there exists a submodule N of M such that K is a supplement of N in M. Any module M is amply supplemented if for any submodules A, B of M with $M = A + B$ there exists a supplement P of A such that $P \subseteq B$. The module M is called *weakly supplemented* if for every

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submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$.

Note that every supplement submodule of any module M is coclosed in M, and if M is weakly supplemented then every coclosed submodule of M is a supplement in M (see [2, Lemma 1.1].

Let M be a module. M is called a *lifting module* if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$, equivalently, for every submodule N of M there exist submodules K, K' of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \ll K'$. By [3, Proposition 4.8], the module M is lifting if and only if M is amply supplemented and every supplement (that is, coclosed) submodule of M is a direct summand.

Smith and Tercan [4] studied the following property for a module M:

(P_n) For every submodule K of M such that K can be written as a finite direct sum $K_1 \oplus \cdots \oplus K_n$ of complements $K_1, \cdots, K_n$ of M, every homomorphism $\alpha : K \longrightarrow M$ can be lifted to a homomorphism $\beta : M \longrightarrow M$.

Following this idea Santa-Clara and Smith [5] are concerned with the study of *self-c-injective modules*, i.e., modules M that satisfy (P_1). As a dual notion to the notion of self-c-injective modules we introduce the following definition:

Let M_1 and M_2 be modules. The module M_2 is M_1-*cc-projective* if every homomorphism $\alpha : M_2 \longrightarrow M_1/K$, where $K \leq_{cc} M_1$, can be lifted to a homomorphism $\beta : M_2 \longrightarrow M_1$. Clearly, if M_2 is M_1-projective, then M_2 is M_1-cc-projective. A module M is called *self-cc-projective* when it is M-cc-projective. Lifting modules are an example of modules with this property (see Proposition 2.2). On the other hand, every self-cc-projective module need not be lifting $(\mathbb{Z}_{\mathbb{Z}})$.

We prove general properties of self-cc-projective modules and find sufficient conditions for a direct sum of two self-cc-projective modules to be self-cc-projective. Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of relatively projective modules M_i with M amply supplemented. We prove that M is self-cc-projective if and only if each M_i is self-cc-projective (see Theorem 2.8)

2 cc-Projectivity

Lemma 2.1. *Let M be a module and let K be a coclosed submodule of M. If M/K is M-cc-projective, then K is a direct summand of M.*

Proof. By hypothesis, there exists a homomorphism $\alpha : M/K \longrightarrow M$ that lifts the identity $1 : M/K \longrightarrow M/K$. It is not hard to see that $M = K \oplus \alpha(M/K)$, so that K is a direct summand of M. $\square$

Proposition 2.2. *The following are equivalent for an amply supplemented module M.*

(i) M is lifting.

(ii) Every module is M-cc-projective.

(iii) For every coclosed submodule K of M, M/K is M-cc-projective.

Proof. (i)$\Rightarrow$(ii) Let N be any module. Let $\alpha : N \longrightarrow M/K$ be any homomorphism with $K \leq_{cc} M$. Since every coclosed submodule of M is a direct summand of M, now the proof is clear. Obviously, (ii) implies (iii). That (iii) implies (i) follows by Lemma 2.1. $\square$

The Prüfer p-group $Z(p^\infty)$ is a lifting Z-module. Therefore by Proposition 2.2, $Z(p^\infty)$ is self-cc-projective, but it is not self-projective.

Corollary 2.3. *The following are equivalent for any semiperfect ring R.*

(i) Every right R-module M is R-cc-projective.

(ii) For every coclosed right ideal I of R, R/I is R-cc-projective.

Lemma 2.4. *Let M_1 and M_2 be modules. If M_1 is M_2-cc-projective, then for every coclosed submodule N of M_2, M_1 is N-cc-projective. Moreover, if M_2 is weakly supplemented, then for every coclosed submodule N of M_2, M_1 is (M_2/N)-cc-projective.*

Proof. Let N be a coclosed submodule of M_2. Clearly, every coclosed submodule of N is a coclosed submodule of M_2. Therefore it is obvious that M_1 is N-cc-projective. Let us prove that M_1 is (M_2/N)-cc-projective and assume that M_2 is weakly supplemented. Let X/N be a coclosed submodule of M_2/N. By [2, Lemma 1.4(2)], X is coclosed in M_2. Let $f : M_1 \longrightarrow (M_2/N)/(X/N) \cong M_2/X$. Since M_1 is M_2-cc-projective, it can be easily seen that M_1 is (M_2/N)-cc-projective. $\square$

Lemma 2.5. *Let M and $\{N_i \mid i \in I\}$ be modules. Then $\oplus_{i \in I} N_i$ is M-cc-projective if and only if N_i is M-cc-projective, for every $i \in I$.*

Proof. The proof follows as for projectivity (see for example, [3, Proposition 4.32]. $\square$

The modules M_1 and M_2 are *relatively cc-projective* if M_i is M_j-cc-projective, for every $i, j \in \{1, 2\}$, $i \neq j$.

Corollary 2.6. *Let M_1 and M_2 be modules. If $M_1 \oplus M_2$ is self-cc-projective, then M_1 and M_2 are both self-cc-projective and relatively cc-projective. In particular, a direct summand of a self-cc-projective module is self-cc-projective.*

Proof. By Lemmas 2.4 and 2.5. □

Let M_1 and M_2 be modules. The module M_1 is *small M_2-projective* if every homomorphism $f : M_1 \longrightarrow M_2/A$, where A is a submdule of M_2 and $Im f \ll M_2/A$, can be lifted to a homomorphism $g : M_1 \longrightarrow M_2$. It is clear that if the module M_1 is M_2-projective then M_1 is small M_2-projective. Since the Z-module $Z/2Z \oplus Z/8Z$ is not lifting (see [1, Corollary 2]), which is amply supplemented, there exists a Z-module N such that N is not $Z/2Z \oplus Z/8Z$-cc-projective by Proposition 2.2 although N is both $Z/2Z$-cc-projective and $Z/8Z$-cc-projective. In this vein we prove the following lemma.

Lemma 2.7. *Let M_1 and M_2 be modules with $M = M_1 \oplus M_2$ amply supplemented such that M_1 is small M_2-projective. If any module N is M_2-cc-projective and M_1-projective, then it is M-cc-projective.*

Proof. Let $K \leq_{cc} M$ and consider the homomorphism $\alpha : N \longrightarrow M/K$ and the natural epimorphism $\pi : M \longrightarrow M/K$. Since M/K is amply supplemented, by [2, Proposition 1.5], there exists a submodule H/K of M/K such that $H/K \leq (K+M_1)/K, (K+M_1)/H \ll M/H$ and $H/K \leq_{cc} M/K$. Note that by [2, Lemma 1.4(2)], $H \leq_{cc} M$. Since $K + M_1 = H + M_1$, $(H+M_1)/H \ll M/H$. Therefore, there exists a submodule H' of H such that $M = H' \oplus M_2$ by [2, Lemma 2.4 or Proposition 2.6]. As M_2 and M/H' are isomorphic, N is M/H'-cc-projective. Let β be the epimorphism from M/K to M/H defined by $\beta(m + K) = m + H$ for all $m + K \in M/K$ and π_1 the epimorphism from M/H' to $M/H \cong (M/H')/(H/H')$ defined by $\pi_1(m + H') = m + H$ for all $m + H' \in M/H'$. Since N is M/H'-cc-projective, there exists a homomorphism $g : N \longrightarrow M/H'$ such that $\pi_1 g = \beta\alpha$. Now, consider the following homomorphisms:

$$N \xrightarrow{g} M/H' \overset{f}{\cong} M_2 \xrightarrow{i_1} M \xrightarrow{\pi} M/K$$

where i_1 is the inclusion map and f is the isomorphism from M/H' to M_2. Then we have the homomorphism $\pi i_1 f g : N \longrightarrow M/K$. Take any element n in N, and suppose $\alpha(n) = m' + K$ and $g(n) = m + H'$ with $m, m' \in M$. Therefore, $\pi_1 g(n) = \beta\alpha(n)$ implies that $m - m' \in H$. Write $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. Now,

$$(\pi i_1 f g - \alpha)(n) = \pi i_1 f g(n) - \alpha(n) =$$

$$= \pi i_1 f(m + H') - (m' + K) = \pi i_1 f(m_1 + m_2 + H') - (m' + K) =$$

$$= \pi i_1(m_2) - (m' + K) = \pi(m_2) - (m' + K) =$$

$$= m_2 + K - (m' + K) = m_1 + m_2 - m' - m_1 + K$$

implies that $Im(\pi i_1 fg - \alpha) \subseteq (H + M_1)/K = (K + M_1)/K$. Consider the inclusion map $i_2 : (K + M_1)/K \longrightarrow M/K$. Since $Im(\pi i_1 fg - \alpha) \subseteq Im(i_2) = (K + M_1)/K$, there exists a homomorphism $\gamma : N \longrightarrow (K + M_1)/K$ such that $i_2\gamma = \pi i_1 fg - \alpha$. Let $\pi_2 : M_1 \longrightarrow (K + M_1)/K$ be the natural epimorphism. Since N is M_1-projective, γ can be lifted to a homomorphism $\phi : N \longrightarrow M_1$. Consider, finally, the homomorphism $\theta = i_1 fg - \phi : N \longrightarrow M$. Let $n \in N$. Then

$$\pi\theta(n) = \pi(i_1 fg - \phi)(n) = \pi i_1 fg(n) - \pi\phi(n) =$$

$$= \pi i_1 fg(n) - \alpha(n) + \alpha(n) - \pi_2\phi(n) = (\pi i_1 fg - \alpha)(n) + \alpha(n) - i_2\gamma(n) = \alpha(n).$$

Therefore, α can be lifted to the homomorphism θ and N is M-cc-projective.
$\square$

We can now prove the following theorem.

Theorem 2.8. *Let $M_1, \cdots, M_n$ ($n \in \mathbb{N}$) be relatively projective modules with $M = M_1 \oplus \cdots \oplus M_n$ amply supplemented. Then M is self-cc-projective if and only if M_i is self-cc-projective, for every $i \in \{1, \cdots, n\}$.*

Proof. By Lemmas 2.5 and 2.7, using induction. $\square$

References

[1]   D. Keskin, *Finite direct sums of (D1)-modules*, Tr.J. of Mathematics, **22**, 85-91 (1998).

[2]   D. Keskin, *On lifting modules*, Comm. Algebra, **28**(7), 3427-3440 (2000).

[3]   S. H. Mohamed and B. J. Müller, *Continuous and discrete modules*, London Math. Soc. LNS 147 Cambridge Univ. Press, Cambridge (1990).

[4]   P.F. Smith and A. Tercan, *Continuous and quasi-continuous modules*, Houston Journal Math., **18**, 339-348 (1992).

[5]   C. Santa-Clara and P.F. Smith, *Modules which are self-injective relative to closed submodules*, Algebra and its Applications (Athens, OH, 1999), Contemp. Math., **259**, 487-499, Amer. Math. Soc., Providence, RI (2000).

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DECOMPOSITIONS OF ASSOCIATIVE RINGS

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Abstract

We introduce the notion of a quiver of a ring associated with an ideal and obtain a decomposition theorem in case when this ideal is T-nilpotent. We prove that right perfect domains are piecewise semi-primary.

1. Introduction

Let A be an associative ring with $1 \neq 0$, R be the Jacobson radical of the ring A and $Pr(A)$ be the prime radical of A.

As usual, a ring A is called *decomposable* if it decomposes into a direct product of two rings, otherwise the ring is indecomposable.

Definition 1.1 A ring A is called *finite decomposable* ring, or simply FD-ring, if A is a direct product of a finite number of indecomposable rings.

Obviously, right Noetherian rings and semi-perfect rings are FD-rings.

The next theorem is well-known (see, for example, [AF]). It is suitable for us to formulate this theorem in the following form:

Theorem 1.2 *Every finite decomposable ring A has a unique decomposition into a finite direct product of indecomposable rings, i.e. if*

$$A = B_1 \times \ldots \times B_s = C_1 \times \ldots \times C_t$$

are two such decompositions then $s = t$ and there exists a permutation σ of
$\{1, \ldots, s\}$ *such that* $B_i = C_{\sigma(i)}$ $(i = 1, \ldots, s)$.

Definition 1.3 Following Gabriel a finite oriented graph is called *quiver*. A quiver without multiple arrows and multiple loops is called a *simply laced quiver*.

Denote $VQ = \{1, \ldots, s\}$ - a set of the all vertices of the quiver Q and AQ – a set of the all arrows of Q. Then we write $Q = \{VQ, AQ\}$ and $[Q] = (t_{ij})$ is adjacency matrix of Q.

Let $M_n(R)$ be the set of all real matrices of order n. Denote $P_\tau = \sum_{i=1}^{n} e_{i\tau(i)}$ where τ is a permutation of the letters $1, 2, \ldots, n$ and e_{ij} are corresponding matrix units. Clearly, $P_\tau^T P_\tau = P_\tau P_\tau^T$. The matrix P_τ is a permutation matrix.

Definition 1.4 A matrix $B \in M_n(R)$ is called *permutational reducible* if there exists a permutation matrix P_τ such that

$$P_\tau^T B P_\tau = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix},$$

where B_1 and B_2 are square matrices of the order less then n.

Proposition 1.5 *For any matrix $B \in M_n(R)$, there exists a permutation matrix P_τ such that*

$$P_\tau^T B P_\tau = \begin{pmatrix} B_1 & B_{12} & \ldots & B_{1t} \\ 0 & B_2 & \ldots & B_{2t} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & B_t \end{pmatrix},$$

where the matrices $B_1, B_2, \ldots, B_t$ are permutational irreducible.

Proof. Consider an arbitrary matrix $B \in M_n(R)$. If it is permutational reducible then there exists a permutation matrix P_1 such that

$$P_1^T B P_1 = \begin{pmatrix} C & X \\ 0 & D \end{pmatrix}.$$

If some of matrices C and $\mathcal{D}$ is permutational reducible, then obviously the matrix B can be transformed by means a permutation matrix P_2 to the form:

$$P_2^T B P_2 = \begin{pmatrix} K & X & Y \\ 0 & L & Z \\ 0 & 0 & M \end{pmatrix}.$$

If anyone of the matrices K, L, M is permutational reducible, then this process can be continued. In several steps, we obtain the proof of proposition.

Let Q be a quiver and $VQ = \{1, \ldots, s\}$. If an arrow σ connects the vertex i with the vertex j then i is called the *beginning* and j the *end* of σ. It will be denoted as $\sigma : i \to j$.

A path of quiver Q from a vertex i to a vertex j is an ordered set of k arrows such that the beginning of each arrow is the end of the previous one, the vertex i is the beginning of σ_1, while the vertex j is the end of σ_k.

Definition 1.6 A quiver is called *strongly connected* if there is a path between any two of its vertices. By convention, one-pointed graph without arrows will be considered as strongly connected quiver.

Proposition 1.7.[Lan, ch.9]. *A quiver Q is strongly connected if and only if the adjacency matrix $[Q]$ is permutational irreducible.*

From Propositions 1.5 and 1.7 we obtain such assertion.

Proposition 1.8 *There exists a numeration of the vertices of the quiver Q such that*

$$[Q] = \begin{bmatrix} B_1 & B_{12} & \ldots & B_{1m} \\ 0 & B_2 & \ldots & B_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & B_m \end{bmatrix},$$

where the matrices $B_1, \ldots, B_m$ are permutational irreducible, i.e. corresponding to the strongly connected components of the quiver Q.

Definition 1.9 The numeration of the vertices of Q will be called *standard* if $[Q]$ has form as in the proposition 1.8.

2. Quiver associated with an ideal

Let J be a two-sided ideal of a ring A contained in the Jacobson radical R of A such that the idempotents can be lifted modulo J.

Definition 2.1 The factor-ring A/J will be called the *J-diagonal* of the ring A.

In particular, if $J = Pr(A)$, then $Pr(A) \subset R$ and the idempotents can be lifted modulo $Pr(A)$.

Definition 2.2 The factor-ring $A/Pr(A)$ is called the *diagonal* of the ring A.

Definition 2.3 A ring A is called a *ring with finitely decomposed J-diagonal*, or simply $FD(J)$-ring, if its J-diagonal A/J is a FD-ring.

If $J = Pr(A)$ we have such definition.

Definition 2.4 A ring A is called a *ring with finitely decomposed diagonal*, or simply FDD-ring, if its diagonal $A/Pr(A)$ is a FD-ring.

For arbitrary $FD(J)$-ring A we will build the quiver $Q[A, J]$.

Consider the J-diagonal of the $FD(J)$-ring A: $\bar{A} = A/J = \bar{A}_1 \times \ldots \times \bar{A}_t$, where all rings $\bar{A}_1, \ldots, \bar{A}_t$ are indecomposable and $\bar{1} = \bar{f}_1 + \ldots + \bar{f}_t$ is the corresponding decomposition of $\bar{1} \in \bar{A}$ into a sum of mutually orthogonal central idempotents, i.e. $\bar{f}_i \bar{A} \bar{f}_i = \bar{f}_i \bar{A} = \bar{A} \bar{f}_i = \bar{A}_i$ for $i = 1, \ldots, t$. Put $W = J/J^2$. Establish the correspondence between idempotents $\bar{f}_1, \ldots, \bar{f}_t$ and vertices $1, \ldots, t$ connecting a vertex i with a vertex j by an arrow with the beginning at i and the end at j if and only if $\bar{f}_i W \bar{f}_j \neq 0$. The obtained finite oriented graph $Q(A, J)$ will be called the quiver associated with the ideal J.

Taking into account Theorem 1.2, one can easily see that the quiver $Q(A, J)$ of the $FD(J)$-ring A is defined uniquely up to a renumeration of the vertices and $Q(A, J) = Q(A/J^2, W)$.

By definition the quiver $Q(A, J)$ is the simply-laced quiver the adjacency matrix $[Q(A, J)]$ is the $(0, 1)$-matrix.

Definition 2.5 A quiver $Q(A, Pr(A))$ of FDD-ring A will be called the *prime quiver* of the ring A.

Suppose that J be a two-sided ideal of a ring A contained in the Jacobson radical R of $FD(J)$-ring A such that the idempotents can be lifted modulo J. Let $\bar{A} = A/J = \bar{A}_1 \times \ldots \times \bar{A}_t$ be a decomposition of $\bar{A}$ into a direct product of indecomposable rings $\bar{A}_1, \ldots, \bar{A}_t$ and let $\bar{1} = \bar{f}_1 + \ldots + \bar{f}_t$ be the corresponding decomposition of $\bar{1} \in \bar{A}$ into a sum of mutually orthogonal idempotents.

By [Lam, Ch.3] the idempotents $\bar{f}_1, \ldots, \bar{f}_t$ can be lifted modulo J preserving the orthogonality: $1 = f_1 + \ldots + f_t$, where $f_i f_j = \delta_{ij} f_j$ and $\bar{f}_i = f_i + J$ $(i, j = 1, \ldots, t)$.

Let $A_{ij} = f_i A f_j$ and $J_i = f_i J f_i$ $(i, j = 1, \ldots, t)$. Then we have the following two-sided Peirce decomposition of A and J:

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1t} \\ A_{21} & A_{22} & \ldots & A_{2t} \\ \ldots & \ldots & \ldots & \ldots \\ A_{t1} & A_{t2} & \ldots & A_{tt} \end{bmatrix}, \qquad (1)$$

$$J = \begin{bmatrix} J_1 & A_{12} & ... & A_{1t} \\ A_{21} & J_2 & ... & A_{2t} \\ ... & ... & ... & ... \\ A_{t1} & A_{t2} & ... & J_t \end{bmatrix} . \qquad (2)$$

Definition 2.6 The two-sided Peirce decomposition of $FD(J)$-ring A will be called J-*standard*, if $Q(A, J)$ has a standard numeration of its vertices.

Definition 2.7 The two-sided Peirce decomposition of FDD-ring A will be called *standard*, if $PQ(A)$ has a standard numeration of its vertices.

3. Decomposition of rings with T-nilpotent ideal.

A ring A is called *semi-primary* if the factor ring A/R is artinian and R is nilpotent.

A ring A is called *semi-perfect* if the factor ring A/R is artinian and the idempotents can be lifted modulo R [B].

Obviously, every semi-primary ring is semi-perfect.

We'll give an important definition of T-nilpotentency introduced by H.Bass [B].

Definition 3.1 A set S is called *right (resp. left) T-nilpotent* if for any sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements of S, there exists a positive integer k depending on the sequence such that $a_k a_{k-1} \ldots a_1 = 0 \quad (a_1 \ldots a_{k-1} a_k = 0)$. A set S is called T-*nilpotent* if it is right and left T-nilpotent.

Clearly, any T-nilpotent ideal (right, left, two-sided) is a nil-ideal.

A ring A is called *right perfect (left perfect)* if R is right (left) T-nilpotent and A/R is artinian. A right and left perfect ring is called *perfect*.

Obviously, every right (left) perfect ring is semi-perfect.

Theorem 3.2 [Ka, §11.5] *For any right ideal $\mathcal{I}$ in a ring A, the following conditions are equivalent:*
(1) $\mathcal{I}$ is right T-nilpotent.
(2) A right A-module M, that satisfies the equality $M\mathcal{I} = M$, is equal to zero.
(3) For any nonzero right A-module M it holds $M\mathcal{I} \neq M$.
(4) $A^I \mathcal{I} \neq A^I$, where A^I is a free module of a countable rank.

Lemma 3.3 *If J be a two-sided right T-nilpotent ideal of a ring A, then eJe is right T-nilpotent ideal of a ring eAe for every nonzero idempotent $e \in A$.*

Proof. Obviously, a set eJe is a two-sided ideal of a ring eAe. Let $a_1, a_2, \ldots$ be a sequence of elements of eJe. Since $eJe \subset J$, then for some k we have $a_k a_{k-1} \ldots a_1 = 0$.

Theorem 3.4 *The following conditions are equivalent for a ring A with T-nilpotent ideal J:*
(1) A is indecomposable.
(2) The factor ring A/J^2 is indecomposable.

Proof. $(1) \implies (2)$. Suppose that the factor ring $\bar{A} = A/J^2 = \bar{A}_1 \times \bar{A}_2$ and $\bar{1} = \bar{f}_1 + \bar{f}_2$ is a corresponding decomposition of the identity $\bar{1}$ of the ring $\bar{A}$ into a sum of orthogonal central idempotents of $\bar{A}$. Since J^2 is a nil-ideal then there exist idempotents $f_1, f_2 \in A$ such that $1 = f_1 + f_2$ and $\bar{f}_1 = f_1 + J^2$, $\bar{f}_2 = f_2 + J^2$.

Consider the two-sided Peirce decomposition of the ring A corresponding to the decomposition $1 = f_1 + f_2$:

$$A = \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix},$$

where $A_i = f_i A f_i$ $(i = 1, 2)$, $X = f_1 A f_2$, $Y = f_2 A f_1$.

Since $\bar{f}_1 \bar{A} \bar{f}_2 = 0$ and $\bar{f}_2 \bar{A} \bar{f}_1 = 0$, then $X \subset J^2$ and $Y \subset J^2$, where from $X = f_1 J^2 f_2$ and $Y = f_2 J^2 f_1$.

Computing J^2, we obtain:

$$J^2 = \begin{pmatrix} J_1^2 + XY & J_1 X + X J_2 \\ Y J_1 + J_2 Y & J_2^2 + Y X \end{pmatrix}.$$

Since $X = f_1 J^2 f_2$ and $Y = f_2 J^2 f_1$ then $X = J_1 X + X J_2$ and $Y = Y J_1 + J_2 Y$. Since J is T-nilpotent then due to lemma 3.3 and theorem 3.2 we get that $X = 0$ and $Y = 0$. Therefore $A = A_1 \times A_2$ and the implication $(1) \implies (2)$ is proved.

The inverse implication $(2) \implies (1)$ is obvious.

Using theorems 3.3, 3.4 and standard two-sided Peirce decomposition of the FDD-ring A with T-nilpotent prime radical, we can prove the following theorem:

Theorem 3.5 *Let A be an FDD-ring. The prime quiver of an FDD-ring A with the T-nilpotent prime radical $Pr(A)$ is connected if and only if the ring A is indecomposable.*

Definition 3.6 A ring A with finite decomposable diagonal will be called *connected* if the prime quiver $PQ(A)$ of FDD-ring A is connected.

Taking into account that the prime radical of the right Noetherian ring is nilpotent [Lam, Ch.3], one can obtain the following result.

Corollary 3.7 *A right Noetherian ring has a unique decomposition into a finite direct product of connected rings.*

4. Right perfect piecewise domains are semi-primary

Proposition 4.1 [B], [F, ch. 22, Th.22.9] *If a ring A is right (left) perfect, then every nonzero left (right) A-module has nonzero socle.*

Denote $M_n(\mathcal{D})$ a ring of all square matrices of the order n over the division ring $\mathcal{D}$.

Theorem 4.2 *If A is a semi-prime semi-perfect indecomposable ring and $A \not\simeq M_n(\mathcal{D})$, then $soc_A A = soc A_A = 0$.*

Proof. Obviously, we can assume that A is a reduced ring, i.e. A/R is a finite direct product of division rings. We'll show that $S_r = soc A_A$ equals zero. If $S_r \subset R$ then $S_r R = 0$ and $S_r^2 = 0$. Now $S_r \not\subset R$. Let $A = P_1 \oplus \ldots \oplus P_s$ be a decomposition of A into a direct sum of indecomposable projective modules and $U_i = P_i/P_i R$, $i = 1, \ldots, s$, are simple and $s > 1$.

Then from $S_r \not\subset R$ we obtain that at less one of the modules $P_1, \ldots, P_s$ (for example, $P_s = e_s A$ and $e_s^2 = e_s$) is simple.

Denote $f = 1 - e_s$ and $e = e_s$. We have such two-sided Peirce decomposition:

$$A = \begin{pmatrix} fAf & fAe \\ eAf & eAe \end{pmatrix}.$$

Obviously, $eAf = 0$ and so

$$\mathcal{I} = \begin{pmatrix} 0 & fAe \\ 0 & 0 \end{pmatrix}.$$

is the nonzero ideal of A (A – the indecomposable ring) and $\mathcal{I}^2 = 0$. We have contradiction and $S_r = 0$. Analogously, $S_l = 0$. Consequently, $A = P_s$ and in the general case, we have $A = P_s^n$ and $A \simeq End_A P_s^n = M_n(E(P_s))$, where $\mathcal{D} = E(P_s)$ is the division ring. Theorem is proved.

Corollary 4.3 *Every semi-prime right (left) perfect ring is a semi-simple artinian ring.*

Corollary 4.4 *The Jacobson radical and the prime radical of right (left) perfect ring are coincides.*

We will give a definition of piecewise domain [GS] in a semi-perfect case.

Definition 4.5 A semi-perfect ring A is called a *piecewise domain* if every nonzero homomorphism of indecomposable projective A-modules is monomorphism

Proposition 4.6 [GS] *A prime radical of a piecewise domain is nilpotent.*

Proposition 4.7 [Kir1],[Kir3] *A semi-perfect semi-hereditary ring is a piecewise domain.*

From proposition 4.6 and corollary 4.4 we have the next theorem.

Theorem 4.8 *A piecewise right perfect domain is a semi-primary ring.*

This theorem is a generalization of one theorem of Teply [T].

Definition 4.9 [P] A quiver without oriented cycles is called *acyclic*.

Theorem 4.10 [KSY, Theorem 3.3] *The prime quiver $PQ(A)$ of a semiperfect piecewise domain A is an acyclic simply laced quiver.*

Theorem 4.11 *If J be a two-sided right T-nilpotent ideal of $FD(J)$-ring and a quiver $Q(A, J)$ is acyclic, then J is nilpotent.*

Proof. Let t be a number of the vertices of a quiver $Q(A, J)$. Now we will prove by induction by t that an ideal J is nilpotent.

In the case $t = 1$ $J^2 = J$ and by Theorem 3.2 $J = 0$.

Let $t > 1$. We will suppose that a numeration of the vertices of the quiver $Q(A, J)$ is standard. Then a vertex t is a sink.

Let

$$A = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1t} \\ A_{21} & A_{22} & ... & A_{2t} \\ ... & ... & ... & ... \\ A_{t1} & A_{t2} & ... & A_{tt} \end{bmatrix}$$

be the J-standard two-sided Peirce decomposition and $1 = f_1 + ... + f_t$ is the corresponding decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents.

Denote $F_t = f$ and $e = 1 - f$. Let $J_1 = eJe$ and $fJf = J_2$, $X = eAf$ and $Y = fAe$.

Consider the two-sided Peirce decomposition of the ideal J:

$$J = \begin{pmatrix} J_1 & X \\ Y & J_2 \end{pmatrix} .$$

Computing J^2, we obtain:

$$J^2 = \left(\begin{array}{cc} J_1^2 + XY & J_1 X + X J_2 \\ Y J_1 + J_2 Y & J_2^2 + Y X \end{array} \right).$$

Since the vertex t is a sink we have that $f_t W f_i = 0$ for $i = 1, \ldots, t$ and consequently $J_2^2 + Y X = J_2$ and $Y J_1 + J_2 Y = Y$. By Theorem 3.2, $J_2 = Y X$ and $Y = J_2 Y$. So $Y = J_2 Y = Y X Y \subset Y J_1$ and $Y = 0$.

Therefore, $J = \left(\begin{array}{cc} J_1 & X \\ 0 & J_2 \end{array} \right)$. By induction, J_1 and J_2 are nilpotent ideals, and so J is nilpotent. Theorem is proved.

Let J be as above.

Corollary 4.12 *A degree of a nilpotentency of an ideal J is less or equal t, where t is a number of the vertices of a quiver $Q(A, J)$.*

From Theorems 4.10 and 4.11 we have obtained another proof of Theorem 4.8.

References

[AF] Anderson F., K.Fuller, Rings and Categories of Modules, Graduate Texts in Math. 13, Springer-Verlag, New York - Heidelberg - Berlin, (1973).

[B] Bass H., *Finitistic dimension and homological generalization of semi-primary rings*, Trans. AMS, **95**(1960), 466-488.

[F] Faith C., Algebra II, Ring Theory, Springer-Verlag, Berlin Heideberg New York, (1976).

[GS] Gordon R., L.W.Small , *Piecewise domains*, J.Algebra, **23**(1972), 555-564.

[Ka] Kasch F., Moduln und Ringe, B.G.Tenbner Stuttgart, (1977).

[Kir1] Kirichenko V.V., *Hereditary and semi-hereditary serial rings*, Zap. Nauch. Sem. LOMI USSR, v.114, (1982), 137-147, J. of Soviet Math., v.27, N4, (1984), 2933-2941.

[Kir2] Kirichenko V.V., *Semi-perfect rings and their quivers*, Anal. Stiint. Univ. Ovidius, Constanza, 4(1996), 89-96.

[Kir3] Kirichenko V.V., *Semi-perfect semi-distributive rings*, Algebras and Representations Theory, 3(2000), 81-98.

[KSY] Kirichenko V.V., Samir Valio, Yu.V.Yaremenko , *Semi-perfect rings and their quivers*, Infinite Groups and Related Algebraic Topics, Institute of Mathematics NAS Ukraine, 1993, pp. 438-456.

[Lam] Lambek J., Lectures on Rings and Modules, Blaisdell-Ginn, Wal-tham - Toronto - London, (1966).

[Lan] Lankaster P., Theory of matrices, Academic Press. New York - London, (1969).

[T]   Teply M., *Right hereditary, right perfect rings are semiprimary*, Advances in Ring Theory, (1997), pp.313-316.

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SPECIAL SUBGROUPS OF SKEW LINEAR GROUPS

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Abstract

Let D be an infinite division algebra of finite dimension over its centre $Z(D) = F$ and n is a positive integer. The structure of maximal subgroups and finitely generated subnormal subgroups of skew linear groups are investigated. In particular, assume that N is a normal subgroup of $GL_n(D)$ and M is a maximal subgroup of N containing $Z(N)$. It is shown that if $M/Z(N)$ is finite, then N is central.

Let n be a positive integer and D be an infinite division algebra of finite dimension over its centre. In this note we investigate the structure of some particular subgroups of $GL_n(D)$. We recall that finite subgroups of $GL_1(D)$ are completely classified by Amitsur in [1], and there exist also some results concerning the case $n > 1$ in the literature (cf. [17]). Here we are interested in the structure of infinite subgroups of $GL_n(D)$. The structure of subnormal subgroups of $GL_n(D)$ is investigated in [2-3], and [13]. Here, we present shorter proofs for some of the results given in those papers, and we shall then apply the results to present some interesting facts about the behaviour of maximal subgroups of $GL_n(D)$ with $n \geq 1$. More precisely, let D be an infinite division algebra of finite dimension over its centre F and let n be a positive integer. Given a normal subgroup N of $GL_n(D)$, assume that M is a maximal subgroup of N containing $Z(N)$. It is shown that if $M/Z(N)$ is finite, then N is central. It is also proved that if M is a maximal subgroup of $GL_n(D)$ containing F^* and $[M : F^*] < \infty$, then $D = F$. In this direction we also show that if D is algebraic over its centre F, then the finiteness of M implies that of D. For more recent results on the structure of subgroups of $GL_n(D)$ one may consult [2], [4], [7-9], and [12]. We begin the material of this note with

Lemma 1. *Given a division ring D with centre F, assume that M is a maximal subgroup of D^*. Then either $Z(M) = F^* \cap M$ or $M \cup \{0\}$ is a maximal division subring of D.*

Proof. By Proposition 1 of [3], we know that either $F^* \subset M$ or the derived group $D' \subset M$. If $D' \subset M$, then M is normal in D^*. Now, using Proposition 4 of [11], we obtain $Z(M) = F^* \cap M$. If $F^* \subset M$ and D' is not contained in M, then take the division ring $F(M)$ generated by F and M. By maximality of M in D^*, we have either $F(M)^* = M$ or $D = F(M)$. If the last case occurs, then it is easily checked that $Z(M) = F^* \cap M$. Otherwise, $F(M)^* = M$ or equivalently $M \cup \{0\}$ is a division subring of D which is obviously maximal.

To prove our next result, we shall use the following theorems:

Theorem A. *Let D be a division ring with centre F. If the derived group D' is radical over F, then D is commutative.*(cf. [9]).

Theorem B. *Let D be a division ring with centre F, and assume that M is a maximal subgroup of D^* containing F^*. If $[M : F^*] < \infty$, then $D = F$.*(cf. [3]).

Theorem C. *Let D be a division ring with centre F and put $A = M_n(D)$, $A^* = GL_n(D)$, and $A' = SL_n(D)$, where $n \geq 1$. Then we have*

(1) *If $A'/Z(A')$ is torsion, then $D = F$.*

(2) *If each element of A' is algebraic over F, then A^* is algebraic over F.*(cf. [10]).

Theorem 2. *Let D be a division algebra algebraic over its centre F and n is a positive integer. Assume that M is a maximal subgroup of $GL_n(D)$. If M is finite, then so is D.*

Proof. We consider two cases:

Case 1. Assume that $n = 1$. We know, by Propositon 1 of [3], that either $F^* \subset M$ or $D' \subset M$. If $D' \subset M$, then D' is torsion and thus, by Theorem A, we obtain $D = F$. So M is a maximal subgroup of F^*. Now, take an element $x \in F^*\backslash M$, then $F^* = <x, M>$, where $M = <x_1, \cdots, x_r>$. This shows that F^* is finitely generated. Now, by a result of [2], we conclude that F^* is finite and so the result is established in this case.

Case 2. Assume now that $n \geq 2$. Therefore, either $F^* \subset M$ or $SL_n(D) \subset M$. If $SL_n(D) \subset M$, then $SL_n(D)$ is torsion and so, by Theorem C, we conclude that $D = F$. Take an element $\lambda \in F^*$ and consider the diagonal matrix B whose diagonal entries are $\lambda, \lambda^{-1}, 1, \cdots, 1$. It is clear that $B \in SL_n(D)$ and so $B^r = I$ and consequently $\lambda^r = 1$. Therefore, any element of

F^* satisfies the equation $x^r = 1$ and so F must be finite and the result follows in this case. If $F^* \subset M$, then F is finite and so D is algebraic over a finite field. Thus, by a theorem of Jacobson (cf. [5]), we conclude that $D = F$ is finite, and the proof is complete.

In the theory of groups, there are infinite groups in which each proper subgroup has a prime order (cf.[14]). Thus, in a group G if a maximal subgroup is finite we may not conclude that G is finite. But for a linear group the situation is different as the following result shows. The idea of the following proof is due to M. Mahmudi.

Theorem 3. *Let G be a linear group that is not simple, and M be a maximal subgroup of G. If M is finite, then so is G.*

Proof. Assume on the contrary that G is infinite. Since M is finite we may easily conclude that G is finitely generated. Since G is not simple we have $G' \neq G$. If the derived group G' is contained in M, then M is normal in G and consequently $G/M \cong Z_p$ for some prime number p. This contradicts our assumption that G is infinite. So, there exist normal subgroups that are not contained in M and let N be one of those normal subgroups. Then we have $G = MN$ and so we obtain $G/N \cong M/M \cap N$. Thus $[G : N] \leq |M| := r$. Assume that $x \in G \backslash M$ and put $M = \{x_1, \cdots, x_r\}$ and consider $r + 1$ elements $x, x_1, \cdots, x_r$. By Malcev's Approximation Theorem (cf.[18]), there exist a finite field k and a homomorphism ϕ from G into $GL_r(k)$ such that $\phi(x), \phi(x_1), \cdots, \phi(x_r)$ are distinct and so $[G : Ker\phi] \geq r$. Since $Ker\phi$ is infinite we conclude that $Ker\phi$ is not contained in M. This contradiction establishes the result.

The above theorem may be used to give a short proof of the following result which is first appeared in [3].

Corollary 4. *Let D be a division algebra of finite index over its centre F, and assume that M is a maximal subgroup of D^* containing F^*. If $[M : F^*] < \infty$, then $D = F$.*

Proof. Since the index of D over F is finite we may assume that D is a linear group. We know that F^* is closed in D^* with the Zariskii topology. Thus, by a result of Chevalley (cf. [18]), D^*/F^* is a linear group. Now, since M/F^* is maximal and finite in D^*/F^*, by Theorem 3, we conclude that D^*/F^* is finite. Finally, use Kaplansky's Theorem (cf. [6]) to obtain $D = F$.

We now turn to investigate the structure of finitely generated subnormal subgroups of $GL_n(D)$. Let D be a division algebra of finite dimension n over its centre F. Then each element of D may be viewed as an element of $M_n(F)$

by means of the regular representation. If D is finitely generated as a ring, then we may view D as a finitely generated $\mathbb{Z}$-algebra or $\mathbb{Z}_p$-algebra, where p is the characteristic of D, i.e., $D = < d_1, \cdots, d_r >$. Take an element $a \in F$. Since each element of D may be written as a sum and product of d_i, the representation of a in $M_n(F)$ is of the form aI, where I is the unit matrix of $M_n(F)$. Denote by R the ring generated by all entries of the matrices of representations of d_i. Then, we have $a \in R$ and since a is arbitrary we conclude that $R \subset F$ and consequently $R = F$. If $CharF = 0$, then we have $\mathbb{Z} \subset \mathbb{Q} \subset F$, where $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the field of rational numbers, respectively. Since F is finitely generated as a $\mathbb{Z}$-algebra we conclude that it is finitely generated as $\mathbb{Q}$-algebra. This implies that F is finitely generated as a $\mathbb{Q}$-module. Thus $\mathbb{Q}$ is finitely generated as a $\mathbb{Z}$-module which is nonsense. So we may assume that $CharF = p > 0$. Therefore, F is finitely generated as a $\mathbb{Z}_p$-module. Thus, F is a finite extension of $\mathbb{Z}_p$, i.e., F is finite. Now, since D is of finite dimension over F we conclude that D is finite by Wedderburn's Theorem. Therefore, we have shown that if D is finitely generated as a ring, then D is finite. This is a useful observation and will be used in the proofs of later results.

The next result was first appeared in [3] and its proof was very long. Here we shall give a short proof of the result using the theory of rings with polynomial identities.

Corollary 5. *Let D be a division algebra of finite index over its centre F and N be a normal subgroup of D^*. If N is finitely generated, then N is central.*

Proof. If N is non-central, by a theorem of Scott (cf. [16]), we conclude that N is not abelian. Thus, there exist elements $a, b \in N$ such that $ab \neq ba$, put $c = ba - ba$. Since c is algebraic over F we have $c^n + a_{n-1}c^{n-1} + \cdots + a_0 = 0$ where $a_i \in F$. Denote by R the ring generated by N and $a_{n-1}, \cdots, a_1, a_0^{-1}$. Since a_i is central we conclude that R is normal in D. Let M be a proper maximal left ideal of R. Since R is normal in D, by Cartan- Brauer-Hua Theorem (cf. [6]), we may conclude that M is also a maximal right ideal in R and so $E := R/M$ is a division ring. Since D is of finite index we may assume that D is embedded in $M_n(F)$, where n is the dimension of D over F. Now, by Amitsur-Levitzki Theorem (cf. [15]), D satisfies the polynomial S_{2n} and so does E. Thus, by Kaplansky's Theorem on primitive rings with polynomial identities (cf. [15]), E is of finite dimension over its centre. Now, since R is finitely generated, we may conclude that E is finitely generated. Therefore, by the remark made before the theorem, E is finite and so E is commutative by Wedderburn's theorem, i.e., $(a + M)(b + M) = (b + M)(a + M)$. This means that $c = ab - ba \in M$. On the other hand, we have $c^{-1} \in R$ since $a_0^{-1} \in R$.

Thus, $1 = cc^{-1} \in M$ which is in contradiction to the fact that M is proper, and so the result follows.

Corollary 6. *Let D be a division algebra of finite index over its centre F and N be a normal subgroup of D^*. If $N \cap D'$ is finitely generated, then N is central.*

Proof. Assume that N is not central. If $D' \subset N$, then $N \cap D' = D'$ and so D' is finitely generated. By Theorem 5, we conclude that D' is central. Now, by Theorem A, we obtain $D = F$ which contradicts the fact that N is non-central. Otherwise, $N \cap D'$ is normal in D^* and so, by Theorem 5, $N \cap D'$ is central. But this contradicts a result of Scott (cf. [16]) which asserts that the intersection of non-central normal subgroups of D^* is non-central and this completes the proof.

Corollary 5 is also true for finitely generated subnormal subgroups of D^*. In fact, the following more general result is shown in [13]:

Theorem D. *Let D be an infinite division algebra of finite dimension over its centre F. Assume that N is a subnormal subgroup of $GL_n(D)$ with $n \geq 1$. If N is finitely generated, then $N \subset F^*$.*

Using the above results one is able to strengthen Theorem 2 in the following form

Theorem 7. *Let D be an infinite division algebra of finite dimension over its centre F and n is a positive integer. Assume that M is a maximal subgroup of $GL_n(D)$ containing F^*. If $[M : F^*] < \infty$, then $D = F$.*

Proof. Let $x_1, \ldots, x_t$ be the representatives for cosets of F^* in M, i.e., $M = F^* x_1 \cup \ldots \cup F^* x_t$. Then, we have $M = < x_1, \ldots, x_t > F^*$, where $< x_1, \ldots, x_t >$ is the group generated by $x_1, \ldots, x_t$. Take $x \in GL_n(D) \backslash M$. By maximality of M, we obtain $GL_n(D) = < x_1, \ldots, x_t, x > F^*$. Put $H = < x_1, \ldots, x_t, x >$. Thus, $GL_n(D) = HF^*$ and consequently we have $SL_n(D) = H' \subset H$, i.e., H is normal in $GL_n(D)$. Now, by Theorem D, we conclude that $H \subset F^*$, i.e., $D^* = F^*$ which implies that $D = F$.

Finally, we are now in a position to prove a more general form of Theorem D as the following

Theorem 8. *Let D be an infinite division algebra of finite dimension over its centre F and n be a positive integer. Assume that N is a normal subgroup of $GL_n(D)$ and M is a maximal subgroup of N containing $Z(N)$. If $M/Z(N)$ is finite, then N is central.*

Proof. Let $x_1, \ldots, x_t$ be the representatives for cosets of $Z(N)$ in M, i.e.,

$M = Z(N)x_1 \cup \ldots \cup Z(N)x_t$. Then, we have $M =< x_1, \ldots, x_t > Z(N)$, where $< x_1, \ldots, x_t >$ is the group generated by $x_1, \ldots, x_t$. Take $x \in N \backslash M$. By maximality of M, we obtain $N =< x_1, \ldots, x_t, x > Z(N)$. Put $H =< x_1, \ldots, x_t, x >$. Thus, $N = HZ(N)$ and consequently we have $N' = H' \subset H$, i.e., H is normal in N since $Z(N) = N \cap F^*$ by Proposition 4 of [11]. Thus H is normal in N and so H is subnormal in $GL_n(D)$. Now, by Theorem D, we conclude that $H \subset F^*$, i.e., $N = HZ(N) \subset F^*$ which completes the proof.

It is believed that all the above results are true when D is a general division ring not necessarily of finite dimension over its centre.

References

[1] Amitsur, S. A., *The Finite Subgroups of the Multiplicative Group of a Division Ring*, Trans. Amer. Math. Soc., **80** (1995), 361-386.

[2] Akbari, S., Mahdavi-Hezavehi, M., *Normal Subgroups of $GL_n(D)$ Are not Finitely Generated*, Proc. Amer. Math. Soc., **128**(2000), 1627-1632.

[3] Akbari, S., Mahdavi-Hezavehi, M., Mahmudi, M. G., *Maximal Subgroups of $GL_1(D)$*, Journal of Algebra **217**(1999), 422-433.

[4] Goncalves, J. Z., *Free Groups in Subnormal Subgroups and the Residual Nilpotence of the Group of Units of Group Rings*, Can. Math. Bull. **27**(1984), 365-370.

[5] Jacobson, N., *Structure of Rings*, Coll. Pub., Vol. 37, Amer. Math. Soc., Providence, R.I., 1956.

[6] Lam, T.Y., *A First Course in Non-commutative Rings*, GTM, No. 131, Springer-Verlag, (1991).

[7] Lichtman, A. I., *On Normal Subgroups of the Multiplicative Group of Skew Fields Generated by a Polycyclic-by Finite Group*, J. Algebra, **78** (1982), 548-577.

[8] Lichtman, A. I., *On Subgroups of the Multiplicative Group of Skew Fields*, Proc. Amer. Math. Soc., **63** (1977), 15-16.

[9] Mahdavi-Hezavehi, M., *On Derived Groups of Division Rings*, Comm. Algebra, **23**(3), 913-926, (1995).

[10] Mahdavi-Hezavehi, M., and Akbari, S., *Some special subgroups of $GL_n(D)$* , Algebra Colloquium 5:4 (1998), 361-370.

[11] Mahdavi-Hezavehi, M., *Commutator subgroups of finite dimensional division algebras*, Rev. Roumaine Math. Pures Appl., 43(1998), 9-10, 853-867.

[12] Mahdavi-Hezavehi, M., Akbari, S., Mehraabaadi, M., and Hajie-Abolhassan, H., *On Derived Groups of Division Rings II*, Comm. Algebra, Vol. 23, No. 8, (1995), 2881-2887.

[13] Mahdavi-Hezavehi, M., Mahmudi, M. G., Yasamin, S. *Finitely Generated Subnormal Subgroups of $GL_n(D)$ Are Central*, Journal of Algebra (to appear).

[14] Olshanskii, A. Yu., *Infinite groups with cyclic subgroups*, Soviet Math. Dokl., **20**(1979) 343-346.

[15] Rowen, L., *Ring Theory*, Volume II, Academic Press, INC, 1988.

[16] Scott, W. R., *Group Theory*, Dover Publication, INC, 1987.

[17] Shirvani, M., Wehrfritz, B. A. F., *Skew Linear Groups*, LMS, Lecture Note Series, No. 118, (1986).

[18] Wehrfritz, B. A. F., *Infinite Linear Groups*, Springer-Verlag, (1973).

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INTEGRAL REPRESENTATIONS OF FINITE GROUPS AND GALOIS STABILITY

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Abstract

For a given algebraic number field F we consider a normal extension E/F of finite degree d and finite abelian subgroups $G \subset GL_n(E)$ of a given exponent t. We assume that G is stable under the natural action of the Galois group of E/F and consider fields $E = F(G)$ that are obtained via adjoining all matrix coefficients of all matrices $g \in G$ to F. It is proved that under some reasonable restrictions for n any E can be realized as $F(G)$, while if all coefficients of matrices in G are algebraic integers, there are only finitely many fields $E = F(G)$ for prescribed integers n and t or prescribed n and d. Some related results and conjectures are considered.

1   Introduction

In this paper some arithmetic problems for representations of finite groups over algebraic number fields and arithmetic rings under the ground field extensions are presented.

We consider some Galois extension E/F of finite degree d with the Galois group Γ for a field F of characteristic 0 and a finite abelian subgroup $G \subset GL_n(E)$ of the given exponent t, where we assume that G is stable under the natural coefficientwise Γ-action.

Throughout the paper O_E is the maximal order of E and $F(G)$ denotes a field that is obtained via adjoining to F all matrix coefficients of all matrices $g \in G$.

The main objective of this paper is to prove the existence of abelian Γ-stable subgroups G such that $F(G) = E$ provided some reasonable restrictions

for the fixed normal extension E/F and integers n, t, d hold and to study the interplay between the existence of Γ-stable groups G over algebraic number fields and over their rings of integers. Some recent results from [1], [5], [6] are presented in section 2.

The results related to the Galois stability of finite groups in a situation similar to ours arise in the theory of definite quadratic forms and Galois cohomologies of certain arithmetic groups if F is an algebraic number field and G is realized over its maximal order ([2], see also [8]). In our context we study whether a given field E normal over F can be realized as a field $E = F(G)$ in both cases $G \subset GL_n(E)$ and $G \subset GL_n(O_E)$, and if this is so what are the possible degrees n of matrix representations and the structure of G. Some similar questions for Γ-stable orders in simple algebras are considered in [9], see also [10] for some applications.

We give a positive answer to the first question: we prove that any finite normal field extension E/F can be obtained as $F(G)/F$ if $n \geq \phi_E(t)d$ where $\phi_E(t) = [E(\zeta_t) : E]$ is the generalized Euler function and ζ_t is a primitive t-root of 1. An explicit construction of these fields is given in Theorem 2 in sections 3 and 4. In fact, we construct some Galois algebras in the sense of [4], and we establish the lower bounds for their possible dimensions. We show (see Theorem 3 in section 3) that the restrictions for the given integers n, t, and d in Theorem 2 can not be improved.

The situation becomes different if E is an algebraic number field and all matrix coefficients of $g \in G$ are algebraic integers.

The existence of any Galois stable subgroups $G \subset GL_n(O_E)$ such that $F(G) \neq F$ is a rather subtle question. In particular, for $F = \mathbf{Q}$ all fields $F(G)$ whose discriminant is divisible by an odd prime must contain non-trivial roots of 1 [1], [5].

The paper is organized as follows. We discuss representations of finite Galois stable groups G over integers of algebraic number fields and their realization fields in section 2. Some general conjectures are given. But since they are reduced in [7], [5] to considering abelian groups G (and even elementary abelian G of prime exponent p), the further results of this paper deal mainly with representations of abelian groups G. We state an existence criterion, a finiteness theorem and some related results from [1], [5] and [6]. In section 3 we state the results of the similar nature for representations of abelian groups G over fields and their realization fields, their proofs are given in section 4.

Notation

We denote $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{Q}$ the fields of complex, real and rational numbers. $\mathbf{Z}$ is the ring of rational integers. $GL_n(R)$ denotes the general linear group over a ring R. $[E : F]$ denotes the degree of the field extension E/F. Throughout this

paper we write Γ for Galois groups, $\sigma, \gamma \in \Gamma$ for the elements of Γ. $\Gamma_j(\mathbf{p}) \subset \Gamma$ denotes the j-th ramification subgroup of a prime ideal $\mathbf{p}$. Finite groups are usually denoted by capital letters G, H, and their elements by small letters, e.g. $g \in G$, $h \in H$. We write ζ_t for a primitive t-root of 1. We denote by $\phi_K(t) = [K(\zeta_t) : K]$ the generalized Euler function for a field K. I_m stands for a unit $m \times m$-matrix. $\det M$ is the determinant of a matrix M. If G is a finite linear group, $F(G)$ stands for a field obtained by adjoining to F all matrix coefficients of all matrices $g \in G$. For Γ acting on G and any $\sigma \in \Gamma$ and $g \in G$ we write g^σ for the image of g under σ-action. $\dim_K A$ denotes the dimension of K-algebra A over the field K. $M_n(R)$ is the full matrix algebra over a ring R. O_K denotes the maximal order of a number field K.

2   Integral representations stable under the Galois action

Let K be a totally real algebraic number field with the maximal order O_K, G an algebraic subgroup of the general linear group $GL_n(\mathbf{C})$ defined over the field of rationals $\mathbf{Q}$. Because of the embedding of G to $GL_n(\mathbf{C})$ the intersection $G(O_K)$ of $GL_n(O_K)$ and $G(K)$, the subgroup of K-rational points of G, can be considered as the group of O_K-points of an affine group scheme over $\mathbf{Z}$, the ring of rational integers. Assume G to be definite in the following sense: the real Lie group $G(\mathbf{R})$ is compact. The problem which is interesting for applications to arithmetic groups and quadratic forms is the question: does the condition $G(O_K) = G(\mathbf{Z})$ always hold true?

This problem is easily reduced to the following conjecture from the representat theory: let $K/\mathbf{Q}$ be a finite Galois extension of the rationals and $G \subseteq GL_n(O_K)$ be a finite subgroup stable under the natural action of the Galois group $\Gamma := Gal(K/\mathbf{Q})$. Then there is the following

Conjecture 1. *If K is totally real, then $G \subseteq GL_n(\mathbf{Z})$.*

In [5] 2 steps of reduction for Conjecture 1 are proved:

— We can assume that G is an abelian group of exponent p for some integer p (see [7], Proposition 1, [5], Proposition 1 and also Proposition 6).

— We can assume that G is irreducible under conjugation in $GL_n(\mathbf{Q})$ (see [5] and Theorem B below for a more general approach).

In fact, the first step allows to consider G to be a finite commutative $(\mathbf{Z}/p\mathbf{Z})\Gamma$-module for some prime p. It is also possible to assume that $K/\mathbf{Q}$ is unramified outside p for this prime p (see [3]).

There are several reformulations and generalizations of Conjecture 1. Consider an arbitrary not necessarily totally real finite Galois extension K of the rationals $\mathbf{Q}$ and a free $\mathbf{Z}$-module M of rank n with basis $m_1, \ldots, m_n$. The group $GL_n(O_K)$ acts in a natural way on $O_K \otimes M \cong \bigoplus_{i=1}^n O_K m_i$.

The finite group $G \subseteq GL_n(O_K)$ is said to be of A-type, if there exists a decomposition $M = \bigoplus_{i=1}^k M_i$ such that for every $g \in G$ there exists a permutation $\Pi(g)$ of $\{1, 2, \ldots, k\}$ and roots of unity $\epsilon_i(g)$ such that $\epsilon_i(g)gM_i = M_{\Pi(g)i}$ for $1 \leq i \leq k$. The following conjecture generalizes (and would imply) Conjecture 1:

Conjecture 2. *Any finite subgroup of $GL_n(O_K)$ stable under the Galois group $\Gamma = Gal(K/\mathbf{Q})$ is of A-type.*

For totally real fields K Conjecture 2 reduces to Conjecture 1.

In fact, Conjecture 2 implies that any finite Γ-stable subgroup of $GL_n(O_K)$ is contained in $GL_n(O_E)$ for some cyclotomic extension $E/\mathbf{Q}$. In particular, Conjecture 2 implies, in the virtue of Kronecker–Weber theorem, that the commutator subgroup Γ' of Γ acts trivially on G.

Both conjectures are true in the case of Galois field extension $K/\mathbf{Q}$ with odd discriminant. Also some partial answers are given in the case of field extensions $K/\mathbf{Q}$ that are unramified outside 2.

Let $F(G)$ denote the field obtained via adjoining to F the matrix coefficients of all matrices $g \in G$. The following result was obtained in [1] (see also [5] for the case of totally real fields).

Theorem 1. *1) Let K be a finite Galois extension of $\mathbf{Q}$ with an odd discriminant, and G be a finite subgroup of $GL_n(O_K)$ that is stable under the natural action of the Galois group Γ of the field K. Then G is of A-type.*

2) Let $K = \mathbf{Q}(\mathbf{G})$ and G be a group satisfying the conditions of 1). Furthermore, let the discriminant of K be even but divisible by at least one odd prime///, and K contains no roots of 1 $\zeta \neq \pm 1////$. Then G is of A-type.

3) Let $G \subset GL_n(O_K)$ be a finite Γ-stable subgroup and $K = \mathbf{Q}(G)$. Let all primes $p \neq 2$ be unramified in K, $K \neq \mathbf{Q}$. In this case we can assume that G is an abelian subgroup of exponent 2. Let us suppose that one of the following conditions is fulfilled:

(1) for any central primitive idempotent $\varepsilon \in KG$ all coefficients of the matrix 2ε are contained in the valuation ring $O_\mathbf{p}$ of some prime divisor $\mathbf{p}$ of 2 in the ring O_K for at least one $\mathbf{p}$;

(2) j-th ramification group $\Gamma_j(\mathbf{p})$ of the ideal $\mathbf{p}$ is distinct from $\{1\}$ for the index j which is equal to the ramification index e of $\mathbf{p}$;

(3) there is an even integer j such that $\Gamma_j(\mathbf{p}) \neq \Gamma_{j+1}(\mathbf{p})$ (note that condition (2) is a particular case of (3));

(4) $e = 2q$, q is odd and Γ is distinct from its commutant and $\sqrt{-1}$ is not contained in K;

(5) $e = 2^t$.

Then G is of A-type.

Corollary 1. Conjecture 1 is true for totally real Galois extensions $K/\mathbf{Q}$ of degree $[K : \mathbf{Q}] \leq 480$. If the generalized Riemann hypothesis for the zeta function of the number field K is true, then Conjecture 1 holds also for totally real fields $K/\mathbf{Q}$ with $[K : \mathbf{Q}] \leq 960$.

Corollary 2. If all totally real fields $\mathbf{Q}(\zeta_{2^m} + \zeta_{2^m}^{-1})$ have class number 1, then Conjecture 1 holds for arbitrary solvable extensions $K/\mathbf{Q}$.

Corollary 3. Conjecture 2 is true if the degree of the Galois extension $K/\mathbf{Q}$ is less than 288.

Let us formulate a criterion for the existence of an integral realization of an abelian group G with properties introduced above.

Let E, L be finite extensions of a number field F. Let O'_E, O'_F, O'_L be semilocal rings that are obtained by intersection of valuation rings of all ramified prime ideals in the rings O_E, O_F, O_L. If $F = \mathbf{Q}$ we can define O_F to be the intersection of F and O_E. Let $w_1, w_2, \ldots, w_d$ be a basis of O'_E over O'_F, and let D be a square root of the discriminant of this basis. By the definition $D^2 = \det[Tr_{E/F}(w_i w_j)]_{ij}$. It is known that $D = det[w_m^{\sigma_k}]_{k,m}$. Let us suppose that some matrix $g \in GL_n(E)$ has order t $(g^t = I_n)$ and all Γ-conjugates g^γ, $\gamma \in \Gamma$ generate a finite subgroup $G \subset GL_n(E)$ of exponent t. Let $\sigma_1 = 1$, $\sigma_2, \ldots, \sigma_d$ denote all automorphisms of the Galois group Γ of E over F. Assume that $L = E(\zeta_{(1)}, \zeta_{(2)}, \ldots, \zeta_{(n)})$ where $\zeta_{(1)}, \zeta_{(2)}, \ldots, \zeta_{(n)}$ are the eigenvalues of the matrix g. We shall reserve the same notations for certain fixed extensions of σ_i to L. Automorphisms of L over F will be denoted $\sigma_1, \sigma_2, ..., \sigma_r$, $r > d$. Theorem 2 below implies the existence of the group G provided $n \geq \phi_E(t)[E : F]$. Let $E = F(G)$ be obtained by adjoining to F all coefficients of all $g \in G$. For an appropriate set of d eigenvalues $\zeta_{(1)}, \zeta_{(2)}, \ldots, \zeta_{(d)}$ which depends on the primitive idempotents of algebra LG the following Theorem is true (it is proved in [5], see also [1]):

Theorem A. Let $G \subset GL_n(E)$ be irreducible under $GL_n(F)$-conjugation. Then G is conjugate in $GL_n(F)$ to a subgroup of $GL_n(O'_E)$ if and only if all determinants

$$
D_k = \det \begin{vmatrix}
w_1 & \cdots & w_{k-1} & \zeta_{(1)} & w_{k+1} & \cdots & w_d \\
w_1^{\sigma_2} & \cdots & w_{k-1}^{\sigma_2} & \zeta_{(2)}^{\sigma_2} & w_{k+1}^{\sigma_2} & \cdots & w_d^{\sigma_2} \\
\vdots & & & & & & \\
w_1^{\sigma_d} & \cdots & w_{k-1}^{\sigma_d} & \zeta_{(d)}^{\sigma_d} & w_{k+1}^{\sigma_d} & \cdots & w_d^{\sigma_d}
\end{vmatrix}
$$

are divisible by D in the ring O'_L.

In this theorem G is Γ-stable and generated by g and all $g^\gamma, \gamma \in \Gamma$ but this condition is not very restrictive for 2 reasons. Firstly, any Γ-stable subgroup $H \in GL_n(E)$ contains subgroups like G. And by Theorem 3 below, if H is a minimal subgroup of exponent t with the property $E = F(H)$, then H is just of the form given in Theorem A.

The proof of Theorem A is constructive. It is based on the commutativity of the L-algebra LG, the L-span of G, and uses a system of linear equations that arises from simultaneous diagonalization of commuting matrices

$$g = \sum_{i=1}^d w_i B_i, g^\sigma = \sum_{i=1}^d w_i^\sigma B_i, \ \sigma \in \Gamma,$$

whose solutions are the eigenvalues of commuting matrices B_i, $i = 1, 2, \ldots, d$.

In fact, it is proved that the eigenvalues of B_j, $j = 1, \ldots, d$ are just the elements of the set $\{(D_j D^{-1})^\gamma, \ \gamma$ are varying in the Galois group of $L/F\}$.

We also use the fact that each semisimple matrix $B \in GL_n(F)$ is conjugate in $GL_n(F)$ to a matrix from $GL_n(O'_F)$ if and only if all its eigenvalues are contained in O'_L (see [5]):

Lemma 1. *1) Let all eigenvalues λ_i, $i = 1, 2, \ldots, n$ of a semisimple matrix $B \subset GL_n(F)$ be contained in the ring O'_L for some field $L \supset F$. Then B is conjugate in $GL_n(F)$ to a matrix that is contained in $GL_n(O'_F)$.*

2) Conversely, if a matrix B is contained in $GL_n(O'_F)$, then its eigenvalues are contained in O'_L.

We note that the reduction to the case of an irreducible group G is motivated by the following easy lemma [5]:

Lemma 2. *If $G \subset GL_n(E_1)$ is a finite Γ-stable subgroup which has $GL_n(F_1)$-irreducible components $G_1, G_2, \ldots, G_r$, and E_1, F_1 are rings having quotient fields E and F respectively, then $F(G)$ is the composite of fields $F(G_1)$, $F(G_2)$, $\ldots$, $F(G_r)$.*

Theorem A can be used to prove the existence of Γ-stable subgroups $G \subset GL_m(O'_E)$ with the property $F(G) \neq F$ for some integer m. The following Corollary of Theorem A reduces the problem of existence for Γ-stable groups G to the case of $GL_n(F)$-irreducible G.

Theorem B. *If there is an abelian Γ-stable subgroup $G \subset GL_m(O'_E)$ generated by g^γ, $\gamma \in \Gamma$ such that $E = F(G) \neq F$ as above, then $GL_m(F)$-irreducible components $G_i \subset GL_{m_i}(E)$, $i = 1, ..., k$ of G are conjugate in $GL_{m_i}(F)$ to subgroups $G'_i \subset GL_{m_i}(O'_E)$ such that $E = F(G_1)F(G_2)...F(G_k)$. In particular, $F(G_i) \neq F$ for some indices i.*

Proof of Theorem B.

If $G \subset GL_m(O'_E)$ is a group of exponent t and $g = B_1 w_1 + B_2 w_2 + \ldots + B_d w_d$ for a basis $w_1, \ldots, w_d$ of O'_E over O'_F, then $B_i \in M_m(O'_F)$, and it follows from Lemma 1 that the eigenvalues of B_j are contained in O'_L. But eigenvalues are preserved under conjugation, so the latter claim is also true for all components G_i. We can apply Theorem A to $G_i, i = 1, \ldots, k$. It follows that G_i are conjugate to subgroups $G'_i \subset GL_{m_i}(O'_E)$. Now, Lemma 2 implies $E = F(G_1)F(G_2)\ldots F(G_k)$. This completes the proof of Theorem B.

Theorem C. *Let E/F be a normal extension of number fields with Galois group Γ. Let $G \subset GL_n(E)$ be an abelian Γ-stable subgroup of exponent t generated by $g = B_1 w_1 + B_2 w_2 + \ldots + B_d w_d$ and all matrices $g^\gamma, \gamma \in \Gamma$, and let $E = F(G)$. Then G is conjugate in $GL_n(F)$ to $G \subset GL_n(O'_F)$ if and only if all eigenvalues of matrices $B_i, i = 1, \ldots, d$ are contained in O'_L, where $L = E(\zeta_t)$.*

Proof of Theorem C.

Let

$$C^{-1}GC = \begin{vmatrix} G_1 & & * \\ & \ddots & \\ 0 & & G_k \end{vmatrix}$$

for $C \in GL_n(F)$ and irreducible components $G_i \subset GL_{n_i}(E), i = 1, \ldots, k$. Then

$$C^{-1}gC = \begin{vmatrix} g_1 & & * \\ & \ddots & \\ 0 & & g_k \end{vmatrix} = B'_1 w_1 + B'_2 w_2 + \ldots + B'_d w_d$$

for $B'_i = C^{-1} B_i C$. Let us consider F-algebra A generated by all $B'_i, i = 1, \ldots, d$, over F. Since A is semisimple, it is completely reducible. It follows that matrices B'_i are simultaneously conjugate in $GL_n(F)$ to the block-diagonal form. Therefore, G is conjugate in $GL_n(F)$ to a direct sum of its irreducible components G_i. We can apply Theorem A to each of them. Theorem B implies that each G_i is conjugate in $GL_{n_i}(F)$ to $G'_i \subset GL_{n_i}(O'_F)$ if and only if all eigenvalues of matrices $B'_i, i = 1, \ldots, d$ are contained in O'_{L_i}, where $L_i = F(G_i)(\zeta_t)$. But $F(G) = F(G_1)F(G_2)\ldots F(G_k)$ by Lemma 2, and so $L = L_1 L_2 \ldots L_k$. This completes the proof of Theorem C.

Remark. *Theorems A,B,C remain true for some other Dedekind subrings $R \subset L$. They can also be modified for the rings of integers O_E, O_F and O_L provided O_E and O_L have O_F-basises (the latter is always true for $F = \mathbf{Q}$).*

The approach to describe all Γ-stable matrix groups up to $GL_n(R)$-conjugatic for certain Dedekind rings $R \subset E$ can be based on either of Theorems A,B,C

Then $\alpha_{i1}^{\sigma_j} = \alpha_{ij}$ and $\lambda_i = (\zeta_t - 1)\alpha_{i1}$ for $i \neq 1$, and $\lambda_1 = 1 + (\zeta_t - 1)\alpha_{11}$.
So $\lambda_i^{\sigma_j} = (\zeta_t - 1)\alpha_{i1}^{\sigma_j} = (\zeta_t - 1)\alpha_{ij}$ for $i \neq 1$, and $\lambda_1^{\sigma_j} = (\zeta_t - 1)\alpha_{11}^{\sigma_j} + 1 = (\zeta_t - 1)\alpha_{1j} + 1$. Since any linear relation

$$k_1(\lambda_1 - 1) + \sum_{i=2}^{d} k_i\lambda_i = 0, k_i \in F(\zeta_t), i = 1, 2, ..., d$$

implies the linear relation

$$k_1(\lambda_1^{\sigma_j} - 1) + \sum_{i=2}^{d} k_i\lambda_i^{\sigma_j} = 0, k_i \in F(\zeta_t), i = 1, 2, ..., d$$

for all $\sigma_j \in \Gamma$, this would also imply $det W^{-1} = 0$, which is impossible. Therefore, $\lambda_1 - 1, \lambda_2, ..., \lambda_d$ generate the field $E(\zeta_t)$ over $F(\zeta_t)$, and so $B_i - I_d, B_2, ..., B_d$ generate $F(\zeta_t)$-span $F(\zeta_t)[B_1, ..., B_d]$ over $F(\zeta_t)$. Note that B_i can be expressed as a linear combination of $g^{\sigma_i}, i = 1, 2, ..., d$ with coefficients in E: $B_i = \sum_{j=1}^{d} \alpha_{ij}g^{\sigma_j}$. This can be obtained from the system of matrix equations

$$g^{\sigma_j} = \sum_{i=1}^{d} w_i^{\sigma_j} B_i, j = 1, 2, ..., d$$

if we consider B_i as indeterminates. Since G has exponent t, $F(\zeta_t)$ is a splitting field for G, the group generated by all $g^{\sigma}, \sigma \in \Gamma$. Therefore, the dimension of $E(\zeta_t)$-span $E(\zeta_t)G = E(\zeta_t) \otimes_{F(\zeta_t)} F(\zeta_t)G$ over $E(\zeta_t)$ is d, and so $F(\zeta_t)$-dimension of $F(\zeta_t)$-span $F(\zeta_t)G$ is also d.

Let us denote by E' the image of $E(\zeta_t)$ under the regular representation of $E(\zeta_t)/F(\zeta_t)$ over $F(\zeta_t)$. Then $A = E(\zeta_t)G = E(\zeta_t) \otimes_{F(\zeta_t)} F(\zeta_t)G$, the $E(\zeta_t)$-span of G, is the Galois E'-algebra in the sense of [4], that is, it is an associative and commutative separable E'-algebra having a normal basis. We can choose idempotents

$$\varepsilon_i = \frac{1}{\zeta_t - 1}(g^{\sigma_j} - I_d), j = 1, 2, ..., d$$

as a normal basis of A over E' so that $\varepsilon_j = \varepsilon_1^{\sigma_j}$.

We have $F(\zeta_t)G = F(\zeta_t)[< g^{\sigma_1}, ..., g^{\sigma_d} >] = F(\zeta_t)[(g-I_d)^{\sigma_1}, ..., (g-I_d)^{\sigma_d}]$, and $dim_{F(\zeta_t)}F(\zeta_t)G = d$. As the length of the orbit of $M = [m_{ij}] = (g - I_d)$ under Γ-action is d, we can use the coefficients of matrices $M^{\sigma_i}, i = 1, 2, ..., d$ to construct an element $\theta = \sum_{i,j} k_{ij}m_{ij}$, $k_{ij} \in F(\zeta_t)$, which generates a normal basis of $E(\zeta_t)/F(\zeta_t)$. Therefore, for any given $\alpha \in E(\zeta_t)$ we have $\alpha = \sum_i k_i\theta^{\sigma_i}$ for some $k_i \in F(\zeta_t)$.

Therefore, our choice of eigenvalues implies that $F(\zeta_t)(G) = E(\zeta_t)$.

Now, we can apply the regular representation R_F of $F(\zeta_t)$ over F to matrices $M = [m_{ij}]_{i,j}, m_{i,j} \in F(\zeta_t)$ in the following way: $R_F(M) = [R_F(m_{ij})]_{i,j}$. So, using R_F for all components of matrices $B_i \in M_n(F(\zeta_t))$ we can obtain an abelian subgroup $G \subset GL_{n_1}(E), n_1 = [F(\zeta_t) : F]d$ of exponent t which is Γ-stable if we identify the isomorphic Galois groups of the extensions E/F and $E(\zeta_t)/F(\zeta_t)$. We have again $dim_F FG = dim_E EG$, E is again the Galois algebra, and $F(G) = E$. Now, using the natural embedding of G to $GL_n(E), n \geq n_1$, we complete the proof of Theorem 2 in the case 1).

2) In virtue of 1) we can consider the case when the intersection $F_0 = E \cap F(\zeta_t) \neq F$. We can use the regular representation R of E over F. Let $\Gamma_0 = \{\sigma_1', \sigma_2', ..., \sigma_d'\}$ be the set of some extensions of elements $\Gamma = \{\sigma_1, \sigma_2, ..., \sigma_d\}$ to $E(\zeta_t)/F$, and let $w_1 = 1, w_2, ...w_d$ be a basis of E over F. So we can use our previous notation and go through a similar argument as in the part 1) of the proof for construction of $g = \sum_{i=1}^d B_i w_i$ and matrices B_i as the regular representations R_0 of eigenvalues

$$\lambda_i = \frac{det W_i}{det W} = \sum_{j=1}^{\phi_E(t)} \lambda_{ij} \zeta^j, i = 1, 2, ..., d,$$

in the following way: we consider

$$B_i = R_0(\lambda_i) = \sum_{j=1}^{\phi_E(t)} R(\lambda_{ij}) \zeta^j,$$

where R is the regular representation of E over F. We also have $\lambda_1^{\sigma_j'} = \alpha_{1j} + 1, \lambda_i^{\sigma_j'} = \alpha_{ij}$ for $j = 2, ..., d$. Now, if we have any linear relation between the rows of the matrix $[\alpha_{ij}(\zeta_t^{\sigma_j'} - 1)]_{i,j}$, this would imply a linear relation between its columns, and so the columns of $W^{-1} = [\alpha_{ij}]$ are linearly dependent, and $det W^{-1} = 0$ which is a contradiction. So, again we obtain that $\lambda_1 - 1, \lambda_2, ..., \lambda_d$ are linearly independent over F, so $dim_F FG'$ $= dim_F F[B_1 - I_d, B_2, ..., B_d] = dim_E EG' = d$ for G' generated by $g^{\sigma_i}, i = 1, 2, ..., d$. As earlier we can consider the elementwise regular representation $R_E(B_i)$ of matrices B_i in the field extension $E(\zeta_t)/E$. So we obtain $g_0 = \sum_{i=1}^d R_E(B_i) w_i$, and we can take the group G generated by all $g_0^{\sigma_i}, i = 1, 2, ..., d$. Since $[E(\zeta_t) : F] = [E(\zeta_t) : E][E : F] = \phi_E(t)d$, the integer $n = \phi_E(t)d$ coincides with the one required in the formulation of Theorem 2. In this way we can construct a Γ-stable group G that satisfies the conditions of Theorem 2.

This completes the proof of Theorem 2.

Proof of Theorem 3.

We can use the proof of Theorem 2.

Let $G \subset GL_n(E)$ be a group given in the formulation of Theorem 3, and let n be minimal possible. Then we have the following decomposition of E-span $A = EG$:

$$A = \varepsilon_1 A + \varepsilon_2 A + ... + \varepsilon_k A$$

for some primitive idempotents $\varepsilon_1, ..., \varepsilon_k$ of A. ε_i are conjugate under the action of the Galois group $\Gamma = \{\sigma_1, ..., \sigma_d\}$. For if the sum of $\varepsilon_i^{\sigma_j}, j = 1, 2, ..., d$ is not I_n then $I_n = e_1 + e_2$ for $e_1 = \varepsilon_1^{\sigma_1} + ... + \varepsilon_1^{\sigma_d}$ and $e_2 = I_n - e_1$, so e_1, e_2 are fixed by Γ; this implies that e_1, e_2 are conjugate in $GL_n(F)$ to a diagonal form. Since either of 2 components e_iG has rank smaller than n, there is a group satisfying the conditions of Theorem 3 of smaller than n degree.

Therefore, $\varepsilon_i = \varepsilon_1^{\sigma_i}$, $k = d$ and the idempotents $\varepsilon_1, ..., \varepsilon_d$ form a normal basis of A. But the rank of a matrix ε_i is not smaller than $\phi_E(t)$. Indeed, $\varepsilon_i G$ contains an element $\varepsilon_i g$, for some $g \in G$ of order t such that $(\varepsilon_i g)^t = \varepsilon_i$, but $(\varepsilon_i g)^k \neq \varepsilon_i$ for $k < t$. We can find $g \in G$ in the following way. Since $I_n = \varepsilon_1 + ... + \varepsilon_k$ for any $h \in G$ of order t there is ε_j such that $(\varepsilon_j h)^t = \varepsilon_j$, but $(\varepsilon_j h)^k \neq \varepsilon_j$ for $k < t$, and the same property holds true for $\varepsilon_j h$ with any $\sigma \in \Gamma$. Then using the property of normal basis $\varepsilon_k = \varepsilon_1^{\sigma_k}$ we can take $g = h^{\sigma_j^{-1}\sigma_i}$.

So, the irreducible component $\varepsilon_i G$ determines a faithful irreducible representation of a cyclic group generated by g. But if $T : C \to GL_r(E)$ is a faithful irreducible representation of a cyclic group C generated by an element g of order t, its degree r is equal to $\phi_E(t)$. It follows that the rank of matrices ε_i is $\phi_E(t)$. So the dimension of A over E is $\phi_E(t)d$.

If G is generated by g^γ, $\gamma \in \Gamma$ and its order is minimal, Γ-stability implies that g has d conjugates under Γ-action, and so G an abelian group of type $(t, ..., t)$ and order t^m for some positive integer $m \leq d$. This completes the proof of Theorem 3.

References

[1] H.-J. Bartels, D. A. Malinin. *Finite Galois stable subgroups of GL_n*. Manuscripte der Forschergruppe Arithmetik, Mannheim-Heidelberg, 2000, Nr.3, 21 S.

[2] H.-J. Bartels. *Zur Galoiskohomologie definiter arithmetischer Gruppen*. J. reine angew. Math. (1978) v. 298, S. 89–97.

[3] H.-J. Bartels, Y. Kitaoka. *Endliche arithmetischer Untergruppen der GL_n*. J. reine angew. Math. (1980) v. 313, S. 151–156.

[4] V.V. Ishkhanov, B.B. Lur'e, D.K. Faddeev. *The embedding problem in Galois theory.* "Nauka", Moscow (1988), 272 p.

[5] D.A. Malinin. *Galois stability for integral representations of finite groups.* Algebra i analiz (2000), v. 12, p. 106–145 (english translation in St. Petersburg Math. J. v. 12, N 3).

[6] D.A. Malinin. *Galois stability for integral representations of finite abelian groups.* Max-Planck-Institut Preprint 99-128 (1999), 20 S.

[7] D.A. Malinin. *On integral representations of finite groups with Galois action.* Dokl. Russian Acad. Sci. (1996) v.349, N 3, p.303- 305.

[8] J.Rohlfs. *Arithmetische definierte Gruppen mit Galois-operation.* Invent. Math. (1978) v. 48, p. 185–205.

[9] J.Ritter and A.Weiss. *Galois action on integral representations.* J. London Math. Soc. (2) (1992) v. 46, p. 411–431.

[10] J.Ritter and A.Weiss. *Regulators and Galois stability.* Math. Nachr. (1992) v. 158, p. 27–41.

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ON RINGS FOR WHICH EVERY MODULE WITH (S*) IS EXTENDING

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Abstract

A module M is said to satisfy (S*) if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and $Z^*(N/K) = N/K$ where $Z^*(M) = \{m \in M : mR$ is a small module$\}$. We prove that if M is an extending module satisfying (S*) and $Z^*(M)$ is semisimple projective then every submodule of M is extending and M is a locally Artinian serial SI-module. We also characterize H-rings and generalized uniserial rings with $J(R)^2 = 0$ by using the conditions "every extending module satisfies (S*)" and "every module satisfying (S*) is extending".

1   Preliminaries

Throughout this paper, R will be a ring with identity and all modules will be unitary right R-modules.

Let M be a module. The injective hull of M is denoted by $E(M)$, the socle of M by $\mathrm{Soc}(M)$, the singular submodule of M by $Z(M)$ and the radical of M by $\mathrm{Rad}M$. $J(R)$ is the Jacobson radical of R. Let N be a submodule of M ($N \leq M$). If N is essential in M, we write $N \leq_e M$. N is *closed* in M provided N has no proper essential extensions in M. If N is a direct summand in M, we write $N \leq_d M$.

A module M is called an *extending module*, or a *CS-module*, if every submodule of M is essential in a direct summand, or equivalently, if every closed submodule of M is a direct summand. A module M is called a *continuous module* if it is extending and if every submodule isomorphic to a summand of M is a summand. A module M is called a *quasi-continuous module* if it is

extending and if M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a summand of M. (see [8] or [4])

Let M_1 and M_2 be modules. The module M_2 is *essentially M_1-injective* if every homomorphism $\alpha : A \longrightarrow M_2$, where A is a submodule of M_1 and $\operatorname{Ker}\alpha \leq_e A$, can be extended to a homomorphism $\beta : M_1 \longrightarrow M_2$ [4]. The modules M_1 and M_2 are *relatively (essentially) injective* if M_i is (essentially) M_j-injective, for every $i, j \in \{1, 2\}$, $i \neq j$.

Following [21], $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are submodules of M-generated modules. Let M and N be modules. N is called *M-singular* if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$ (see [21] or [4]). Largest M-singular submodule is denoted by $Z_M(N)$, and $Z_M(N) \leq Z(N)$. A module M is called an *SI-module* if every M-singular module is M-injective [4]. These 'SI-modules' need not be 'SI-modules' in the sense of Yousif [18]. A ring R is a *right SI-ring* if every singular (right) R-module is injective. A ring R is called a *right GV-ring* if every simple singular (right) R-module is injective. Clearly SI-rings are GV-ring.

A module M is called *locally Artinian* if every finitely generated submodule of M is Artinian. M is said to be *uniserial* if its submodules are linearly ordered by inclusion. M is said to be a *serial module* if it is expressed as a direct sum of uniserial modules. A ring R is called right (left) serial if R_R (resp. $_RR$) is a serial module. When a ring R is both right and left serial, R is said to be a *serial ring*. [21]

Let N be a submodule of a module M. N is called *small submodule* if whenever $N + L = M$ for some submodule L of M we have $L = M$, and it is denoted by $N \ll M$. The module M is called a *small module* if it is a small submodule of some R-module. M is small if and only if $M \ll \mathrm{E}(M)$ [7]. We put

$$Z^*(M) = \{m \in M : mR \ll \mathrm{E}(mR)\} \ [6]$$

Since $\operatorname{Rad}M$ is the sum of all small submodules in M, $\operatorname{Rad}M \leq Z^*(M)$. $Z^*(M) = M \cap \operatorname{Rad}\mathrm{E}(M)$, $Z^*(N) = N \cap Z^*(M)$ for a submodule N of M and $Z^*(\oplus_I M_i) = \oplus_I Z^*(M_i)$ for any modules M_i and any index set I. If $\varphi : M \longrightarrow K$ is a homomorphism of modules M and K, then $\varphi(Z^*(M)) \leq Z^*(K)$ (see [12]). We call a module M is *cosingular* if $Z^*(M) = M$. A ring R is called *(right) cosingular* if the (right) R-module R is cosingular.

Clearly small modules are cosingular. If R is a right cosingular ring then every R-module is cosingular [11]. For example, since $\mathbb{Z} \ll \mathrm{E}(\mathbb{Z}) = \mathbb{Q}$, every $\mathbb{Z}$-module is cosingular.

A module M is called *lifting* (or a *(D1)-module*) if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M$ [8].

In [13] we define the following property for a module M.

(S*) If for every submodule N of M there exists a direct summand K of M such that $K \le N$ and N/K is cosingular.

A ring R satisfies (S*) if the (right) R-module R satisfies (S*).

In Section 2 we give some results about extending module M which satisfies (S*). We prove that if M is an extending module satisfying (S*) and if $Z^*(M)$ is semisimple projective then every submodule of M is extending and M is a locally Artinian serial SI-module (Theorem 2.10). In Sections 3 and 4 we deal with properties of a ring R when every extending module satisfies (S*), and, every module with (S*) is extending. It is proved that every injective R-module is lifting if and only if R is right perfect and every extending R-module satisfies (S*) (Theorem 3.2). It is also proved that R is a generalized uniserial ring with $J(R)^2 = 0$ if and only if R is right perfect and every R-module with (S*) is extending (Theorem 4.5).

2 $Z^*(M)$ is Semisimple Projective

Lifting modules satisfy (S*). But the converse is not true, for example, let R denote the ring of integers $\mathbb{Z}$. Since $Z^*(R) = R$, R is cosingular and hence satisfies (S*). But R is not lifting [8, p.56]. Clearly, a module M satisfying (S*) is lifting if $Z^*(M) \ll M$.

A ring R is semiperfect if and only if the right (left) R-module R is lifting [8]. Hence semiperfect rings satisfy (S*). If R is right self-injective, then $Z^*(R_R) = J(R) \ll R$. It follows that right self-injective rings with (S*) are semiperfect (see also [13, Corollary 4.12]).

In [13] it was proved that a module M satisfies (S*) if and only if for every submodule N of M, M has a decomposition $M = M_1 \oplus M_2$ such that $M_1 \le N$ and $N \cap M_2$ is cosingular if and only if for every submodule N of M, N has a decomposition $N = N_1 \oplus N_2$ such that $N_1 \le_d M$ and N_2 is cosingular [13, Lemma 3.1]. Any submodule of a module M with (S*) satisfies (S*) [13, Lemma 3.2]. And also it can be shown that if M satisfies (S*) then $M/Z^*(M)$ is semisimple (see [13, Proposition 3.10]).

Lemma 2.1. [13, Corollary 3.6] *Let M be a module satisfying (S*). Then there is a decomposition $M = M_1 \oplus M_2$ such that M_1 is semisimple with $Z^*(M_1) = 0$ and $Z^*(M_2) \le_e M_2$.*

Extending modules satisfying (S*) have the following decomposition.

Proposition 2.2. *Let M be an extending module. M satisfies (S^*) if and only if every submodule of M is a direct sum of an extending module and a cosingular module.*

Proof. Suppose that M satisfies (S^*). Let $N \leq M$. Then $N = N_1 \oplus N_2$ where $N_1 \leq_d M$ and N_2 is cosingular. It follows that N_1 is extending. Conversely, suppose that every submodule of M is a direct sum of an extending module and a cosingular module. Let L be any submodule of M. Then $L = L_1 \oplus L_2$ for some extending module L_1 and cosingular module L_2. Since L_1 is extending, there exists a direct summand K of M such that $L_1 \leq_e K$. Then $K \cap L_2 = 0$ and $L = K \oplus L_2$. Hence M satisfies (S^*). $\qquad\square$

From now on we investigate some injectivity properties of modules M such that $Z^*(M)$ is semisimple projective which we need for the proof of Theorem 2.10. They will also help us to determine whether the modules which is a direct sum of a semisimple and an extending module are extending.

Proposition 2.3. *Let M be a module with $M/Z^*(M)$ semisimple. Then every module K with $Z^*(K) = 0$ is M-injective.*

Proof. Let K be a module with $Z^*(K) = 0$. Let $Z^*(M) := A$. Then K is M/A-injective. Let $N \leq M$ and $\varphi : N \longrightarrow K$ be a homomorphism. Let $X = \mathrm{Ker}\varphi$. Note that

$$(A \cap N)/(A \cap X) \cong (A \cap N) + X/X \leq N/X \cong \mathrm{Im}\varphi \leq K$$

Then $Z^*((A \cap N)/(A \cap X)) = 0$. On the other hand, since A is cosingular and the class of cosingular modules is closed under submodules and homomorphic images, $(A \cap N)/(A \cap X)$ is cosingular. This implies that $A \cap N = A \cap X$. Define $\theta : (N + A)/A \longrightarrow K$, $\theta(n + A) = \varphi(n)$, $(n \in N)$. Then θ is well defined and a homomorphism. By hypothesis, θ can be extended to a homomorphism $\alpha : M/A \longrightarrow K$. Now define $\beta : M \longrightarrow K$ by $\beta = \alpha\pi$ where $\pi : M \longrightarrow M/A$ is the canonical projection. It is easy to check that β extends φ. Thus K is M-injective. $\qquad\square$

Corollary 2.4. *Let M be a module satisfying (S^*). Then M is a direct sum of two relatively injective modules.*

Proof. By Lemma 2.1, $M = M_1 \oplus M_2$ such that M_1 is semisimple with $Z^*(M_1) = 0$ and $Z^*(M_2) \leq_e M_2$. Then M_2 is M_1-injective because M_1 is semisimple. Since M_2 satisfies (S^*), $M_2/Z^*(M_2)$ is semisimple. Hence by Proposition 2.3, M_1 is M_2-injective. $\qquad\square$

Proposition 2.5. *Let K and M be modules such that $Z^*(K)$ is semisimple projective, M satisfies (S^*). Then K is essentially M-injective.*

Proof. Let $N \leq M$ and a homomorphism $f : N \longrightarrow K$ with $\operatorname{Ker} f \leq_e N$. M has a decomposition $M = M_1 \oplus M_2$ where $M_1 \leq N$ and $N \cap M_2$ is cosingular. Then $N = M_1 \oplus (N \cap M_2)$. Let $m \in M, m = a + b$ where $a \in M_1$ and $b \in M_2$. We define $g : M \longrightarrow K$, $m \mapsto f(a)$. Clearly g is well-defined. Let $n \in N$ and $n = a + b$ where $a \in M_1$ and $b \in M_2$. Then $g(n) = f(a)$. Since $f(N \cap M_2) \leq Z^*(K)$ and $Z^*(K)$ is semisimple projective,

$$f(N \cap M_2) \cong (N \cap M_2)/(N \cap M_2 \cap \operatorname{Ker} f) \cong ((N \cap M_2) + \operatorname{Ker} f)/\operatorname{Ker} f$$

is projective. Hence $\operatorname{Ker} f \leq_d (N \cap M_2) + \operatorname{Ker} f$. In addition $(N \cap M_2) + \operatorname{Ker} f = N \cap (M_2 + \operatorname{Ker} f)$, $\operatorname{Ker} f \leq_e N$ and $\operatorname{Ker} f \leq N \cap (M_2 + \operatorname{Ker} f) \leq N$. Hence $\operatorname{Ker} f \leq_e (N \cap M_2) + \operatorname{Ker} f$. This implies that $N \cap M_2 \leq \operatorname{Ker} f$. Since $b \in N \cap M_2$, $f(b) = 0$. Thus $f(n) = g(n)$ for all $n \in N$. f extends to g. $\qquad\square$

Note that if R is a right GV-ring then $Z^*(M)$ is semisimple projective for every (right) R-module M [11]. Hence we have the following corollary.

Corollary 2.6. *Let R be a right GV-ring and M an R-module which satisfies (S^*). Then every R-module is essentially M-injective.*

Proposition 2.7. *Let $M = M_1 \oplus M_2$ where $Z^*(M_1)$ is semisimple projective and M_2 is cosingular. Then M_2 is extending if and only if $Z^*(M)$ is extending.*

Proof. Let $M = M_1 \oplus M_2$ where $Z^*(M_1)$ is semisimple projective and M_2 is cosingular. Then $Z^*(M) = Z^*(M_1) \oplus M_2$. Since M_2 satisfies (S*) and $Z^*(Z^*(M_1)) = Z^*(M_1)$ is semisimple projective; by Proposition 2.5 $Z^*(M_1)$ is essentially M_2-injective. Also M_2 is $Z^*(M_1)$-injective and $Z^*(M_1)$ is extending. Hence if M_2 is extending then $Z^*(M)$ is extending by [16, Theorem 8]. Conversely if $Z^*(M)$ is extending, M_2 is extending, because any direct summand of extending modules is extending [8]. $\qquad\square$

Santa-Clara and Smith [15] proved that if R is a right SI-ring then $M_1 \oplus M_2$ is extending, for every semisimple R-module M_1 and every extending R-module M_2, and they also proved if every singular semisimple R-module is injective then $M_1 \oplus M_2$ is extending for every semisimple R-module M_1 and every quasi-continuous R-module M_2. Now we consider when $M_1 \oplus M_2$ is extending for every semisimple R-module M_1 and every extending R-module M_2 over a right GV-ring.

Proposition 2.8. *Let $M = M_1 \oplus M_2$ where M_1 is semisimple, M_2 is extending. If $Z^*(M_1)$ is projective and M_2 satisfies (S^*), then M is extending.*

Proof. Since $Z^*(M_1)$ is semisimple projective and M_2 satisfies (S*), M_1 is essentially M_2-injective. Hence by [16, Theorem 8], M is extending. $\qquad\square$

Corollary 2.9. *If R is a right GV-ring, then $M = M_1 \oplus M_2$ is extending for every semisimple R-module M_1 and every extending R-module M_2 with (S^*).*

Now we prove our theorem.

Theorem 2.10. *Let M be an extending module which satisfies (S^*). If $Z^*(M)$ is semisimple projective then every submodule of M is extending and M is a locally Artinian serial SI-module.*

Proof. Since $M/Z^*(M)$ is semisimple, M/K is semisimple for every essential submodule K of M. On the other hand $Z_M(M) \cap \operatorname{Rad}(M) = 0$ since $\operatorname{Rad}(M)$ is semisimple projective. Hence by [4, 17.2], M is an SI-module. Let $N \leq M$. Then $N = K \oplus L$ where K is extending and L is cosingular by Proposition 2.2. Since $Z^*(M)$ is semisimple projective, L is semisimple projective. Hence by Proposition 2.8, N is extending. Thus every submodule of M is extending.

    To show that M is locally Artinian, let F be a finitely generated submodule of M. Since F satisfies (S^*), $F/Z^*(F)$ is semisimple. Then $F/\operatorname{Soc}(F)$ is semisimple. By [4, 5.15], F has descending chain condition (dcc) on essential submodules. Since F is finitely generated extending and dcc on essential submodules F is Artinian [4, 18.7]. Hence M is locally Artinian.

Since M is a locally Noetherian extending module, M is a direct sum of uniform submodules U by [4, 8.3]. We claim that all U are uniserial. Let K be any nonzero finitely generated submodule of U. If we prove that $K/\operatorname{Rad}K$ is simple then by [21, 55.1], U is uniserial. Let L be a submodule of K such that $\operatorname{Rad}K$ is a proper submodule of L. Since K satisfies (S^*), there exists a direct summand K' of K such that $K' \leq L$ and L/K' is cosingular.

If $K' = 0$ then L is cosingular. By hypothesis, L is semisimple projective. If $\operatorname{Rad}K \neq 0$, since U is uniform we have $\operatorname{Rad}K = L$, a contradiction. If $\operatorname{Rad}K = 0$, K is semisimple by [21, 31.2]. This implies that $L = K$.

If $K' \neq 0$, $K' = K = L$ since K is uniform. Thus $\operatorname{Rad}K$ is a maximal submodule of K. Hence M is a direct sum of uniserial submodules, i.e. M is a (right) serial module. $\square$

A module P is called *M-projective* if for any module N with an epimorphism $\pi : M \to N$ and homomorphism $\theta : P \to N$, there exists a homomorphism $\theta' : P \to M$ such that $\pi\theta' = \theta$ [8]. A module $P \in \sigma[M]$ is called *projective in $\sigma[M]$* if it is K-projective for every $K \in \sigma[M]$. The module M is called *hereditary in $\sigma[M]$* (or *self-hereditary*) if every submodule of M is projective in $\sigma[M]$. [4]

Corollary 2.11. *Let M be an extending module which satisfies (S^*). Assume that $Z^*(M)$ is semisimple projective.*
(i) If M is quasi-projective then $Z_M(M) = 0$,
(ii) If M is projective in $\sigma[M]$ then M is hereditary in $\sigma[M]$.

Proof. By Theorem 2.10 and [4, 17.3].   □

Corollary 2.12. *Let R be a right GV-ring. If M is an extending R-module which satisfies (S^*) then every submodule of M is extending and M is a locally Artinian serial SI-module.*

If we apply Theorem 2.10 to a ring R, we have immediately

Corollary 2.13. *Let R be a right extending ring which satisfies (S^*). If $Z^*(R_R)$ is semisimple projective then every right ideal of R is extending and R is a right Artinian right serial right SI-ring with $J(R)^2 = 0$.*

Proof. Since $J(R) \leq Z^*(R_R)$, $J(R)^2 = J(R)J(R) \leq J(J(R)) \leq J(Z^*(R_R)) = 0$.   □

If R is a ring satisfying the conditions of Corollary 2.13, R need not be left serial:

Example 2.14. Let $R = \begin{bmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$ where $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and the fields of complex numbers, respectively. Then R is a (right and left) Artinian SI-ring, $J(R)^2 = 0$ and R is right extending, right serial but not left extending or left serial [3, Example 3.1], [2, Examples (b)]. So R satisfies the conditions of Corollary 2.13 but R is not left serial.

On the other hand if M is an extending module which satisfies (S*), every submodule of M need not be extending (compare Proposition 2.2):

Example 2.15. Let $K = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ for any prime p. Let $M = E(K)$. Then M is extending by [8, Proposition 2.1]. Since M is a $\mathbb{Z}$-module, $M = Z^*(M)(= \text{Rad}M)$. Then M satisfies (S*) but $Z^*(M)$ is neither semisimple nor projective, since projective modules have maximal submodules. Also by [4, p.58], K is not extending.

If M is an extending module with $Z^*(M)$ semisimple projective, M does not satisfy (S*) in general:

Example 2.16. Let V be an infinite dimensional (right) vector spaces over a field F. Let $R = \text{End}(V_F)$, the endomorphism ring of V_F. Then R is regular right self-injective (see [5, Proposition 2.23]). This implies that R is (right) extending and $Z^*(R_R) = J(R) = 0$ but it does not satisfy (S*).

3   Every extending module satisfies (S*)

The condition that every extending module is lifting characterized by Oshiro (see Theorem 4.2). In the light of this condition, in this section, we will investigate rings which have the condition that every extending module satisfies (S^*). Note that there are extending modules which are not satisfy (S^*) (see Example 2.16).

Lemma 3.1. *The following are equivalent for a ring R.*
(i) Every R-module satisfies (S),*
(ii) Every extending R-module satisfies (S),*
(iii) Every quasi-injective R-module satisfies (S),*
(iv) Every injective R-module satisfies (S),*
(v) Every R-module is a direct sum of an extending module and a cosingular ·
module,
(vi) Every R-module is a direct sum of an injective module and a cosingular
module.

Proof. (i)⇔ (iv) ⇔ (vi) were proved in [13, Theorem 4.2]. (i)⇒ (ii) ⇒ (iii)⇒ (iv) Clear. (ii)⇔ (v) By Proposition 2.2.   □

Artinian serial rings are called a *generalized uniserial ring.* Oshiro [9] called a ring R is a *right H-ring* if every injective (right) R-module is lifting. He proved that if R is a generalized uniserial ring then it is a right and left H-ring [9, Theorem 4.5]. Also if R is a quasi-Frobenius (QF-)ring, R is right and left H-ring [9, Theorem 4.3] .

Theorem 3.2. *The following are equivalent for a ring R.*
(i) R is a right H-ring,
(ii) R is right perfect and every extending R-module satisfies (S),*
(iii) R is right perfect and the injective hull of every semisimple module satisfies
(S)*
(iv) R is right perfect and $E((R/J(R))^{N_0})$ satisfies (S).*

Proof. (i)⇒(ii) Since R is a right H-ring, R is right perfect and every injective module satisfies (S^*). Hence by Lemma 3.1, (ii) holds.
(ii)⇒(iii)⇒(iv) Clear.
(iv)⇒(i) Since $E((R/J(R))^{N_0})$ is injective, $E((R/J(R))^{N_0})$ is lifting. So by [17, Theorem 2.14], R is a right H-ring.   □

4   Every module with (S*) is extending

Since every $\mathbb{Z}$-module is cosingular, every $\mathbb{Z}$-module satisfies (S^*). It is well-known that there are $\mathbb{Z}$-modules which are not extending, for example $\mathbb{Z}/p\mathbb{Z} \oplus$

$\mathbb{Z}/p^3\mathbb{Z}$ (Example 2.15). So we conclude that every module with (S*) is not extending in general. Therefore in this part of this work we consider the condition "every module with (S*) is extending". On the other hand this condition implies that "every lifting module is extending" which was characterized by Oshiro (see Theorem 4.2).

First we investigate more stronger conditions that every module with (S*) is injective (or quasi-injective).

Theorem 4.1. *The following are equivalent for a ring R.*
(i) $R/Z^*(R_R)$ is semisimple and every R-module with (S) is injective.*
(ii) R is right perfect and every R-module with (S) is quasi-injective.*
(iii) R is right perfect and every lifting R-module is quasi-injective.
(iv) R is semisimple.

Proof. (iii) $\Leftrightarrow$ (iv) was proved in [17, Proposition 2.12].
(iv)$\Rightarrow$ (i), (ii) Clear.
(ii)$\Rightarrow$ (iii) Since every lifting module satisfies (S*), it is clear.
(i)$\Rightarrow$ (iv) If (i) holds then every simple R-module is injective, i.e. R is a right V-ring. Then $Z^*(R_R) = R \cap \mathrm{RadE}(R) = 0$ ([21, 23.1]). Hence R is semisimple. □

Theorem 4.2. [10, Theorem 2] *The following are equivalent for a ring R.*
(i) Every extending R-module is lifting,
(ii) Every quasi-injective R-module is lifting,
(iii) Every quasi-projective R-module is extending,
(iv) R is right perfect and every lifting R-module is extending,
(v) R is a generalized uniserial ring.

Proposition 4.3. *Let R be any ring. If every R-module M with $Z^*(M) \leq_e M$ is extending then every R-module with (S*) is extending.*

Proof. Let M be a module satisfying (S*). Then $M = M_1 \oplus M_2$ where M_1 is semisimple with $Z^*(M_1) = 0$ and $Z^*(M_2) \leq_e M_2$. By hypothesis, M_2 is extending. By the proof of Proposition 2.4, M_i's are relatively injective, $i \in \{1, 2\}$. Then by [4, Proposition 7.10], M is extending. □

A module M is called *semiperfect* if every homomorphic image of M has a projective cover (see [8] or [21]).

Proposition 4.4. *Let R be a semiperfect ring such that every R-module with (S*) is extending. Assume M is a semisimple R-module with projective cover P. Then P is extending.*

Proof. Let M be a semisimple R-module with the projective cover P. Since R is semiperfect every simple R-module has a projective cover [8]. So by [21, 42.4(4)], P is semiperfect. Hence P is lifting by [8, Corollary 4.43]. By hypothesis P is extending. $\square$

Theorem 4.5. *The following statements are equivalent for a ring R.*
(i) R is right perfect and every R-module with (S^*) is extending,
(ii) R is semiperfect with $J(R^{(\mathbb{N})}) \ll R^{(\mathbb{N})}$ and every R-module with (S^*) is extending.
(iii) R is right perfect and every R-module M with $Z^*(M) \leq_e M$ is extending.
(iv) Every R-module is extending.
(v) Every R-module is lifting.
(vi) R is a generalized uniserial ring with $J(R)^2 = 0$.

Proof. (iv)$\Leftrightarrow$ (v)$\Leftrightarrow$ (vi) were proved in [17, Proposition 2.13]
(i)$\Rightarrow$ (ii) It is clear.
(ii)$\Rightarrow$ (i) Since R is semiperfect, $R/J(R)$ is semisimple. By hypothesis, we have that $R^{(\mathbb{N})}$ is the projective cover of $(R/J(R))^{(\mathbb{N})} \cong R^{(\mathbb{N})}/J(R)^{(\mathbb{N})} = R^{(\mathbb{N})}/J(R^{(\mathbb{N})})$. So by Proposition 4.4, $R^{(\mathbb{N})}$ is extending. By [4, 11.13], R is right perfect.
(i)$\Rightarrow$ (iv) If (i) holds then every lifting R-module is extending. By Theorem 4.2, R is a generalized uniserial ring. Hence R is a right H-ring. By Theorem 3.2 and Lemma 3.1, every R-module satisfies (S^*). Thus every R-module is extending.
(iv)$\Rightarrow$ (iii) It is clear.
(iii)$\Rightarrow$ (i) By Proposition 4.3. $\square$

A ring R is called a *left Kasch ring* if every simple left R-module can be embedded in $_R R$. For example, QF-rings are left Kasch rings.

Pardo and Yousif in [14, Theorem 2.2] proved that a ring R is left extending and right Kasch $\Leftrightarrow$ R is semiperfect left continuous with $\text{Soc}(R_R) \leq_{eR} R$. Also Yousif in [19, Proposition 1.21] proved that R is right self-injective $\Leftrightarrow$ $R/J(R)$ is a regular ring, idempotents lift modulo $J(R)$, $(R \oplus R)_R$ is extending and $J(R) = Z(R_R)$.

Theorem 4.6. Assume that R_R satisfies (S^*) and every R-module with (S^*) is extending. If R is a left Kasch ring, then R is semiperfect right self-injective.

Before proving Theorem 4.6 we give the following proposition.

Lemma 4.7. [21, 41.14] *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$. M_1 is M_2-projective if and only if for every submodule N of M such that $M = N + M_2$, there exists a submodule N' of N such that $M = N' \oplus M_2$.*

Proposition 4.8. *If a ring R satisfies (S*), then $R \oplus R$ satisfies (S*).*

Proof. Let $L \leq R \oplus R$. $R = A_1 \oplus A_2$ where A_2 is cosingular. $R \oplus R = A_1 \oplus A_2 \oplus R = L + (A_2 \oplus R)$

Assume $R \cap (L + A_2) = 0$. $L \cap A_2 \leq A_2$ and A_2 satisfies (S*). Then $A_2 = C_1 \oplus C_2$, $C_1 \leq L \cap A_2, C_2 \cap L \cap A_2 = L \cap C_2$ is cosingular. $R \oplus R = A_1 \oplus A_2 \oplus R = (A_1 \oplus C_1) \oplus (C_2 \oplus R) = L + (C_2 \oplus R)$. Since $A_1 \oplus C_1$ is $C_2 \oplus R$-projective [8], there exists $L' \leq L$ such that $R \oplus R = L' \oplus C_2 \oplus R$ by Lemma 4.7. $L \cap (C_2 \oplus R) \leq C_2 \cap (L + R) = L \cap C_2$ is cosingular. Hence $R \oplus R$ satisfies (S*).

If $R \cap (L + A_2) \neq 0$, then $R = B_1 \oplus B_2$, $B_1 \leq R \cap (L + A_2)$ and that $R \cap (L + A_2) \cap B_2 = B_2 \cap (L + A_2)$ is cosingular. $R \oplus R = A_1 \oplus B_1 \oplus A_2 \oplus B_2 = L + (R \oplus A_2) = L + B_1 + B_2 + A_2 = L + (A_2 \oplus B_2)$. Since $A_1 \oplus B_1$ is $A_2 \oplus B_2$-projective, then $R \oplus R = L' \oplus (A_2 \oplus B_2)$ where $L' \leq L$. $L \cap (A_2 \oplus B_2)$ is cosingular since $B_2 \cap (L + A_2)$ and $A_2 \cap (L + B_2)$ are cosingular. It follows that $R \oplus R$ satisfies (S*). $\qquad\square$

Proof. of Theorem 4.6 Since R is an extending and left Kasch ring, R is a semiperfect right continuous ring by [14, Theorem 2.2]. Since R is right continuous, $J(R) = Z(R_R)$ (see [4, 2.12]). On the other hand since R_R satisfies (S*), $(R \oplus R)_R$ satisfies (S*) by Proposition 4.8. So $(R \oplus R)_R$ is extending. Hence by [19, Proposition 1.21] R is right self-injective. $\qquad\square$

A ring R is called a right PF-ring (pseudo-Frobenius) if it is right self-injective right Kasch.

Corollary 4.9. *Assume that R satisfies (S*) and every R-module with (S*) is extending. If R is a right and left Kasch ring then R is a right PF-ring.*

Example 4.10. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ be the ring of upper triangular matrices over a field F. Then R is an (right and left) extending Artinian serial SI-ring with $J(R)^2 = 0$ which is neither a left nor right Kasch ring, and $J(R) \neq Z(R_R) = 0$ (see [2], [4, 13.6] and [20, Example 4]). Note that by Theorem 4.5, every R-module is extending; and R is not right self-injective.

References

[1] I. Al-Khazzi, P.F. Smith, *Classes of modules with many direct summands*, J.Austral.Math.Soc., 59 (1995), 8-19.

[2] G. Bacella, *Generalized V-rings and von Neumann regular rings*, Rend. Sem. Mat. Padova, 72 (1984), 117-133.

[3] G.F. Berkinmeier, J.K. Park, *When is the CS-condition hereditary?* Comm. Alg. 27(8) (1999), 3875-3885.

[4] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, Extending modules (Pitman RN Mathematics 313, Longman, Harlow 1994).

[5] K.R. Goodearl, Ring Theory (Marcel-Dekker, 1976).

[6] M. Harada, *Non-small modules and non-cosmall modules*, In Ring Theory: Proceedings of the 1978 Antwerp Conference, F.Van Oystaeyen, ed.NewYork: Marcel Dekker, (1979), 669-690.

[7] W.W. Leonard, *Small modules*, Proc.Amer.Math.Soc., 17 (1966), 527-531.

[8] S.H. Mohamed and B.J. Müller, Continuous and discrete modules (Cambridge University Press, NewYork 1990).

[9] K. Oshiro, *Lifting modules, extending modules and their applications to QF-rings*, Hokkaido Math. J., 13(1984), 310-338.

[10] ――――, *Lifting modules, extending modules and their applications to generalized uniserial rings*, Hokkaido Math.J., 13(3) (1984), 339-346.

[11] A.Ç.Özcan, *Some characterizations of V-modules and rings*, Vietnam J.Math., 26(3) (1998), 253-258.

[12] ――――, *Modules with a zero radical of their injective hull*, Hacettepe Bull. of Natural Sci. and Eng., Series B, 27 (1998), 45-49.

[13] ――――, *Modules with small cyclic submodules in their injective hulls*, Comm. Alg., to appear.

[14] J.L.G. Pardo, M.F. Yousif, *Semiperfect min-CS rings*, Glasgow Math. J., 41 (1999), 231-238.

[15] C. Santa-Clara, P.F. Smith, *Extending modules which are direct sums of injective modules and semisimple modules*, Comm. Alg. 24(11) (1996), 3641-3651.

[16] C. Santa-Clara, *Extending modules with injective or semisimple summands*, J.Pure and App. Alg., 127 (1998), 193-203.

[17] N. Vanaja and V.M. Purav, *Characterizations of generalized uniserial rings in terms of factor rings*, Comm. Alg., 20(8) (1992), 2253-2270.

[18] M.F. Yousif, *SI-modules*, Math. J. Okayama Univ., 28 (1986), 133-146.

[19] ――――, *On continuous rings*, J. Algebra, 191 (1997), 495-509.

[20] ——, *CS rings and Nakayama permutations*, Comm. Algebra, 25(12) (1997), 3787-3795.

[21] R. Wisbauer, Foundations of Module and Ring Theory (Gordon and Breach, Reading 1991).

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should be an equilibrium between:

- Pure mathematics,
- Pure mathematics in Physics, etc. and
- Applicable mathematics.

We should always keep in mind that the applicat:ons of tomorrow are based on the foundations of today.

There is no room at the Universities for the traveling salesman; he does belong into the realm of the Polytechnica.

Humboldtian ideals

Let me pause a moment to recall the lives of both Alexander and Wilhelm von Humboldt and of their ideas.

The brothers Alexander – he lived from 1769 to 1859 – and Wilhelm – he lived from 1767 to 1835 – von Humboldt pass on to us two biographies which supplement each other rather than contradicting each other, and although they diverge temporarily – not only geographically – they sooner or later touch each other and even unite.

In 1787 the brothers entered the University of Frankfurt an der Oder: According to his mother's wish Alexander prepared for a career in government service, but he soon followed his inclination towards natural sciences. In 1792 he received a mining license from the Mining School of Freiburg. His brother got his degree in Göttingen in philosophy and linguistics.

The milestone in 1796, which marks a turning point – on the basis of the common origin of being brought up together – was Gottlob Johann Christian Knuth. After the early death of their father in 1779 Knuth took care of the financial matters of their mother and after her death in 1796 he executed the will of their parents:

- Knuth made the family man Wilhelm the landlord of various houses in Berlin-Tegel,

- the bachelor Alexander he made a rich man, so that he could realize his boyhood dream of traveling around the world. The fruits of his travels were so manifold that it took the scientific world decades to catch up on the vast amount of new knowledge gained by him.

With the early death of their mother the Humboldt brothers were able to each realize the dreams they had until then nourished in secret, and the trails of their lives could develop freely and take their remarkable directions.

In the years 1794-1796 the family of Wilhelm von Humboldt already held close contact with Friedrich Schiller and his wife. Both brothers were intrigued by Goethe's "unitarian thinking" (Ganzheitsdenken); i. e. the whole universe – including its various inhabitants, humans, animals and plants – form a unity; they do not consist of separated entities like birds, plants, etc., instead their symbiosis influences them mutually. Both brothers integrated this "Ganzheitsdenken" individually in their research:

- Alexander demonstrated this in his studies of nature, in his "world-painting", first depicted in 1880 in his considerations of nature – "Ansichten der Natur" – and later deepened this in his "Kosmos-lectures" in Berlin in 1827/28.

- Wilhelm applied this "Ganzheitsprinzip" – i. e. the idea that the world forms an entity – in his general and comparing studies of languages.

Both required in science – not only natural science – a careful observation of the details, which must then be put together – like in a puzzle – to form the global picture.

This is the essence of what one understands – concerning research under the Humboldtian idea of research; not merely the local view, but the various detailed local views, which must be put together into a global view.

However – except at the Universities – this scientific attitude and, in particular, the vast practically unlimited number of objects of research it would require, soon became untimely. The spirit of "positivistic" science, i. e., the source of all human knowledge is the "given", ignores everything that does not have its origin in sensual observations; i. e. it ignores things such as regularities, various categories, structures, etc. This spirit, which now took over, ignored Humboldt's ideology and later led the brothers into isolation.

The ideology of the Humboldt brothers' "Ganzheitsdenken", the consideration of our world as an entity, has recently experienced a strong reactivation inside the Universities as well as outside them – even in politics – with view to numerous problems such as the climate, the rain forests, pollution, etc.

In 1799 Alexander von Humboldt started, after obtaining the permission by King Charles IV of Spain, a 5-year trip through South America and Mexico, investigating

the botany, the zoology,
the vegetation and the geology

of these countries.

Back home he published his findings combined with a global theory about the relations between vegetation, wildlife and landscape formations in three books

- The equinoctial plants (1805),

- Ideas for a Geography of plants and a Nature Picture of the Tropics (1805) and

- Views of Nature with Scientific Commentaries (1808).

Johann Wolfgang von Goethe wrote about Alexander von Humboldt:

"Alexander von Humboldt has no equal in
knowledge and vital learning.
Whatever the subject of discussion,
he is completely at home in it,
and gems of wisdom are pouring forth.
He is like a fountain with a vast number of outlets."

Humboldt's ambition was

"TO SURVEY NATURE AS A WHOLE, AND TO PRODUCE EVIDENCE OF THE
INTERPLAY OF NATURAL FORCES".

If we talk about Humboldt's ideas and ideals on research we mean: a broad and not narrow-minded knowledge about the sciences. In our days, however, we should be more modest due to the immense increase in knowledge. For mathematicians we mean:

- a broad knowledge in the realm of mathematics;
  in order to be able to develop a general theory for the problems,
  which may come from outside mathematics (applied mathematics) or
  from inside mathematics,

- to see connections between various phenomena (inside and/or outside) of mathematics,

- to find common patterns in various different problems in applications or inside mathematics.

In this connection let me quote Lagrange

> "As long as algebra and geometry proceeded along separate paths,
>   their advance was slow and their applications limited.
> But when these sciences joined company, they drew from each
>   other fresh vitality and
> hence-forward marched on at a rapid pace towards perfection."

The same applies to other fields of mathematics and their applications.

After having returned from his journeys to South America, Alexander von Humboldt lectured about his observations and theories at Scientific Institutions in Paris and Berlin.

If we talk about Humboldt's ideas and ideals of teaching, we mean a symbiosis between research and teaching – here with teaching I do no mean undergraduate courses – these can be taught much better by teaching assistants – I mean courses for eager students:

- teaching the results of research to the graduate students; this has a two-fold merit:

    - it shows the students what good mathematical research is;
    - it forces the teacher to cut the research into small and lucid pieces, so that the students can grasp the idea and the arguments; this often also gives the Professor * new insight,

- pointing out the connections between the various areas of mathematics;

- streamlining the course, by concentrating on the essentials both in the selection of the topics and in the structure of the proofs.

These things, however, can only be achieved properly by teachers with a strong backbone with respect to research.

At this place I would like – based on my own experience – to point out that a teacher in his class is the better, the more he is challenged by the material; so I would like to **require** that

> good researchers teach demanding courses which challenge them,
>   the other courses could be taught by
>   teaching assistants or special teachers,
>   who are not required to do research,
>   but who are engaged in didactics, i. e. teaching.

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*Let me stress that with the word "Professor" I do identify a classical Professor at a University – less a teacher at a Polytechnicum or a College

Let me summarize "HUMBOLDTIAN RESEARCH AND TEACHING:

- We do need a broad knowledge in our field to see the connections; do not be narrow minded; always keep the "ENTITY" in mind.

- Try to build a theory on the "observations" and "examples"; always ask "why"?

- Teaching (advanced courses) and research should form a "unity"; they are intimately related.

For an outstanding mind like Humboldt's it was possible to keep track of and understand the various sciences of those days. This is impossible today, so we have to modify the postulate to:

Postulate 0.2.   • Do not be narrow-minded,

- look around you and see the connections.

Teaching at Universities

Let us recall that the Latin word "Professor" means "public teacher", so in its Latin origin it has nothing to do with research. The modern usage of the word "Professor" is synonymous to

- academic teacher;

- scholar, scientific investigator.

Let us briefly look back at the Universities in Humboldt's time:

- At that time there were only few and engaged students – something that is missing to a large extent today.

- The few textbooks which were available were far behind the present state of research.

- The Professor in those days would work interactively with the students. He would present the new results to the students in reformulated ways, so that it would be suitable for the mental capacity of the students. The dialogue with the students would often bring new light to some aspects of the topic under consideration.

There was an intimate interaction between "teaching" and "research". Today, at the "mass Universities", the above hypotheses are non longer valid:

- We have too many students, who – to a large part – are not engaged in science; many of them go to the University because they expect to get a better job, to have more respect in society or to earn more money. Often they want to get their degree (Diploma) with a minimum amount of work. They do not attend the lectures to understand to subject, but only in order to be able to pass the examination with the least amount of work.
I should stress that there are always exceptions – but not so many.

- The material taught in the first semesters (undergraduate studies) is available in good textbooks, written – to some extent – by excellent "teachers".

- So there is no need in the undergraduate courses for scientifically highly qualified teachers. It would be a loss of time and energy for the excellent scholar to have him teach to disinterested undergraduate students.

- The scholars should mainly teach courses on their special field of research where they can unfold their knowledge and talent. They should also teach the undergraduate courses for a few selected excellent and engaged students.

So let us analyze the two levels of University teaching more closely:

Undergraduate Teaching

This includes the elementary undergraduate courses like "calculus" and "linear algebra". It also includes the "mathematics courses" for other faculties such as economics, engineering, etc.

The requirements of the teachers here are more on the side of DIDACTICS than on HIGH LEVEL RESEARCH. They should in general not be given by full Professors, who have obtained their position mainly because of their outstanding scientific research.

Let me pause to talk about "Universities" in Germany. Many of those institutions which call themselves "Universities" nowadays used to be Polytechnica; in the course of our "academic revolution", which started around 1967, they all became Universities. The Polytechnica wanted to be called "Universities", since this name carries the flavor of more importance. So they were named

"Universities", but without changing their character as Polytechnica. In these Polytechnical Universities, the engineers still dominate all important committees. For example, the "mathematics Faculty" has to give each year 4 different courses of "calculus for engineers". Until recently the engineering faculties did insist on these courses being taught by "full Professors" from the mathematics faculty. The only point I can see in such a request is that it makes the engineering faculties feel more important when these elementary mathematics courses are taught by full Professors.

I myself feel completely over-qualified, teaching elementary calculus to a class of 300 – 500 students the majority of whom are not really interested in learning mathematics. You hear 300 pencils drop on the table when the word "proof" is used – but who only want to know the "rules" according to which they can solve routine problems and pass the examinations.

Teaching such courses cannot be the duty of a full Professor who is actively and successfully involved in mainstream mathematical research and who leads his disciples to becoming good mathematicians.

To teach these courses we have to have people with a high level of didactical talent and a solid knowledge of elementary mathematics – i. e. mathematics taught in undergraduate courses. They should like mathematics and they should be able to motivate the students and communicate some of the appeal of mathematics to the students.

Undergraduate mathematics is LEARNING – and I stress "learning" – the elementary techniques of mathematics. One can compare the learning and understanding of such elementary mathematics with learning the techniques of playing the piano: You have to practice playing the scales. For this you do not need Arthur Rubinstein as a teacher. Here the teacher has to be able to motivate you and he must be a master of these technical skills.

Here comes another postulate

Postulate 0.3. For undergraduate teaching we need people with didactical talent who can motivate the students to master elementary mathematics. They must be skillful artists in this "routine" mathematics.

These parts of mathematics are surely enough for teaching in

- Colleges,
- Pedagogical High Schools,
- Polytechnica and
- undergraduate studies.

Let me ask the PROVOCATIVE QUESTION:

How many students really do need more mathematics?

Graduate Teaching

Let us assume that the students have successfully passed the undergraduate mathematical studies, i. e. they know the elementary skills and basic facts of mathematics in, say,

Analysis, Algebra, Geometry, Topology,
Partial Differential equations, Numerical mathematics,
and last but not least with a
mathematical Computer system such as "Mathematica" or "Maple".

The graduate teaching should assume that these foundations are known and present with the students – this is often wishful thinking at German Universities.

The teaching of more skills and more techniques – no matter of how sophisticated they are does not belong to the graduate courses at a University. These are the main topics at "Polytechnica", and "Colleges", whose primary goals is to prepare the students to a **special** occupation; so they need to have high skills in the direct applications of mathematics in mind.

The graduate teaching of mathematics at Universities should have the following main goals, which I formulate as

Postulate 0.4. The goals of graduate teaching are:

- To see connections and get a feeling of mathematics, be it in pure or applicable mathematics.

- They should learn the general theories necessary to understand these connections.

- They should be trained to apply what they have learned in a field A to problems in a field B.

- They should be trained to be creative at solving problems.

- They should learn to develop a theory.

The teaching of such courses and the guidance to be given to the students in order to reach these goals require an excellent scholar, who can also inspire in

the students his love and enthusiasm for mathematics. It cannot be done by teachers who do not have a good insight into and overview of mathematics, and who have not done important research.

Summarizing we get yet another

Postulate 0.5. At the Universities we should have two types of teachers:

- The one who teaches the undergraduate and elementary graduate course and whose strength is didactics and the ability to motivate students to work hard,

- The Professor in the classical sense who does both important research and leads the students to the frontiers of research and presents them with the interplay of various fields and aspects of mathematics.

If one agrees with this, the image of a Professor has to be modified both in public and in the minds of the politicians.

It is en vogue – at least in Germany – to consider Professors as being disinterested in teaching the students and some of them even being lazy – this latter statement was formulated in an interview by the chairman of the assembly of all rectors of German Universities (HRK). These politicians want the professors to be the slaves of the students:
the courses – please, they have to be taught in such a way as to make the students happy – a joke every 5 minutes, not demanding courses and easy examinations.

There is a tendency in Germany of introducing student evaluation and making the budget of the faculty dependent of the results. Student evaluation is not bad eo ipso, but it has to be done with great care:
Can a student really judge at the end of a course whether the teacher was successful, or can this only be done in retrospect, after one has an overview? How about asking after 5 years in praxis – i. e. in a job – the most successful businessmen what they think about their University teachers now.

Maybe the politicians who favor this sort of evaluation will next ask the children at the age between 10 and 16 every year what they think about the education by their parents. The outcome of this evaluation by the children will then determine the amount of support for the children (Kindergeld) the parents get from the government. What a generation of people will such a policy generate!

When I look back at my school days and try to evaluate my teachers, I think that I have learned most – not only concerning the subject but also concerning life – from those teachers whom as students we somehow disliked because they

were stern, austere and tough in their requirements.
I think that a similar phenomenon occurs with our students.

With my students I have made the following empirical observation:

- The number of failing students in the first 4 semesters is almost the same whether you give a demanding course or whether you make it very easy for the students.

- In the student evaluation there is a strong correlation between the questions:

    - Did the Professor go too fast and did he explain things properly? and

    - How much time did you spend trying to understand the lectures and doing the exercises?

I personally do not believe in student evaluations, I believe in what my students achieve in life.

Teaching Oriented towards Practice

We hear more and more often the request from the politicians and also from some Rectors and Professors of Universities that

THE TEACHING AT THE UNIVERSITIES SHOULD BE
MORE ORIENTED TOWARDS PRACTICE ;
this should not be confused with oriented towards Applications.

For brevity I shall say *"practice oriented"* to mean *"oriented towards very special problems which arise in praxis"*.

The above request is a nonsense:

- Already the word "universitas = the totality, the whole" is a contradiction to "practice oriented";
for this kind of teaching we have Colleges and Polytechnica (Fachhochschulen).

- With "practice oriented" teaching and research we are doing *"the work for the industry.*
By paying the money for an Assistant, who does the research for an industry project, the industry saves the multi-fold amount of money.

The assistant works for 4 years on a research project the industry needs for their production. The professor is given a new computer and some money, which he can use for his work. One should think about the amount of money the industry saves this way, the expenses being covered by the taxpayer.
This surely cannot be the purpose of teaching and research at a University.

- "Praxis oriented" teaching and research deals with the techniques which are used **today**; but the life of working men in industry lasts 35 years, where new techniques are developed; these new techniques must be learned, understood and applied.

- So, **what we have to teach our students** is

    - how to learn by themselves and to digest inwardly what they have learned;
    - we have to teach them the foundations and the essentials of applications;
    - we have to teach them the foundations of (pure) mathematics in order to make them flexible, to teach them to work on a hard theoretical problem.

The flexibility of the mind and the potential of reacting flexibly is extremely important in our rapidly changing society, when people are laid off and have to be re-educated for another occupation. An open mind for flexibility is not promoted by praxis oriented teaching, but rather by intensive occupation with theoretical mathematics, both in the pure and applied sector of mathematics.

For the relation between applied and pure mathematics, we should follow Felix Klein:

"...We do not want that the pendulum,
which in former decades
may have inclined too much towards the abstract side,
should now swing to the other extremes;
we would rather pursue the middle course."

So I **require** that

- "Praxis oriented" teaching and research – i. e. oriented towards very special problems which arise in praxis – should be banned form the Universities,

- it should be obligatory to learn the fundamental areas of pure mathematics (algebra, analysis and topology),

- the areas of applied mathematics to be taught should concentrate on general principles and techniques, with special applications as exercises,

- the students have to be taught to learn rigorously new and nontrivial theories – both in pure and applied mathematics.

In public mathematics is often identified with helping to "solve very special problems that arise in praxis". This however is only a negligible part of mathematics. We should thus also promote – in public and as propaganda – another very important aspect of mathematics, the study of the abstract theories of applied and pure mathematics, which sharpens the logical thinking and hence is important for everybody.

This aspect of mathematics has been held in high respect over the centuries by several outstanding people:

- **Plato**:

  > "It would be proper then to lay down laws for this branch of science (mathematics) and
  > persuade those about to engage in the most important matters of state
  > to apply themselves to computation, and study it,
  > not in the common vulgar fashion ... for the sake of buying and selling
  > but for the reason that the soul may acquire the facility of turning itself ...
  > to truth and real being."

- **Immanuel Kant**:

  > "The instruction of children should aim gradually to combine 'knowledge and doing' (Wissen und Können).
  > Among all sciences, mathematics seems to be the only one of a kind,
  > to satisfy this aim most completely."

- **Oliver Cromwell**

  > "I would have my son mind and understand
  > business, read a little history, study the mathematics and cosmography;

these are good, with subordination to the things of God ...
These make fit for public services, for which men was born."

In the public picture of mathematics– outside mathematics – and in the thinking of the Professors (in mathematics) these aspects should again gain more weight in our time.

The administrative duties of Professors

The so-called DEMOCRATIZATION of the Universities has led to the crippling reign of the committees, which take up a lot of time that could otherwise be used for research and teaching.

The influence of these committees and the "burocrats" does not only not promote outstanding achievements, but, which is worse, it hampers them.

In Germany it is for example possible – and it has indeed happened [†] – that a chancellor (the supervisor of the "non-academic" staff) in a University misuses his powers and bends the law [‡] to harm research units. Such an interference of non-academics staff into academic matters opens the door to regulation of research and a classification of the academic staff into "personae gratae" and "personae non gratae", a situation Germany has experienced with pain some 65 years ago. Part of the censorship which existed then apparently still exists in the mind of the Rektor Fritsch from the former "Technische Hochschule Stuttgart" University, who forced the dean of Mathematics – without consulting me – to close my University-homepage in March 2001; the reason was that I had put my letter of resignation to the minister of science in Baden-Württemberg [§], which Rektor Fritsch apparently did not like, into my homepage. What an act of democracy!

This reign of the administration has created the "committee Professor", who is a member of almost every committee, teaches elementary courses and does a little bit of weak research.

But this is the professor our politicians seem to like and whom they promote:

- Since he is weak in research, he does not have a strong scientific backbone.

- Since he is a committee Professor he needs to be elected, so he does not have strong views, he voices the opinion of the "mediocre majority".

[†] It was done by the chancellor Schwarze of the University of Stuttgart in 1997 ff.

[‡] Here we want to note, that we do not use "bending of the law" as terminus technicus in the German sense of "Gesetzesbeugung". We–as a mathematician without knowledge in law–use it as a translation of the German expression "Umgehung von Vorschriften".

[§] The letter can be found on my private Web-site: "http://www.roggenkampmath.de".

- Since he most likely has political ambitions, his mind is open to the opinions of the ministers and superior political administrators and not so much to scientific arguments.

Such a "committee Professor" wants to make his University attractive by achieving the following:

- The majority of students should finish their studies in a relatively short time. Since the quality of the average student entering the Universities has gone down continuously – some politicians want to raise the number of people with a higher education – this goal can only be achieved if the level of the courses and the academic requirements of the examinations is lowered more and more.

- The student evaluations of the teaching quality of their professors should be most favorable. This implies that the professor leaves out the solid foundations in his lectures and makes a more or less intelligent joke every five minutes.

- In order to raise the teaching quality, the faculties will get less money if the student evaluation of the teaching of their professors is poor.

- The students should also be attracted by cheap housing nearby, close skiing possibilities, good sporting facilities, etc.

- The tuition fee and other fees should be very low.

This is not a utopia, but reality at some places in Germany.
Such a "committee Professor" does not care about the classical measurements of the quality of a University such as high international ranking and reputation of the Professors in research and teaching. He does not care about the status the former students achieve in their life.

Though the technical universities have changed their name, they still remain "Technische Hochschulen". The prime example has given the rector Prof. Dr. Pritschow from the University of Stuttgart in an article in the "Böblinger Kreiszeitung" from April 12th 2000 where he talks about the burden of being a rector. "The one who must be rector for 6 years [before it was only 4 years] has to say good-bye to research. What may be possible with Anglists and representatives of soft fields is impossible in the technical fields." What an arrogance and ignorance this rector shows by naming Anglistics and the other humanities soft fields.

Why do students go to Universities like Harvard, MIT, Oxford, Cambridge, Göttingen or Bonn? Surely not for the above-mentioned reasons.

The next postulate applies mainly to the Universities, it only partly applies to Polytechnica and it does not apply at all to Colleges.

Postulate 0.6. WE HAVE TO MAKE OUR UNIVERSITIES ATTRACTIVE BY THE INTERNATIONAL SCIENTIFIC REPUTATION OF THE PROFESSORS, WHO DO THE TEACHING OF THE HIGH-LEVEL COURSES.
WE HAVE TO MAKE IT CLEAR TO OUR POLITICIANS THAT THESE ARE THE QUALITIES WHICH RAISE THE REPUTATION OF A UNIVERSITY AND CONSEQUENTLY MAKE IT ATTRACTIVE.
WE ALSO HAVE TO POINT OUT TO THE POLITICIANS THE IMPORTANCE OF MATHEMATICS FOR THE GENERAL EDUCATION OF THE MIND.

Let me narrate a real-life story about politics and politicians:

There was a small conservative state. After the election the positions of the ministers had to be filled. As is usual in politics the choice does not only depend on the qualification for this job – most of our ministers are "universal" – but on the merits such a candidate has accumulated by his work for the reigning party. The situation was not different in our small state. There was an assistant at a certain University, who had been an assistant for decades without having obtained a Ph. D. yet; he was, so to speak, a "lifetime" assistant – a slave to the full Professor on whom he depended. However, this assistant had been very active in the now ruling party. Since he had been in academics – the position of an assistant is the lowest academic position at our Universities – the party made this assistant the Minister for "Research" and as such he became the boss of all University Professors in our little state. Let me repeat: This state now has a Minister for "Research" who did not get a Ph. D. during his elongated work as an assistant at his University and who had been subordinate to Professors all his life; now he is their boss – not because of scientific excellence but because of his "party work".

Let me point out what kind of policy we are consequently having in this state; a policy which is also somehow supported by the public opinion:

- Teaching should be the main occupation of Professors.

- The Professors should be present several days a week in order to be available for the students – even during the vacation. I myself was never able to prove any result in my office in the University, since there something or somebody always distracted me. I myself have worked at home all my life.

- The salary will depend on the success in teaching – whatever "success in teaching" is.

- The Professors should do whatever is possible – except raising the scholarly quality – in order to attract more students, in particular foreign students.

There are recently tendencies of paying the professors according to their achievements in research and teaching. In an interview in the Stuttgarter Zeitung from April 12th 2000, the rector of the University of Stuttgart Prof. Dr. G. Pritschow had all the answers: [according to which criteria this additional salary should be given he answered:] "in research this is simple. Here the Professor can be judged according to his additional achievements in getting sponsored." [So next time we see our Professors in the lectures wearing T-shirts with "drink coca-cola" printed on them.] Asked further on by the interviewer then the bel-arts would be handicapped compared to the engineers, Pritschow answers: "Yes, possibly. However the professors in literature could engage in finding sponsors"... [could you imagine Goethe wearing a T-shirt with the imprimé "support Faust" and going from door to door]. Also for judging the achievements in teaching, the rector had good suggestions: "In early times this was easy. Then the students had to pay "Hörgelder" [it must be Hörergelder]. [This meant that students in order to attend the lectures of a professor had to pay him a certain fee]. "On this the professors did live" [this is only partly correct]... Today the remains of this are the examination fees [in undergraduate courses with an examination the students have to pay a certain fee for the examination of which a certain amount is given to the professor. Old fashioned professors do give this money to their assistants who have done the actual grading work]. With this [the fees for the examination] the professor is payed according to his achievement. This means that the best teacher will be the one who teaches the low level and little demanding courses like "Mathematics for engineering" with some hundred students. Because then he would get a lot of fees for the examinations.

I hope this whole interview was not meant seriously and is an utopia.

What I did not hear from this minister were statements such as:

In order to cope with the requirements of the future as an
industrialized nation
we have to support high quality research
and demanding teaching.

This is a dangerous and very shortsighted policy.

In Germany (February 2000), following an idea from the USA, the politicians had the idea of importing computer specialists from Asian countries in order to supply the industry with highly qualified specialists. In my opinion this shows that the University education at some German Universities is not – like the ones in Asia – up to the standards required by the Industry. This cannot be changed by brain importation – it has to be changed at the roots, the University education.

Reminiscences About Pure vs Applied Mathematics

In "A Mathematician's Apology" (Cambridge Univ. Press 1940), G. H. Hardy (1877 – 1947) writes:

> "Very little of mathematics is useful practically and ...
> that little is comparatively dull ...
> The "real" mathematics of the "real" mathematician,
> the mathematics of Fermat, Euler, Gauss and Riemann is almost
>   wholly useless. ...
> We have concluded that the trivial mathematics is, on the whole,
>   useful
> and that the "real" mathematics, on the whole, is not."

He explains what he means by writing:

> "There is the real mathematics of the real mathematicians, and there is what I will call the 'trivial' mathematics, for want of a better word."

As can be seen here, Hardy gives a black and white picture of mathematics. He indicates what he means by "trivial" mathematics.

> "... It is undeniable that a good deal of elementary mathematics – and I use the word 'elementary' in the sense in which professional mathematicians use it, in which it includes, for example, a fair working knowledge of the differential and integral calculus – has considerable practical utility. ...
> 'But our general conclusion must be that such mathematics is useful as is wanted by a superior engineer or a moderate physicist; and that is roughly the same thing as to say, such mathematics has no particular esthetic merit."

Then he continues, rightfully so,

> "These parts of mathematics are, on the whole, rather dull".

On the other hand, Hardy also sees depth in some parts of applied mathematics, as becomes clear from the following quotation:

> "If the theory of numbers could be employed for any practical and obviously honorable purpose, ... then surely neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications."

which is to say: he sees the possibility of important results in applied mathematics, as was later shown to be the case. Further on he writes:

> "The geometer offers to the physicist a whole set of maps from which to choose. One map, perhaps, will fit the facts better than others, and then the geometry which provides that particular map will be the geometry most important for applied mathematics. I may add that even a pure mathematician may find his appreciation of this geometry quickened, since there is no mathematician so pure that he feels no interest at all in the physical world; ..."

Here Hardy actually describes what happened later on with Einstein and Riemann's non-Euclidian differential geometry. The final quote really shows that Hardy means good and bad mathematics and that he definitely foresees important applications of pure mathematics:

> "One rather curious conclusion emerges, that pure mathematics is on the whole distinctly more useful than applied. A pure mathematician seems to have the advantage on the practical as well as on the esthetic side. For what is useful above all is *technique*, and mathematical technique is taught mainly through pure mathematics."

Hardy would include, though – in order to force external reality into his model – theoretical physics, which I have labeled "Mathophysical Art" into his canon of "real" mathematics. He writes

> "I count Maxwell and Einstein, Eddington and Dirac among "real" mathematicians.
> The great modern achievements of applied mathematics have been in relativity and quantum mechanics,

> and these subjects are, at present at any rate, almost as "useless"
>   as the theory of numbers."

This was in 1940. Things have changed tremendously over the last 50 years. In view of the present development, Henri Poincaré (1854-1912) was much more far-sighted and realistic than Hardy:

> "The scientist does not study nature because it is useful;
> he studies it because he delights in it, and he delights in it because
>   it is beautiful
> If nature were not beautiful, it would not be worth knowing,
> and if nature were not worth knowing, then life would not be worth
>   living."

I think here Poincaré expresses what I understand as a good University Professor:
He has to get these ideals of Poincaré down to pass them on to his students. But this is only possible if the Professor is surrounded by eager students whose horizon is wider than just passing the next examination.

Although I am with my whole heart a "pure mathematician" I cannot and I do not want to let Hardy's point of view stand as my view.

From the time of Plato's academy through the middle ages and into the rise of the post-medieval Universities, mathematics had always been central: The classical scholastic curriculum included

- trivium: logic, grammar and rhetoric the more advanced

- quadrivium: geometry, astronomy, arithmetic and music (harmonic relationships)

Here mathematics would serve two purposes, which I think should also be considered to be important today:

- The study of mathematics sharpens the logical thinking and is thus useful in everyday life.

- The study of mathematics is necessary for solving problems in everyday life.

In the late nineteenth century the abstract approach, already held in high esteem by the classical (Greek) mathematicians, was being applied again with more enthusiasm, and new standards of rigor were emerging:

- The algebraic approach was applied to geometry and topology,

- analytic function theory was in full bloom.

In many areas mathematics was running so far ahead of applications that it was widely assumed that most of these fields would never have any (application). This was in particular true for certain classical areas such as number theory and algebra.

In retrospect – and this is very important in our approach to the curriculum in Universities – the best mathematics consistently found very important applications, but often not until many decades later.

Let us list a few ¶:

- Riemann's "clearly inapplicable" non-Euclidian Differential Geometry from the 1850's became the basis of Einstein's General theory of Relativity some 60 years later, which is now in the space age a vital part of concrete calculations.

- The purely abstract field of group theory and representation theory from 1850 – 1900 became a vital part of particle physics and coding theory in 1930/40.

- Finite fields, invented by Evariste Galois around 1800, which were considered to be the purest of pure mathematics, have become, starting in 1950, the basis for the design of error correcting codes.
  These are now indispensable in everything from

    – computer data storage to

    – deep space communication to preserve fidelity to

    – recording music on Compact disks.

- George Boole's nineteenth-century invention of formal logic has become the basis for electronic switching theory and for digital computer design.

- Number theory has become important in cryptography (large prime numbers and prime factorization).

- Topology and in particular knot theory – which would seem to be particularly useless – form extremely important applications in physics (quantum mechanics and super-string theory) and in molecular biology (the knotted structures of nucleic acids and proteins).

¶ These are taken from [Golomb].

- Graph theory – the trivial one-dimensional topology – has blossomed into a major discipline where the boundary between "pure" and "applied" (e. g. transportation problems) is virtually invisible.

There are plenty more such examples where results from the "purest" areas of mathematics have found very important applications.

In this connection I would like to quote from Golomb, S. W. who is both a pure and an applied mathematician , *"Mathematics after 40 years of space age"*, [Golomb]:

"It may still be necessary for some Mathematics Departments to defend themselves from being turned into short-term providers of assistance to other disciplines, which are consumers rather than producers of mathematics; but the basic principle that good 'pure' mathematics is almost certain to have very important applications *eventually* is now widely recognized.

For most mathematicians today, the distinction that matters is between '*good* mathematics' and '*bad* mathematics', not between 'pure' and 'applied' mathematics."

So we have in some sense come back to Hardy's distinction between "real" mathematics and "trivial" mathematics. Except that it is not true that "real" mathematics is useless and that "useful" mathematics is "trivial".

Postulate 0.7. Both in our courses at the Universities and in our research we have to make sure that we are working with "good" mathematics both in "pure" and "applicable" mathematics.

Epilogue

It seems that pure mathematics is currently in a similar situation as in the middle ages, when practicing pure mathematics was not appropriate from the religious point of view. This has not always been so. Let me point out the profound role mathematics has played in the eyes of Goethe, by quoting from Goethe's "Wilhelm Meisters Wanderjahre" (1829), "Betrachtungen im Sinne der Wanderer":

"Just like dialectics, mathematics is an agent of the inner higher intellect; in its execution it is an art."

Mathematics, both pure and applicable – not the applications – in its rigidity and logic should again become – as it has been in times and countries of high

civilization – a vital part of education, both in school and at the Universities. It should be a matter of prestige for a state to promote this kind of mathematics. Engaged teachers should work hard in the Schools and in the Universities to raise the reputation of mathematics in the mind of the politicians and the people. In this spirit Nietzsche wrote in "Frhliche Wissenschaft":

"We want to carry the rigidity and harmony of mathematics into all sciences, as much as possible;

Not in the belief that we understand things that way, but in order to recognize our human relation to the things.

mathematics is the "ultimatima ratio" of the knowledge of human nature" ‖.

We must come back to Universities where outstanding achievements in research are recognized as what they are: results achieved by outstanding personalities. The University Professors are the elite of each country; the future prosperity on the country depends on them. On their education·of the youth is built the new generation of leading administrators, managers, engineers, medical doctors etc.

No country – not even Germany – can keep an outstanding position compared to the other countries if it does not promote demanding research and pretentious teaching.

We, the Professors, have to fight for this, since we cannot get any help from the politicians – they only think from one election to the next; we cannot get help from managers in industry – they think along very selfish lines.

So we are left alone, but united we can be strong and can use our influence to make the Universities better – in the sense of Humboldt.

References

[Archimedes] Archimedes - T. L. Heath, New York, Dover Publications, Inc., 1912

[Euclid] Euklid - T. L. Heath, New York, Dover Publications, Inc., 1956

[Goethe] Goethe, J. W. Erzhlende Dichtung, Deutsche Buchgemeinschaft, Berlin - Darmstadt, 1953

[] Gemoll, Griechisch-Deutsches Schul- und Handwrterbuch, G. Freytag Verlag, Mnchen, 1954

[Golomb] Golomb, S. Mathematics After Forty Years of the Space Age, The Mathematical Intelligencer Vol. 21, No. 4 (1999), 38–44

‖ "Wir wollen die Feinheit und Strenge der Mathematik in alle Wissenschaften hineintreiben, soweit dies nur irgend mglich ist; nicht im Glauben, da wir auf diesem Wege die Dinge erkennen werden, sondern um damit unsere menschliche Relation zu den Dingen festzustellen. Die Mathematik ist nur das Mittel der allgemeinen letzten Menschenkenntnis."

[Grotemeyer] Grotemeyer, K. P. Einfhrung zur DMV-Tagung in Bielefeld, 1991

[Hardy] Hardy, A Mathematician's Apology, with a Foreword by C. P. Snow, Canto edition 1992, Cambridge University Press

[Heinichen] Heinichen, Lateinisch-Deutscher Schulwrterbuch, 8. Aufl. B. G. Teubner, Leipzig, 1909

[Juschkewitsch] Juschkewitsch, A. P. Geschichte der Mathematik in Mittelalter, B. G. Teubner Verlagsgesellschaft, Leipzig, 1964

[Kline] Kline, M. Mathematical Thought from Ancient to Modern Times, Oxford Univ. Press, New York, 1972

[Nietzsche] Nietzsche, F. Frhliche Wissenschaft, Alfred Krner Verlag, Leipzig, 1930

[Platon] Platon - Smtliche Werke, Bd. 2, Verlag Lampert Schneider, Heidelberg, 1958

[Roggenkamp I] Geisteswissenschaftliche Aspekte der Mathematik, Lecture at the session of the Academy of Sciences in Erfurt, appeared in Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Klasse 6 (1994/97), 93–114

[Roggenkamp II] Roggenkamp, K. W. Promoting research – a challenge for Professors, Lecture at "2nd International Conference on Romania and Romanians in contemporary science" in Braşov, Romania in May 1997, and appeared in a shortened version in Newletter of the European Mathematical Society, Issue 28, June 1998, p. 26–31

[Zedler] Zedlers 'Groes vollstndiges Universallexikon', Bd. 19, 2047, Halle/Leipzig, 1739, Nachdruck Graz/Wien, 1961

[Wolffen] Mathematischen Lexikon, Christian Wolffen, Leipzig 1716, Nachdruck Hildesheim, 1965

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Note of the local editor: The ideas and opinions in this paper belong totally to the author.

DERIVED ORDERS AND AUSLANDER-REITEN QUIVERS

Wolfgang Rump

Let R be a complete discrete valuation domain with quotient field K, and let Λ be an R-order (see §1) in a finite dimensional K-algebra A. For a hereditary embedding $v : P_0 \hookrightarrow I_0$ of a projective Λ-lattice P_0 into an injective Λ-lattice I_0 (see [10, 11] or §1 below), a derived order $\delta_v \Lambda$ in $M_2(A)$ can be defined. There is a functor $\partial_v : \Lambda\text{-}\mathbf{lat} \longrightarrow \delta_v \Lambda\text{-}\mathbf{lat}$ which yields an equivalence of quotient categories

$$\tilde{\partial}_v : \Lambda\text{-}\mathbf{lat}/[v^{-1}] \xrightarrow{\sim} \delta_v \Lambda\text{-}\mathbf{lat}/[B],$$

where the ideal $[v^{-1}]$ is generated by finitely many indecomposables, and B is a bijective $\delta_v \Lambda$-lattice. We show in §1 that any such functor ∂_v is a composition of functors ∂_u given by hereditary embeddings $u : P \hookrightarrow I$ such that I/P is uniserial. Then we have a surjective map

$$p : \text{ind}(\Lambda\text{-}\mathbf{lat}) \twoheadrightarrow \text{ind}(\delta_u \Lambda\text{-}\mathbf{lat}) \tag{0}$$

between the isomorphism classes of indecomposable Λ- and $\delta_u \Lambda$-lattices, featuring a "discrete blowing down": Unless $P = I$, there is a single exceptional fiber $p^{-1}(B)$ of cardinality > 1, namely, the finite chain of Λ-lattices between P and I. In [10] we have shown that Zavadskij's differentiation algorithms for partially ordered sets [15] and for tiled orders [16], as well as Simson's generalization to vector space categories [12, 13] can be obtained as special cases of (0).

In this paper we analyse the relationship between the Auslander-Reiten quivers $\mathfrak{A}_\Lambda$ and $\mathfrak{A}_{\delta_u \Lambda}$, provided that the K-algebra $A = K\Lambda$ is semisimple.

(For representations of posets, cf. [14].) To this end, we make use of Iyama's concept of a τ-category [6]. Such categories possess a generalized τ-quiver, with meshes formed by generalized almost split sequences $C' \xrightarrow{u} C \xrightarrow{v} C''$ which we call τ-*sequences*. Here u is only a weak kernel of v, and v is a weak cokernel of u, that is, the condition that u is monic and v epic is dropped. Nevertheless, for given C' or C'', the whole sequence is unique up to isomorphism. When we pass to a quotient category $\overline{C} := \Lambda\text{-}\mathbf{lat}/[\mathcal{N}]$ with respect to an object class $\mathcal{N}$, every almost split sequence $C' \rightarrowtail C \twoheadrightarrow C''$ with $C, C', C'' \neq 0$ in $\overline{C}$ survives as a τ-sequence (not necessarily as an almost split sequence) in $\overline{C}$. If $C' = 0$ in $\overline{C}$, the object $C'' \in \overline{C}$ is in a sense "projective" (Corollary of Proposition 8), and there is a *right* τ-sequence $0 \to C \to C''$ in $\overline{C}$. If the middle term C vanishes in $\overline{C}$, the right τ-sequence in $\overline{C}$ is always of the form $0 \to 0 \to C''$, although C' might be non-zero in $\overline{C}$. For our special quotient categories occuring in $\tilde{\partial}_u$, however, there are at most two indecomposable Λ-lattices C' of that type which can be determined explicitly. Therefore, using the structure of bijective lattices over orders [9], we obtain a rather simple procedure for the determination of $\mathfrak{A}_{\delta_u\Lambda}$. The possible cases are given in Theorem 2.

1   Derived orders

Throughout the following, let R be a complete discrete valuation domain with quotient field K, and let A be a finite dimensional K-algebra. An R-subalgebra Λ of A is said to be an R-*order* in A if $_R\Lambda$ is finitely generated and $K\Lambda = A$. Thus Λ is a free R-module. Conversely, every R-algebra Λ with $_R\Lambda$ finitely generated and free is an R-order in $K \otimes_R \Lambda$. A finitely generated R-free (left) Λ-module E is said to be a (left) Λ-*lattice*. The (additive) category of left (resp. right) Λ-lattices will be denoted by Λ-$\mathbf{lat}$ (resp. $\mathbf{lat}$-Λ). There is a duality

$$\Lambda\text{-}\mathbf{lat} \xrightarrow{\sim} \mathbf{lat}\text{-}\Lambda \tag{1}$$

which associates to any left Λ-lattice E the right Λ-lattice $E^* := \mathrm{Hom}_R(E, R)$. Therefore, $E \in \Lambda$-$\mathbf{lat}$ is said to be *injective* if E^* is projective. A projective and injective Λ-lattice is called *bijective*.

A morphism in an arbitrary category is said to be called *regular* if it is monic and epic. Thus a morphism

$$u : P \to I \tag{2}$$

in Λ-**lat** is regular if and only if u is injective with R-torsion cokernel. To u we associate the class $\mathcal{Z}_u$ of Λ-lattices H such that u admits a factorization

$$u : P \xrightarrow{u'} H \xrightarrow{u''} I \tag{3}$$

with regular u' and u''. If we regard u as an embedding, then the objects of $\mathcal{Z}_u$ are isomorphic to the Λ-lattices between P and I.

If $\mathcal{R}_\Lambda$ denotes the class of regular morphisms in Λ-**lat**, then Λ-**lat** admits a calculus of left and right fractions [5] with respect to $\mathcal{R}_\Lambda$. The localization $(\Lambda$-**lat**$)[\mathcal{R}_\Lambda^{-1}]$ is equivalent to the category A-**mod** of finitely generated left A-modules. In fact, every $E \in \Lambda$-**lat** admits a natural embedding $E \hookrightarrow KE :=$ $K \otimes_R E \in A$-**mod**, and the morphisms $E \to F$ in $(\Lambda$-**lat**$)[\mathcal{R}_\Lambda^{-1}]$ can be regarded as A-linear maps $KE \to KF$. We always assume this identification in what follows. In particular,

$$\mathrm{Hom}_\Lambda(E, F) = \{f \in \mathrm{Hom}_A(KE, KF) \mid f(E) \subset F\}. \tag{4}$$

For a class $\mathcal{M}$ of morphisms in Λ-**lat**, let $[\mathcal{M}]$ denote the ideal generated by $\mathcal{M}$. Regarding objects as identity morphisms, this also applies to object classes $\mathcal{C}$. Thus $[\mathcal{C}]$ consists of the morphisms which factor through a finite direct sum of objects in $\mathcal{C}$. For a regular morphism u in Λ-**lat**, we denote by $[u^{-1}]$ the ideal in Λ-**lat** generated by the morphisms which factor through u^{-1} (if regarded as morphisms in $(\Lambda$-**lat**$)[\mathcal{R}_\Lambda^{-1}]$). Then we have

$$[\mathcal{Z}_u] = [u^{-1}]. \tag{5}$$

We call a regular morphism (2) in Λ-**lat** *hereditary* [10] if it satisfies

(H1) P is projective; I is injective.
(H2) $\mathrm{Hom}_\Lambda(P, \mathrm{Cok}\, u) = \mathrm{Ext}_\Lambda(\mathrm{Cok}\, u, I) = 0$.
(H3) $\mathrm{Ext}_\Lambda(H, L) = 0$ for $H, L \in \mathcal{Z}_u$.

For regular $u : P \hookrightarrow I$ and $E \in \Lambda$-**lat** we define

$$\begin{aligned} E_- &:= E \cap \bigcap \{f^{-1}(P) \mid f \in \mathrm{Hom}_\Lambda(E, I)\} \\ E^+ &:= E + \sum \{f(I) \mid f \in \mathrm{Hom}_\Lambda(P, E)\}. \end{aligned} \tag{6}$$

Thus

$$E_- \subset E \subset E^+. \tag{7}$$

Dually, the regular morphism $u^* : I^* \hookrightarrow P^*$ defines a sublattice and an overlattice for any right Λ-lattice F:

$$F_+ \subset F \subset E^-. \tag{8}$$

The relationship between (7) and (8) can also be expressed by

$$(E^+)^* = (E^*)_+; \quad (E_-)^* = (E^*)^-. \tag{9}$$

Proposition 1.. *If $u : P \hookrightarrow I$ is a regular morphism in Λ-lat satisfying (H1), then (H2) is equivalent to*

$$P_- = P; \quad I^+ = I. \tag{10}$$

Proof. The short exact sequence $P \hookrightarrow I \twoheadrightarrow I/P$ induces short exact sequences

$$\operatorname{Hom}_\Lambda(P, P) \rightarrowtail \operatorname{Hom}_\Lambda(P, I) \twoheadrightarrow \operatorname{Hom}_\Lambda(P, I/P)$$
$$\operatorname{Hom}_\Lambda(I, I) \rightarrowtail \operatorname{Hom}_\Lambda(P, I) \twoheadrightarrow \operatorname{Ext}_\Lambda(I/P, I).$$

Now the assertion follows immediately. $\square$

The closure condition (10) implies

$$P^+ = I; \quad I_- = P. \tag{11}$$

In fact, $P^+ \subset I^+ = I$, and thus $P^+ = I$ since the identity $1 : P \to P$ maps I to I. By duality, this also gives $I_- = P$. In other words, (11) says that there is a natural ring isomorphism

$$\operatorname{End}_\Lambda(P) = \operatorname{End}_\Lambda(I). \tag{12}$$

By [10], Proposition 6 and Proposition 8, we have

Proposition 2.. *Let $u : P \hookrightarrow I$ be a hereditary morphism in Λ-lat. Then Λ^+ and Λ^- are overorders of Λ, and for every $E \in \Lambda$-lat:*

$$E^+ = \Lambda^+ E; \quad E_- = \operatorname{Hom}_\Lambda(\Lambda^-, E); \quad \Lambda_- E^+ \subset E_-. \tag{13}$$

Here $\operatorname{Hom}_\Lambda(\Lambda^-, E)$ has to be regarded as a Λ-lattice in $\operatorname{Hom}_A(A, KE) = KE$. Thus E_- is the largest Λ^--submodule of E. Proposition 2 implies that a hereditary morphism $u : P \hookrightarrow I$ allows to associate to Λ the *derived order*

$$\delta_u \Lambda := \begin{pmatrix} \Lambda^+ & \Lambda^+ \Lambda^- \\ \Lambda_- & \Lambda \end{pmatrix} \subset \mathrm{M}_2(A). \tag{14}$$

Moreover, we have an additive functor

$$\partial_u : \Lambda\text{-lat} \longrightarrow \delta_u \Lambda\text{-lat} \tag{15}$$

given by $\partial_u E := \begin{pmatrix} E^+ \\ E_- \end{pmatrix}$. From (10) and (11) we infer

$$\partial_u(\mathcal{Z}_u) = \begin{pmatrix} I \\ P \end{pmatrix}. \tag{16}$$

The main theorem of [10] is as follows.

Theorem 1.. *Let Λ be an R-order in a finite dimensional K-algebra A, and let $u : P \hookrightarrow I$ be a hereditary morphism in Λ-lat. Then the functor (15) induces an equivalence of quotient categories:*

$$\tilde{\partial}_u : \Lambda\text{-lat}/[\mathcal{Z}_u] \xrightarrow{\sim} \delta_u\Lambda\text{-lat}/[\begin{pmatrix} I \\ P \end{pmatrix}]. \tag{17}$$

In order to apply this theorem, we have to know more about the $\delta_u\Lambda$-lattice $\begin{pmatrix} I \\ P \end{pmatrix}$ and the object class $\mathcal{Z}_u$. Firstly, we have

Proposition 3.. *For a hereditary morphism $u : P \hookrightarrow I$ in Λ-lat, the $\delta_u\Lambda$-lattice $\begin{pmatrix} I \\ P \end{pmatrix}$ is bijective.*

Proof. By duality, it suffices to show that $\begin{pmatrix} I \\ P \end{pmatrix}$ is projective. As the functor (15) is additive and $\begin{pmatrix} I \\ P \end{pmatrix} = \partial_u P$, we only have to prove that ∂_u carries indecompos projective Λ-lattices to projective $\delta_u\Lambda$-lattices. In fact, (14) shows that $\partial_u({}_\Lambda\Lambda)$ is a direct summand of $\delta_u\Lambda$. $\square$

Notice that for any R-order Γ, bijective Γ-lattices B can be *rejected*, i. e. there is an R-order Γ' and a ring homomorphism $\Gamma \to \Gamma'$ with R-torsion cokernel such that every indecomposable Γ-lattice is either a Γ'-lattice or a direct summand of B (see [4], 2.9; [10], Proposition 7).

Now let us turn our attention to $\mathcal{Z}_u$. By (12) there is a decomposition

$$\begin{pmatrix} I \\ P \end{pmatrix} = \begin{pmatrix} I_1 \\ P_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} I_s \\ P_s \end{pmatrix} \tag{18}$$

with I_i, P_i indecomposable for all i. There may be *trivial* direct summands $\begin{pmatrix} I_i \\ P_i \end{pmatrix}$ with $P_i = I_i$. They have no effect to the functor (15). On the other hand, if $\begin{pmatrix} I_i \\ P_i \end{pmatrix}$ is a non-trivial direct summand of (18), then I_i/P_i is indecomposable by [10], Proposition 9, and uniserial by [10], Lemma 3 and Proposition 10. Obviously, the inclusions $u_i : P_i \hookrightarrow I_i$ are again hereditary, and (18) amounts to a decomposition

$$u = u_1 \oplus \cdots \oplus u_s. \tag{19}$$

Proposition 4.. *Let $u : P \hookrightarrow I$ be a hereditary morphism in Λ-lat. Consider a decomposition (19). Then every $H \in \mathcal{Z}_u$ is isomorphic to some $H_1 \oplus \cdots \oplus H_s$ with $H_i \in \mathcal{Z}_{u_i}$ for $i \in \{1, \ldots, s\}$.*

Proof. We may assume that H has no direct summand in common with P. By [10], Corollary of Theorem 1, there is some $H' := H_1 \oplus \cdots \oplus H_s$ with $H_i \in \mathcal{Z}_{u_i}$ such that $H/P \cong H'/P$. By (H3) this isomorphism lifts to homomorphisms $f : H \to H'$ and $g : H' \to H$ with $f, g \in \text{End}_\Lambda(P)$. Thus $1 - gf$ factors through P, whence f is a split monomorphism. $\square$

For an object class $\mathcal{C}$ in an additive category $\mathcal{A}$, let $\text{add}\,\mathcal{C}$ be the full subcategory of $\mathcal{A}$ consisting of the direct summands of finite direct sums of objects in $\mathcal{C}$. By $\text{Ind}\,\mathcal{A}$ we denote the class of indecomposable objects in $\mathcal{A}$, whereas $\text{ind}\,\mathcal{A}$ denotes a system of representatives of the isomorphism classes in $\text{Ind}\,\mathcal{A}$. With

$$\mathcal{H}_u := \text{add}\,\mathcal{Z}_u \tag{20}$$

we clearly have

$$[\mathcal{Z}_u] = [\mathcal{H}_u] = [\text{ind}\,\mathcal{H}_u]. \tag{21}$$

Corollary 1. *Let $u : P \hookrightarrow I$ be a hereditary morphism in Λ-**lat** with a decomposition (19). Then*

$$\text{Ind}\,\mathcal{H}_u = \mathcal{Z}_{u_1} \cup \cdots \cup \mathcal{Z}_{u_s}. \tag{22}$$

In particular, $\text{ind}\,\mathcal{H}_u$ *is finite.*

Proof. The finiteness of $\text{ind}\,\mathcal{H}_u$ follows since for each $u_i : P_i \hookrightarrow I_i$, the Λ-module I_i/P_i is uniserial by [10], Lemma 3 and Proposition 10. $\square$

Corollary 2. *Every hereditary morphism $u : P \hookrightarrow I$ in Λ-**lat** gives rise to a surjective map*

$$p : \text{ind}(\Lambda\text{-}\mathbf{lat}) \twoheadrightarrow \text{ind}(\delta_u\Lambda\text{-}\mathbf{lat}) \tag{23}$$

with finite fibers $p^{-1}(E)$ having just one element if E is not a direct summand of $\binom{I}{P}$.

Proof. This follows by Theorem 1 and (16). $\square$

Remark. In other words, Theorem 1 yields an "almost bijection" (23) between the indecomposables of Λ-**lat** and $\delta_u\Lambda$-**lat**, respectively. There are finitely many exceptional fibers $p^{-1}\binom{I'}{P'}$ corresponding to the indecomposable direct summands $\binom{I'}{P'}$ of $\binom{I}{P}$, and up to isomorphism, the Λ-lattices in $p^{-1}\binom{I'}{P'}$ form a chain $P' = H_0 \subset H_1 \subset \cdots \subset H_l = I'$.

In the following proposition, $l(\ \ )$ denotes the length of a Λ-module.

Proposition 5.. *Every hereditary morphism u in Λ-**lat** has a decomposition*

$$u \cong u_1^{m_1} \oplus \cdots \oplus u_r^{m_r} \tag{24}$$

with $u_i : P_i \hookrightarrow I_i$ hereditary such that the I_i/P_i are uniserial and the composition factors of $I_1 \oplus \cdots \oplus I_r / P_1 \oplus \cdots \oplus P_r$ are pairwise non-isomorphic. Then

$$|\mathrm{ind}\,\mathcal{H}_u| = r + \sum_{i=1}^r l(I_i/P_i). \qquad (25)$$

Moreover, $u_1' := \partial_{u_2 \oplus \cdots \oplus u_r}(u)$ is hereditary in $\delta_{u_2 \oplus \cdots \oplus u_r} \Lambda\text{-lat}$, and

$$\partial_u = \partial_{u_1 \oplus \cdots \oplus u_r} = \partial_{u_1'} \partial_{u_2 \oplus \cdots \oplus u_r}. \qquad (26)$$

Proof. The existence of a decomposition (24) with the mentioned properties follows by [10], Theorem 1 and Proposition 10. Formula (25) says that the Λ-lattices between P_i and I_i are pairwise non-isomorphic. Thus assume that $P_i \subset L \subset H \subset I_i$ with $H \cong L$ and $H \neq L$. An isomorphism $\alpha : H \xrightarrow{\sim} L$ extends to an endomorphism of $H^+ = L^+ = I_i$ with $\alpha(I_i) \neq I_i$. By [10], Proposition 9, this implies $\alpha(I_i) \subset P_i$, whence $P_i = L \cong H = I_i$, in contrast to (H2).

In order to prove (26), define

$$\begin{pmatrix} E^\sharp \\ E_\flat \end{pmatrix} := \partial_{u_2 \oplus \cdots \oplus u_r}(E); \quad \begin{pmatrix} E^\oplus \\ E_\ominus \end{pmatrix} := \partial_{u_1}(E)$$

for any $E \in \Lambda\text{-lat}$. Then

$$E^+ = E^\sharp + E^\oplus; \quad E_- = E_\flat \cap E_\ominus.$$

Furthermore, we may assume that $m_1 = \cdots = m_r = 1$. Then

$$u_1' : \begin{pmatrix} P_1 \oplus I_2 \oplus \cdots \oplus I_r \\ P_1 \oplus P_2 \oplus \cdots \oplus P_r \end{pmatrix} \hookrightarrow \begin{pmatrix} I_1 \oplus I_2 \oplus \cdots \oplus I_r \\ I_1 \oplus P_2 \oplus \cdots \oplus P_r \end{pmatrix}.$$

This implies (26). It remains to show that u_1' is hereditary in $\delta_{u_2 \oplus \cdots \oplus u_r} \Lambda\text{-lat}$. Firstly, u_1' satisfies (H1) since $\partial_{u_2 \oplus \cdots \oplus u_r}$ respects projectives and injectives. Moreover, (10) holds for u_1', whence u_1' satisfies (H2). Finally, [11], Theorem 2 and (26), yields (H3) for u_1'. $\square$

Proposition 5 shows that the equivalences (17) given by Theorem 1 can be decomposed into equivalences $\widetilde{\partial}_u$ with indecomposable u.

Example. Examples of hereditary morphisms in $\Lambda\text{-lat}$ for representation-finite R-orders Λ with $A = K\Lambda$ semisimple are given in [10] and [11]. In the triangular matrix algebra $A = \mathrm{T}_3(K)$, consider the R-order

$$\Lambda = \begin{pmatrix} R & 0 & 0 \\ \mathfrak{p} & R & 0 \\ \mathfrak{p} & \mathfrak{p} & R \end{pmatrix},$$

where $\mathfrak{p} := \operatorname{Rad} R$. If R is a power series ring $k[[t]]$ over a field k, then Λ is wild in the sense of Drozd and Greuel [3]. By [10], Proposition 14, the morphism

$$u : \begin{pmatrix} R \\ \mathfrak{p} \\ \mathfrak{p} \end{pmatrix} \hookrightarrow \begin{pmatrix} R \\ R \\ \mathfrak{p} \end{pmatrix}$$

in Λ-**lat** is hereditary. The derived order $\delta_u \Lambda$ is Morita equivalent to the R-order

$$\delta_u' \Lambda := \begin{pmatrix} R & 0 & 0 & 0 \\ R & R & R & 0 \\ \mathfrak{p} & \mathfrak{p} & R & 0 \\ \mathfrak{p} & \mathfrak{p} & R & R \end{pmatrix}.$$

Hence there is a surjection $p : \operatorname{ind}(\Lambda\text{-}\mathbf{lat}) \twoheadrightarrow \operatorname{ind}(\delta_u' \Lambda\text{-}\mathbf{lat})$ which is one-to-one up to the exceptional fiber $p^{-1} \begin{pmatrix} R \\ R \\ \mathfrak{p} \\ \mathfrak{p} \end{pmatrix} = \left\{ \begin{pmatrix} R \\ \mathfrak{p} \\ \mathfrak{p} \end{pmatrix}, \begin{pmatrix} R \\ R \\ \mathfrak{p} \end{pmatrix} \right\}.$

2 τ-categories

It is well-known [1] that an R-order Λ in A has almost split sequences if and only if A is semisimple. In this case, if u is a hereditary morphism in Λ-**lat**, there is a close relationship between the Auslander-Reiten quivers of Λ and $\delta_u \Lambda$. We shall prove this by means of Iyama's concept of a τ-category [6].

An additive category $\mathcal{C}$ is said to be a *Krull-Schmidt category* ([6]; cf. [8]) if every object in $\mathcal{C}$ is a finite direct sum of objects C with the property that for any $\alpha \in \operatorname{End}(C)$, either α or $1 - \alpha$ is invertible. In other words, $\operatorname{End}(C)$ is a (possibly large) local ring. (Iyama [6] assumes that $\mathcal{C}$ is skeletally small.) Consequently, the Krull-Schmidt theorem holds for decompositions of objects in $\mathcal{C}$. For example, a ring S is semiperfect if and only if the category S-**proj** of finitely generated projective left S-modules is a Krull-Schmidt category. Therefore, a Krull-Schmidt category with finitely many isomorphism classes of indecomposable objects is tantamount to a semiperfect ring. This implies, in particular, that idempotents split in a Krull-Schmidt category.

Let $\mathcal{C}$ be a Krull-Schmidt category. The ideal $\operatorname{Rad} \mathcal{C}$ generated by the non-invertible morphisms between indecomposable objects of $\mathcal{C}$ is called the *radical* of $\mathcal{C}$. A morphism $f : A \to B$ in $\mathcal{C}$ is said to be *right almost split for B* [2] if $f \in \operatorname{Rad} \mathcal{C}$, and every $h : C \to B$ in $\operatorname{Rad} \mathcal{C}$ factors through f. We call f *left*

minimal if every morphism $g : C \to A$ with $fg = 0$ lies in Rad $\mathcal{C}$. A morphism $f : A \to B$ is said to be a *weak kernel* of $g : B \to D$ if $gf = 0$, and every $h : C \to B$ with $gh = 0$ factors through f.

Proposition 6.. *Let $g : C \to A$ be a weak kernel of $f : A \to B$ in a Krull-Schmidt category $\mathcal{C}$. Then g decomposes into $g = \begin{pmatrix} 0 & g_1 \end{pmatrix} : C_0 \oplus C_1 \to A$ such that g_1 is a left minimal weak kernel of f.*

Proof. If g is not left minimal, then there exists an indecomposable object E and a morphism $e : E \to C$ such that $ge = 0$ and $e \notin$ Rad $\mathcal{C}$. Therefore, e has a retraction $r : C \to E$ with $re = 1$. Since idempotents split in $\mathcal{C}$, we get a decomposition $g = \begin{pmatrix} 0 & g' \end{pmatrix} : C = E \oplus C' \to A$, and g' is a weak kernel of f. Now the same argument applies to C', and the requested decomposition $g = \begin{pmatrix} 0 & g_1 \end{pmatrix}$ is obtained by induction. $\qquad\qquad\square$

Proposition 7.. *Let $\mathcal{C}$ be a Krull-Schmidt category. A left minimal weak kernel (of a given morphism), and a left minimal right almost split morphism (for a given object) is unique up to isomorphism.*

Proof. Let $f : A \to B$ and $f' : A' \to B$ be left minimal weak kernels of $g : B \to D$. Then there are morphisms $a : A \to A'$ and $a' : A' \to A$ with $f' = fa'$ and $f = f'a$. Hence $f(1 - a'a) = 0$, and thus $1 - a'a \in$ Rad $\mathcal{C}$. This shows that a is a split monomorphism. By symmetry, $1 - aa' \in$ Rad $\mathcal{C}$, whence a is a split epimorphism. A similar argument applies to right almost split morphisms f. $\qquad\qquad\square$

If g has a weak kernel, we write wker g for the left minimal weak kernel (which exists and is essentially unique by the above propositions). The concept of *left almost split* morphism, *right minimal* morphism, and *weak cokernel* are defined dually. The right minimal weak cokernel of a morphism f will be denoted by wcok f. A morphism of the form wcok f (resp. wker f) will be called a *weak (co-)kernel*.

For an object $C \in \mathcal{C}$, a sequence

$$\tau C \xrightarrow{u} \vartheta C \xrightarrow{v} C \qquad\qquad (27)$$

will be called a *right τ-sequence* if u is left and v right almost split, and $u =$ wker v. Here $u \in$ Rad $\mathcal{C}$ implies that v is left minimal. Therefore, Proposition 7 implies that up to isomorphism, a right τ-sequence is uniquely determined by C. Dually, a sequence

$$C \xrightarrow{u'} \vartheta^- C \xrightarrow{v'} \tau^- C \qquad\qquad (28)$$

with u' left and v' right almost split and $v' = \text{wcok}\, u'$ will be called a *left τ-sequence*. If a right τ-sequence (27) is also a left τ-sequence, we simply speak of a *τ-sequence*. In particular, every almost split sequence (27) is a τ-sequence. Namely, a τ-sequence (27) with C indecomposable is an almost split sequence if and only if it is *exact*, i. e. $u = \ker v$ and $v = \text{cok}\, u$. A Krull-Schmidt category $\mathcal{C}$ such that every object C has a right and left τ-sequence is said to be a *τ-category* [6]. For instance, every R-order Λ in a semisimple algebra gives rise to a τ-category Λ-**lat**. The fundamental property of τ-categories is given by

Proposition 8 ([6], 1.3).. *Let $\mathcal{C}$ be a τ-category. For any indecomposable object C with $\tau C \neq 0$ (resp. $\tau^{-} C \neq 0$) the right (left) τ-sequence starting with C is a τ-sequence.*

Like in the case of almost split sequences, the objects C with $\tau C = 0$ are in a sense "projective":

Corollary. *An indecomposable object P in a τ-category $\mathcal{C}$ satisfies $\tau P = 0$ if and only if every weak cokernel $C \to P$ is a split epimorphism.*

Proof. Let P be indecomposable with $\tau P = 0$, and let $D \xrightarrow{d} C \xrightarrow{c} P$ be a sequence with $c = \text{wcok}\, d$. Suppose that c is not a split epimorphism. Then $c \in \text{Rad}\,\mathcal{C}$, and we get a commutative diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & \vartheta P & \xrightarrow{\ v\ } & P \\
\uparrow & & \uparrow{\scriptstyle e} & & \Big\| \\
D & \xrightarrow{\ d\ } & C & \xrightarrow{\ c\ } & P
\end{array}
$$

with $ed = 0$. Hence $e = pc$ for some $p : P \to \vartheta P$. This implies $(1 - vp)c = 0$, and thus $1 - vp \in \text{Rad}\,\mathcal{C}$. Since $v \in \text{Rad}\,\mathcal{C}$, we get $P = 0$, a contradiction. Conversely, assume that $\tau P \neq 0$. Then $\tau P \to \vartheta P \to P$ is a τ-sequence, whence $\vartheta P \to P$ is a weak cokernel that is not a split epimorphism. $\square$

In contrast to almost split sequences, τ-sequences behave well with respect to quotient categories. Let us call an object S of a Krull-Schmidt category $\mathcal{C}$ a *source object* if every non-zero morphism $C \to S$ is split epic. Equivalently, S is an indecomposable object having a right τ-sequence with $\vartheta S = 0$. Dually, we call S a *sink object* if every non-zero morphism $S \to C$ is split monic. The following result is due to Iyama ([7], 2.1). For the sake of completeness, we include a proof.

Proposition 9.. *Let $\mathcal{C}$ be a τ-category with a full subcategory $\mathcal{N}$. Let (27) be a right τ-sequence in $\mathcal{C}$ with C indecomposable such that C is not a source object in $\mathcal{C}/[\mathcal{N}]$. Then (27) induces a right τ-sequence in $\mathcal{C}/[\mathcal{N}]$.*

Proof. Clearly, $\mathcal{C}/[\mathcal{N}]$ is again a Krull-Schmidt category. Let $\overline{f}$ denote the residue class modulo $\mathcal{N}$ of a morphism f in $\mathcal{C}$. Obviously, the functor $\mathcal{C} \to \mathcal{C}/[\mathcal{N}]$ respects right and left almost split morphisms. Now let $b : A \to \vartheta C$ be given with $\overline{v}\overline{b} = 0$. Then we have a commutative diagram

$$
\begin{array}{ccccc}
\tau C & \xrightarrow{\;u\;} & \vartheta C & \xrightarrow{\;v\;} & C \\
& & \Big\uparrow{\scriptstyle b} & & \Big\uparrow{\scriptstyle c} \\
& & A & \xrightarrow{\;a\;} & N
\end{array}
$$

with $N \in \operatorname{add} \mathcal{N}$. Hence $c \in \operatorname{Rad} \mathcal{C}$, and thus c factors through v, say, $c = vc'$. Therefore, $v(b - c'a) = 0$. Consequently, $b - c'a$ factors through u, whence $\overline{b}$ factors through $\overline{u}$. It remains to prove that $\overline{u}$ is left minimal. Assume that there is a morphism $e : B \to \tau C$ with B indecomposable such that $\overline{u}\overline{e} = 0$ and $\overline{e} \notin \operatorname{Rad}(\mathcal{C}/[\mathcal{N}])$. Then Proposition 8 implies that e is an isomorphism and $\tau C \neq 0$ in $\mathcal{C}/[\mathcal{N}]$. Hence $\overline{u} = 0$, and $\overline{v}$ is a weak cokernel of $\overline{u}$. But this shows that $\vartheta C = 0$ in $\mathcal{C}/[\mathcal{N}]$, i. e. C is a source object in $\mathcal{C}/[\mathcal{N}]$. $\qquad\square$

3   The Auslander-Reiten quiver of a derived order

Now we apply the results of §2 to an R-order Λ and its derived order $\delta_u \Lambda$ with respect to a hereditary morphism $u : P \hookrightarrow I$. By Proposition 5 we may assume w.l.o.g. that u is *indecomposable*, i. e. P (hence I) is indecomposable. For a Λ-lattice E we define the *upper radical* of E as the largest overlattice $F \supset E$ in KE with F/E semisimple:

$$\operatorname{Rad}^{\circ} E := \{x \in KE \mid (\operatorname{Rad} \Lambda)x \subset E\}. \tag{29}$$

For a hereditary morphism $u : P \hookrightarrow I$, consider the following Λ-lattices in $KP = KI$:

$$P' := \Lambda_-(\operatorname{Rad}^{\circ} I); \quad I' = \operatorname{Hom}_\Lambda(\Lambda_-, \operatorname{Rad} P). \tag{30}$$

Proposition 10.. *Let $u : P \hookrightarrow I$ be a hereditary morphism in Λ-$\mathbf{lat}$. Then*

$$\operatorname{Rad}\binom{I}{P} = \binom{I'}{(\operatorname{Rad} P)_-} = \partial_u I'; \quad \operatorname{Rad}^{\circ}\binom{I}{P} = \binom{(\operatorname{Rad}^{\circ} I)^+}{P'} = \partial_u P'. \tag{31}$$

Proof. By duality, it suffices to prove the first pair of equations. Assume w.l.o.g. that u is indecomposable. By [10], Corollary of Proposition 10, the composition factors of the left Λ-module Λ/Λ_- are subquotients of I/P. Hence (H2) implies

$$\Lambda_- I = P; \quad \mathrm{Hom}_\Lambda(\Lambda_-, P) = I. \tag{32}$$

In particular, the second equation gives $I' \subset I$. If $\mathrm{Rad}\binom{I}{P} = \binom{F}{G}$, then $G \subset \mathrm{Rad}\,P$, hence $G \subset (\mathrm{Rad}\,P)_-$. Since $\Lambda_- = \Lambda_+$ is a right Λ^+-lattice, we infer $\mathrm{Rad}\binom{I}{P} = \binom{I'}{(\mathrm{Rad}\,P)_-}$. Thus it remains to show that $(I')_- = (\mathrm{Rad}\,P)_-$. Suppose that $I' \supset P$. Then $I = P^+ \subset I' \subset I$. Hence $\Lambda_- I' = P \not\subset \mathrm{Rad}\,P$, a contradiction. So we have

$$P \not\subset I'; \quad P' \not\subset I. \tag{33}$$

Therefore, $(I')_- \subset I_- = P$, and $(I')_- \neq P$ by (33). Consequently, $(\mathrm{Rad}\,P)_- \subset (I')_- \subset \mathrm{Rad}\,P$, whence $(I')_- = (\mathrm{Rad}\,P)_-$. $\qquad\square$

By Theorem 1 and Proposition 9, the right τ-sequences in Λ-**lat** and $\delta_u\Lambda$-**lat** are related via the right τ-sequences in $\overline{\mathcal{C}} := \delta_u\Lambda\text{-}\mathbf{lat}/[\binom{I}{P}]$, unless they induce the τ-sequence of a source object in $\overline{\mathcal{C}}$. This gives an almost one-to-one correspondence between the almost split sequences in Λ-**lat** and $\delta_u\Lambda$-**lat**. To get the precise relationship we have to locate the bijective object $\binom{I}{P}$ in $\delta_u\Lambda$-**lat**. The classification of bijectives given in [9], Proposition 2, easily generalizes to orders in non-semisimple algebras:

Proposition 11.. *Let Γ be an R-order in a finite dimensional K-algebra, and let B be an indecomposable bijective Γ-lattice. The following (self-dual) cases are possible:*
  (I) $\mathrm{Rad}\,B$ *is injective. Then* $\mathrm{Rad}\,B$ *is indecomposable, and* $\mathrm{Rad}^\circ B$ *is indecomposa‐ projective. Moreover,* KB *is a simple* $K\Gamma$-*module.*
    (a) $\mathrm{Rad}\,B \cong B$. *Then* $\Gamma = \Gamma_0 \times \Gamma_1$ *with a maximal order* Γ_0 *in a simple K-algebra such that* $B \in \Gamma_0$-**lat**.
    (b) $\mathrm{Rad}\,B \not\cong B$. *Then there are irreducible morphisms* $\mathrm{Rad}B \hookrightarrow B \hookrightarrow \mathrm{Rad}^\circ B$
  (II) $\mathrm{Rad}\,B = E_1 \oplus E_2$ *with* $E_1, E_2 \neq 0$. *Then* $\mathrm{Rad}^\circ B = F_1 \oplus F_2$ *with* $E_i \subset F_i$, *and the* KF_i *are simple for* $i \in \{1,2\}$.
    (a) $E_1 \cong E_2$. *Then there is one almost split sequence* $E_1 \hookrightarrow B \twoheadrightarrow F_2$ *with middle term* B.
    (b) $E_1 \not\cong E_2$. *Then there are two almost split sequences* $E_1 \hookrightarrow B \twoheadrightarrow F_2$ *and* $E_2 \hookrightarrow B \twoheadrightarrow F_1$.
  (III) $\mathrm{Rad}\,B$ *is indecomposable non-injective. Then* $\mathrm{Rad}(\mathrm{Rad}^\circ B) = \mathrm{Rad}\,B$, *i. e. there is a submodule C of $\mathrm{Rad}^\circ B$ with $B + C = \mathrm{Rad}^\circ B$ and $B \cap C = \mathrm{Rad}\,B$.*

(a) *All these complements C are isomorphic to B. Then KB is simple, and there is an almost split sequence* $\operatorname{Rad} B \rightarrowtail B^2 \twoheadrightarrow \operatorname{Rad}^\circ B$.

(b) *There exists a complement $C \not\cong B$. Then there is an almost split sequence* $\operatorname{Rad} B \rightarrowtail B \oplus C \twoheadrightarrow \operatorname{Rad}^\circ B$.

There are no other irreducible maps or almost split sequences containing B than the mentioned ones.

In all cases except (Ia) there exists an overorder Γ' of Γ such that B is the only indecomposable Γ-lattice that is not a Γ'-lattice. For bijectives $\binom{I}{P}$ arising from hereditary morphisms $P \hookrightarrow I$, case (Ia) is characterized as follows.

Proposition 12.. *Let $u : P \hookrightarrow I$ be a hereditary morphism in Λ-**lat**. The following are equivalent:*

(a) $\binom{I}{P} \cong \operatorname{Rad}\binom{I}{P}$ (a') $\binom{I}{P} \cong \operatorname{Rad}^\circ\binom{I}{P}$

(b) $I' \cong I$ (b') $P' \cong P$

(c) $I' \in \mathcal{H}_u$ (c') $P' \in \mathcal{H}_u$

(d) $I \cong \operatorname{Rad} P$ (d') $P \cong \operatorname{Rad}^\circ I$.

Proof. Assume w.l.o.g. that u is indecomposable. Since $\binom{I}{P}$ is bijective, (a) is equivalent to (a'). Therefore, it is enough to prove the following implications.

(a) $\Rightarrow$ (b): This is an immediate consequence of Proposition 10.

(b) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (d): $I' \in \mathcal{H}_u$ implies $I' = (I')^+ \cong I$. Hence there is an endomorphism $e : I \twoheadrightarrow I' \hookrightarrow I$ of I. By (33) we have $e \in \operatorname{Rad} \operatorname{End}_\Lambda(I)$, and [10], Proposition 9, implies $e(I) \subset P$. Therefore, we get $I \cong I' = \operatorname{Rad} P$.

(d) $\Rightarrow$ (a): If $I \cong \operatorname{Rad} P$, then P is the unique minimal overlattice of $\operatorname{Rad} P$. Hence (33) gives $I' = \operatorname{Rad} P$. So there is an isomorphism $e : I \xrightarrow{\sim} I'$ which maps $P = I_-$ to $(I')_- = (\operatorname{Rad} P)_-$. By Proposition 10, this implies (a). $\qquad\square$

Proposition 13.. *If an indecomposable hereditary morphism $u : P \to I$ in Λ-**lat** satisfies the equivalent properties of Proposition 12, then the algebra $A = K\Lambda$ has a decomposition $A = A_0 \times A_1$ with A_0 simple such that KP is a simple A_0-module, and the Λ-lattices in KP form a chain.*

Proof. We show first that every Λ-lattice H with $P \subset H \subset I$ has a unique maximal submodule. For $H = P$ this is clear. For $H \neq P$, every maximal submodule H' strictly contains $\operatorname{Rad} P$. Therefore, $\operatorname{Rad} P \cong I$ implies $H' \supset P$, whence H' is unique. Now the argument can be repeated. So we infer that every Λ-lattice L in $S := KP$ with $KL = S$ has a unique maximal

sublattice. Hence the Λ-lattices L with $KL = S$ form a chain. Assume that $_A S$ is not simple. Then there is a non-zero proper submodule N of S. Since every Λ-lattice between $(\operatorname{Rad} R)P$ and P has a unique maximal submodule, we have $N \cap P \subset (\operatorname{Rad} R)P$, a contradiction. Thus $_A S$ is simple. Moreover, $S = KP = KI$ implies that S is projective and injective. Now let Q be any indecomposable projective A-module. Then every non-zero A-linear map $Q \to S$ (resp. $S \to Q$) splits. Hence it is an isomorphism. Consequently, there is a two-sided decomposition $A = A_0 \times A_1$ such that A_0 is a simple K-algebra, and S is a simple A_0-module. $\qquad\square$

Example. Consider the R-order

$$\Lambda := \begin{pmatrix} R & \mathfrak{p} \\ R & R \end{pmatrix}\!\!-\!\!R$$

with $\mathfrak{p} := \operatorname{Rad} R$. There are only 4 indecomposable Λ-lattices, namely,

$$P_1 = \begin{pmatrix} R \\ R \end{pmatrix}; \quad P_2 = \begin{pmatrix} \mathfrak{p} \\ R \end{pmatrix}\!\!-\!\!R; \quad I = \begin{pmatrix} \mathfrak{p} \\ R \end{pmatrix}; \quad R$$

The hereditary morphism $u : P \hookrightarrow I$ with $P := \binom{\mathfrak{p}}{\mathfrak{p}} \cong P_1$ satisfies $I \cong \operatorname{Rad} P$. We get

$$\delta_u \Lambda = \begin{pmatrix} R & \mathfrak{p} & R & R \\ \mathfrak{p}^{-1} & R & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ R & \mathfrak{p} & R & R \\ R & \mathfrak{p} & R & R \end{pmatrix} \times \begin{pmatrix} R & R \\ \mathfrak{p} & R \end{pmatrix},$$

a product of a maximal order Γ in $\mathrm{M}_4(K)$ with $\binom{I}{P}$ as indecomposable Γ-lattice, and a hereditary order in $\mathrm{M}_2(K)$.

For an R-order Λ in a semisimple K-algebra, let $\mathfrak{A}_\Lambda$ denote the Auslander-Reiten quiver of Λ. More generally, the left and the right τ-sequences of a τ-category $\mathcal{C}$ define a valued translation quiver $\mathfrak{A}(\mathcal{C})$. We call it the τ-*quiver* of $\mathcal{C}$. Thus $\mathfrak{A}(\Lambda\text{-}\mathbf{lat}) = \mathfrak{A}_\Lambda$. By Proposition 9, the τ-quiver of a quotient category of Λ-**lat** is immediately given by $\mathfrak{A}_\Lambda$. The relationship between $\mathfrak{A}_\Lambda$ and $\mathfrak{A}_{\delta_u \Lambda}$ for a given hereditary morphism u can be described by Theorem 2 below. We need the following auxiliary result.

Proposition 14.. *Let Λ be an R-order in a semisimple K-algebra, and let $u : P \hookrightarrow I$ be a hereditary morphism in Λ-lat. Then $\tau(\Lambda^-) \in \mathcal{H}_u$. Conversely, every indecomposable non-projective Λ-lattice E with $\tau E \in \mathcal{H}_u$ is a direct summand of Λ^-.*

Proof. Choose a generating system $\varphi_1, \dots, \varphi_n$ of the R-module P^*, and consider $\varphi := (\varphi_1, \dots, \varphi_n) \in (P^*)^n$. Proposition 4, applied to $P^n \hookrightarrow I^n$, gives

$H := (\varphi\Lambda + (I^*)^n)^* \in \mathcal{H}_u$. The epimorphism $q : \Lambda_\Lambda \oplus (I^*)^n \twoheadrightarrow H^*$ with $q(a \oplus \psi) := \varphi a + \psi$ induces a short exact sequence

$$\mathrm{Hom}_\Lambda(H^*, \Lambda) \xrightarrow{q^*} \Lambda \oplus \mathrm{Hom}_\Lambda((I^*)^n, \Lambda) \xrightarrow{p} \Lambda^-$$

in Λ-**lat**, where $q^*(f) = f(\varphi) \oplus f$, and $p(a \oplus f) = a - f(\varphi)$. This shows that $\Lambda^- = \tau^- H \oplus Q$ with $Q \in \Lambda$-**proj**. Hence $\tau(\Lambda^-) = \tau\tau^- H \in \mathcal{H}_u$.

The projection of $(P^*)^n$ to the i-th coordinate yields an exact sequence $H^* \cap (P^*)^{n-1} \hookrightarrow H^* \twoheadrightarrow \varphi_i\Lambda + I^*$ that splits by (H3). Furthermore, there is a (split) exact sequence

$$(\varphi_1\Lambda + \cdots + \varphi_{s-1}\Lambda + I^*) \cap (\varphi_s\Lambda + I^*) \hookrightarrow (\varphi_1\Lambda + \cdots + \varphi_{s-1}\Lambda + I^*) \oplus (\varphi_s\Lambda + $$
$$\twoheadrightarrow \varphi_1\Lambda + \cdots + \varphi_s\Lambda + I^*$$

for any $s \in \{2, \ldots, n\}$. By a suitable choice of $\{\varphi_1, \ldots, \varphi_n\}$ this shows that every indecomposable Λ-lattice in $\mathcal{H}_u$ is a direct summand of H. Therefore, the equation $\Lambda^- = \tau^- H \oplus Q$ completes the proof. $\qquad\square$

Theorem 2.. *Let Λ be an R-order in a semisimple K-algebra A, and let $u : P \hookrightarrow I$ be an indecomposable hereditary morphism in Λ-**lat**. Let $\overline{\mathfrak{A}}$ denote the τ-quiver of $\overline{\mathbb{C}} := \delta_u\Lambda$-**lat**$/[\binom{I}{P})] \approx \Lambda$-**lat**$/[\mathbb{Z}_u]$. The following cases are possible:*

(Ia) *$P' \cong P$. Then $\mathfrak{A}_{\delta_u\Lambda} = \overline{\mathfrak{A}} \amalg \mathfrak{A}_0$, where $\mathfrak{A}_0$ consists of the single vertex $\binom{I}{P}$ together with a loop $\binom{I}{P} \to \binom{I}{P}$.*

Assume that we are not in this case. Then $P', I' \notin \mathcal{H}_u$, i. e. they correspond to vertices $\partial_u P'$ and $\partial_u I'$ in $\overline{\mathfrak{A}}$. The Auslander-Reiten quiver $\mathfrak{A}_{\delta_u\Lambda}$ is obtained from $\overline{\mathfrak{A}}$ by inserting the bijective $B := \binom{I}{P}$ between $\partial_u P'$ and $\partial_u I'$. Possible cases are:

(Ib) *P' is projective. Then there are two irreducible maps $\partial_u I' \to B \to \partial_u P'$.*

(II) *$P' = E_1 \oplus E_2$ with KE_1, KE_2 simple. Then $\partial_u P' = \partial_u E_1 \oplus \partial_u E_2$ and $\partial_u I' = \mathrm{Rad}(\partial_u E_1) \oplus \mathrm{Rad}(\partial_u E_2)$, and B is a diagonal between $\partial_u P'$ and $\partial_u I'$.*

    (a) *$E_1 \cong E_2$. There is one almost split sequence $\mathrm{Rad}(\partial_u E_1) \rightarrowtail B \twoheadrightarrow \partial_u E_2$.*

    (b) *$E_1 \not\cong E_2$. There are two almost split sequences $\mathrm{Rad}(\partial_u E_1) \rightarrowtail B \twoheadrightarrow \partial_u E_2$ and $\mathrm{Rad}(\partial_u E_2) \rightarrowtail B \twoheadrightarrow \partial_u E_1$.*

(III) *P' is indecomposable non-projective. Then there is an almost split sequence $I' \rightarrowtail E \twoheadrightarrow P'$. Let $\partial_u E = B^s \oplus C$ with s maximal.*

    (a) *$E \in \mathcal{H}_u$. There is an almost split sequence $\partial_u I' \rightarrowtail B^2 \twoheadrightarrow \partial_u P'$.*

    (b) *$E \notin \mathcal{H}_u$. There is an almost split sequence $\partial_u I' \rightarrowtail B \oplus C \twoheadrightarrow \partial_u P'$.*

Proof. Note first that $\partial_u(_\Lambda\Lambda)$ is a direct summand of $\delta_u\Lambda$. Thus if P' is projective, then $\partial_u P'$ is also projective. Assume that P' is decomposable, say, $P' = E_1 \oplus E_2$. Then $KP' = KP$ implies that $E_1, E_2 \notin \mathcal{H}_u$. Hence $\partial_u P'$ is decomposable. By Proposition 11 we infer that $\partial_u P'$ is non-projective, and so P' is non-projective. Therefore, we have shown that case (I) is characterized by the property that P' is projective.

Conversely, let $\partial_u P'$ be projective, hence indecomposable. Then P' is also indecomposable. Suppose P' is non-projective. Then there exists an almost split sequence $H \rightarrowtail L \twoheadrightarrow P'$ in Λ-**lat**. Since $K(\partial_u P')$ is simple by Proposition 11, we infer that KP' is simple. By Proposition 12, we have $P' \notin \mathcal{H}_u$, and Propositions 10 and 11 imply that $\partial_u P'$ is a source object in $\overline{\mathcal{C}}$. Consequently, $L \in \mathcal{H}_u$. Moreover, $\delta_u\Lambda = \partial_u\Lambda \oplus \partial_u\Lambda^-$ implies that P' is a direct summand of Λ^-. Hence $H \in \mathcal{H}_u$ by Proposition 14. Since $H = \tau P'$ is indecomposable, this implies $L = H_1 \oplus H_2$ with $H_i \in \mathcal{Z}_u$. Therefore, $H \rightarrowtail L$ gives two irreducible maps $e_i : H \to H_i$ which are injective since KH is simple. We may assume the e_i as inclusions. Then $H^+ \subset H_i^+$ and $H^+ \cong H_i^+ \cong I$ for $i \in \{1, 2\}$. If $H^+ \neq H_i^+$ would hold for one i, there would be a non-invertible $r \in \mathrm{End}_\Lambda(H_i^+)$ with $r(H_i^+) = H^+$. By [10], Proposition 9, this implies $H^+ \subset (H_i)_-$. Since e_i is irreducible, H_i/H is simple. Moreover, $H^+ \neq H_i^+$ implies $H^+ \neq (H_i)_-$. Therefore, the inclusions $H \subset H^+ \subset (H_i)_- \subset H_i$ give $H_i = (H_i)_-$ and $H = H^+ = \mathrm{Rad}(H_i)_-$. Hence $I \cong \mathrm{Rad}\,P$. By Proposition 12, this implies $P' \cong P$, a contradiction. So we have $H^+ = H_i^+$ for $i \in \{1, 2\}$, whence $H_1 = H_2$. But this contradicts the right minimality of $H \hookrightarrow H_1 \oplus H_2$.

Thus we have shown that P' is projective if and only if $\partial_u P'$ is projective. So the cases (Ia) and (Ib) of the theorem correspond to the same cases for the bijective $\delta_u\Lambda$-lattice B according to Proposition 11. This proves the assertions in (I) and (II).

In case (III), there are almost split sequences $\tau P' \rightarrowtail \vartheta P' \twoheadrightarrow P'$ and $\tau(\partial_u P') \rightarrowtail \vartheta(\partial_u P') \twoheadrightarrow \partial_u P'$ where $\tau(\partial_u P') = \partial_u I'$ by Propositions 10 and 11. If $\tau P' \in \mathcal{H}_u$, then P' is a direct summand of Λ^- by Proposition 14. But then $\partial_u P'$ is projective, in contrast to the above. Thus $\tau P' \notin \mathcal{H}_u$. If $\vartheta P' \in \mathcal{H}_u$, then $\partial_u(\tau P')$ is a sink object in $\overline{\mathcal{C}}$. On the other hand, $\partial_u I'$ is the only sink object in $\overline{\mathcal{C}}$. Hence $\tau P' \cong I'$. This settles case (IIIa). Otherwise, $\vartheta P' \notin \mathcal{H}_u$, and the right almost split sequences of P' and $\partial_u P'$ are related via $\overline{\mathcal{C}}$.   □

Example. The R-order

$$\Lambda = \begin{pmatrix} R-R & \mathfrak{p} \times \mathfrak{p} \\ R \times \mathfrak{p} & R-R \end{pmatrix} \subset \mathrm{M}_2(K) \times \mathrm{M}_2(K)$$

has 10 indecomposables, namely, $H_1 := \begin{pmatrix} R \\ R \end{pmatrix}$ and $H_2 := \begin{pmatrix} \mathfrak{p} \\ R \end{pmatrix}$ in the first rational

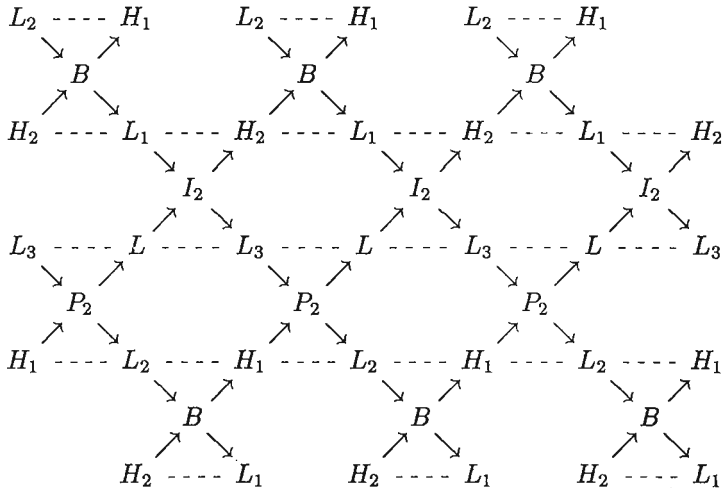component, $L_1 := \binom{R}{\mathfrak{p}}$, $L_2 := \binom{\mathfrak{p}}{R}$, and $L_3 := \binom{R}{R}$ in the second rational component, the projectives $P_1 := \binom{R-R}{R\times\mathfrak{p}}$, $P_2 := \binom{\mathfrak{p}\times\mathfrak{p}}{R-R}$, and injectives $I_1 := \binom{R-R}{R\times R}$, $I_2 := \binom{\mathfrak{p}\times R}{R-R}$, and an indecomposable L of rational length 3. The Auslander-Reiten quiver is a cylinder (with equal vertices to be identified, and dashed lines indicating the almost split sequences):

```
H₂ ---- L₁ ---- H₂ ---- L₁ ---- H₂
    ↘  ↗  ↘  ↗  ↘  ↗  ↘  ↗
      P₁ ---- I₂      P₁ ---- I₂
    ↗  ↘  ↗  ↘  ↗  ↘  ↗  ↘
L₃ ---- L ---- L₃ ---- L ---- L₃
    ↘  ↗  ↘  ↗  ↘  ↗  ↘  ↗
      P₂ ---- I₁      P₂ ---- I₁
    ↗  ↘  ↗  ↘  ↗  ↘  ↗  ↘
H₁ ---- L₂ ---- H₁ ---- L₂ ---- H₁
```
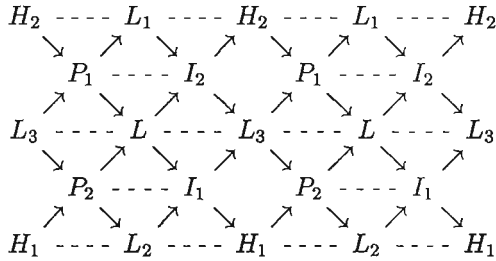
For the hereditary morphism $u : P_1 \hookrightarrow I_1$ we have $P' = H_1 \oplus L_1$ and $I' = H_2 \oplus L_2$, i. e. case (IIb). Removing P_1, I_1 and inserting a bijective B gives the Auslander-Reiten quiver of $\delta_u\Lambda$ which is a torus:

```
L₂ ---- H₁        L₂ ---- H₁        L₂ ---- H₁
  ↘  ↗               ↘  ↗               ↘  ↗
   B                  B                  B
  ↗  ↘               ↗  ↘               ↗  ↘
H₂ ---- L₁ ---- H₂ ---- L₁ ---- H₂ ---- L₁ ---- H₂
     ↘  ↗               ↘  ↗               ↘  ↗
       I₂                 I₂                 I₂
     ↗  ↘               ↗  ↘               ↗  ↘
L₃ ---- L ---- L₃ ---- L ---- L₃ ---- L ---- L₃
  ↘  ↗               ↘  ↗               ↘  ↗
    P₂                  P₂                  P₂
  ↗  ↘               ↗  ↘               ↗  ↘
H₁ ---- L₂ ---- H₁ ---- L₂ ---- H₁ ---- L₂ ---- H₁
     ↘  ↗               ↘  ↗               ↘  ↗
       B                  B                  B
     ↗  ↘               ↗  ↘               ↗  ↘
     H₂ ---- L₁        H₂ ---- L₁        H₂ ---- L₁
```

References

[1] M. Auslander: Isolated singularities and existence of almost split sequences, Springer Lecture Notes in Mathematics, Vol. 1178 (1986), 194-242

[2] M. Auslander, S. O. Smalø: Almost Split Sequences in Subcategories, J. Algebra 69 (1981), 426-454

[3] Yu. A. Drozd, M. G. Greuel: Tame-wild dichotomy for Cohen-Macaulay modules, Math. Ann. 294 (1992), 387-394

[4] Yu. A. Drozd, V. V. Kiričenko: On Quasi-Bass Orders, Izv. Akad. Nauk. SSSR, Ser. Mat. 36 (1972), 328-370 = Math. USSR Izv. 6 (1972), 323-366

[5] P. Gabriel, M. Zisman: Calculus of Fractions and Homotopy Theory, Berlin – Heidelberg – New York 1967

[6] O. Iyama: τ-categories I: Radical Layers Theorem, Preprint

[7] O. Iyama: τ-categories II: Subcategories, Quotient Categories, and Rejection Lemma, Preprint

[8] C. M. Ringel: Tame Algebras and Integral Quadratic Forms, Springer LNM 1099, Berlin – Heidelberg – New York – Tokyo 1984

[9] K. W. Roggenkamp: Gorenstein orders of finite representation type and bijective lattices, in: Representation Theory II, Groups and Orders, Springer LNM 1178 (1986), 243-271

[10] W. Rump: Two-Point Differentiation for General Orders, J. Pure Appl. Algebra, to appear

[11] W. Rump: Differentiation and Splitting for Lattices over Orders, Colloq. Mathematicum, to appear

[12] D. Simson: On Vector Space Categories and Differentiations of Right Peak Rings, in: Proc. Int. Conf. Repr. of Algebras IV, Ottawa, Carleton Univ., 1984

[13] D. Simson: Vector Space Categories, Right Peak Rings, and their Socle Projective Modules, J. Alg. 92 (1985), 532-571

[14] D. Simson: Linear Representations of Partially Ordered Sets and Vector Space Categories, Amsterdam 1992

[15] A. G. Zavadskij: A differentiation with respect to a pair of points, in: "Matrix problems", Kiev 1977, pp. 115-121 (in Russian)

[16] A. G. Zavadskij, V. V. Kiričenko: Semimaximal Rings of Finite Type, Mat. Sbornik 103(145) (1977), 323-345 = Math. USSR Sbornik 32 (1977), 273-291

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MODULES WHOSE EXACT SUBMODULES ARE DIRECT SUMMANDS

Adnan Tercan

Abstract

A module M is called an ES-module provided every exact submodule of M is a direct summand of M. It is shown that if M be the direct sum $M_1 \oplus M_2$ of ES-modules M_1 and M_2 such that M_1 is M_2-injective then M is an ES-module. In consequence it is obtained when the finite direct sums of ES-modules is an ES-module. Moreover some results on the direct sum of two certain CS-modules are generalized to left exact preradicals.

Throughout this paper all rings will have identities and all modules will be unital. Let R be any ring and M a right R-module. Recall that a submodule N of M is called *closed submodule* of M provided M/N is nonsingular and a module M is called *a CLS-module* if every closed submodule of M is a direct summand (see [9]). A functor r from the category of right R-modules to itself is called a *left exact preradical* if it has the following properties

(i) $r(M)$ is a submodule of M for every right R-module M,

(ii) $r(N) = N \cap r(M)$ for every submodule N of a right R-module M, and

(iii) $\phi(r(M)) \subseteq r(M')$ for every homomorphism $\phi : M \longrightarrow M'$, for right R-modules M, M'.

Furthermore, a left exact preradical r is called *radical* if $r(M/r(M)) = 0$ for every right R-module M. It is clear that the singular submodule, socle are left exact preradical and the second singular submodule (or the Goldie torsion submodule) is radical. For an excellent treatment of left exact preradicals the reader is referred to [2] (see, also [8]).

In this paper we generalize CLS-modules in terms of left exact preradicals, for a ring R. We begin by explaining exact submodules.

Definition 1. Let R be any ring, let r be a left exact preradical in the category of right R-modules and let M be a right R-module. A submodule N of M is called *exact submodule* (or *exact in M*) provided $r(M/N) = 0$. Clearly every module M is exact in itself. Moreover, if r is a left exact radical then $r(M)$ is an exact submodule of a module M, and by [2, Proposition 1.1] we have :

Lemma 2. *Let R be a ring and let r be a left exact preradical in the category of right R-modules and let M be a right R-module. Then the intersection of exact submodules of M is also an exact submodule of M.*

Proof. Let $\mathcal{E} = \{X : X \quad is \quad exact \quad in \quad M\}$ and let $I = \cap_{X \in \mathcal{E}} X$. Suppose that N is an exact submodule of M. Then $I \leq N$. Let $N' = N/I$ and $M' = M/I$. Hence $N' \leq M'$ and $r(M'/N') = 0$ i.e. N' is an exact submodule of M'. Thus $r(M')$ is contained in all these modules N'. Therefore $r(M') = 0$. It follows that I is exact in M.

Definition 3. A module M is called an *ES-module* provided every exact submodule of M is a direct summand of M.

The following result shows that ES property is inherited by direct summands.

Lemma 4. *Any direct summand of an ES-module is an ES-module.*

Proof. Suppose $M = K \oplus K'$ for some submodule K, K' of M. Let L be an exact submodule of K. Since

$$M/L \oplus K' = K \oplus K'/L \oplus K' \cong K/L$$

then $r(M/L \oplus K') = 0$. i.e, $L \oplus K'$ is an exact submodule of M which gives that L is a direct summand of M. Then L is a direct summand of K. It follows that K is an ES-module.

Note that a direct sum of ES-modules need not to be ES-module in general. The following example is taken from [9, p.1560].

Example 5. Let M be the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{a/b : a, b \in \mathbb{Z} \quad and \quad b \quad is \quad odd\}$. Let r denote Z_2 i.e. the Goldie torsion theory (see [3] or [8]). Then both $\mathbb{Z}$ and $\mathbb{Z}_2$ are ES-modules. However it is clear that $M_{\mathbb{Z}}$ is not an ES-module. Note also that neither $\mathbb{Z}$ is $\mathbb{Z}_2$-injective nor $\mathbb{Z}_2$ is $\mathbb{Z}$-injective.

We next prove a theorem which was pointed out above.

Theorem 6. *Let a module $M = M_1 \oplus M_2$ be a direct sum of ES-modules M_1, M_2 such that M_1 is M_2-injective. Then M is an ES-module.*

Proof. Let N be an exact submodule of M. Then $r(M/N) = 0$ where r is a left exact preradical for the ring R. Now $M_1/N \cap M_1 \cong M_1 + N/N$ implies $N \cap M_1$ is an exact submodule of M_1. Thus $N \cap M_1$ is a direct summand of M_1 and hence of M. It follows that $N \cap M_1$ is a direct summand of N, so $N = (N \cap M_1) \oplus K$ for some submodule K of N. Let $\pi : M \longrightarrow M_i$, $i = 1, 2$ denote the canonical projections. Let $\alpha = \pi_2 \mid_K$ and $\beta = \pi_1 \mid_K$. Consider the following exact diagram.

$$0 \longrightarrow K \longrightarrow M_2$$
$$\downarrow$$
$$M_1$$

Note that α is a monomorphism and M_1 is M_2-injective. There exists a homomorphism $\varphi : M_2 \longrightarrow M_1$ such that $\varphi\alpha = \beta$. Let

$$L = \{x + \varphi(x) : x \in M_2\}.$$

Then it can be easily checked that L is a submodule of M and $L \cong M_2$. Moreover, $M = M_1 \oplus L$. If $k \in K$ then $k = m_1 + m_2$ for some $m_i \in M_i$, $i = 1, 2$. Then

$$m_1 = \beta(k) = \varphi\alpha(k) = \varphi(m_2);$$

and this implies that $k = \varphi(m_2) + m_2 \in L$. Thus $K \subseteq L$. Since

$$M/N = (M_1/N \cap M_1) \oplus L/K$$

then $r(L/K) = 0$. Hence K is an exact submodule of L. But $L \cong M_2$ so that K is a direct summand of M. It follows that M is an ES-module.

Let n be a positive integer and $M_1, M_2, ..., M_n$ are right R-modules. Recall that these modules are called *relatively injective* if M_i is M_j-injective for all $1 \leq i \neq j \leq n$ see [4].

Theorem 7. *Let R be a ring and M a right R-module such that $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ is a finite direct sum of relatively injective modules*

M_i, $1 \leq i \leq n$. Then M is an ES-module if and only if M_i is an ES-module for each $1 \leq i \leq n$.

Proof. The necessity is clear by Lemma 4. The converse follows by induction on n and using Theorem 6.

Proposition 8. *Let E, M be right R-modules such that E is ES and $r(M) = 0$ for a left exact preradical r in the category of right R-modules. Then any epimorphism $\alpha : E \longrightarrow M$ splits and thus is M an ES-module.*

Proof. Let $K = \ker\alpha$. Then $E/K \cong M$. Hence $r(E/K) = 0$. So K is an exact submodule of E. Therefore α splits. By Lemma 4, M is an ES-module.

Following [8, p. 152] a hereditary torsion theory is called *stable* if the class of torsion modules is closed under injective envelopes. By [8, Proposition 7.3] the Goldie torsion theory is stable. Thus [9, Proposition 8] is a special case of the following result.

Theorem 9. *Let R be a ring and let r be the left exact radical for a stable hereditary torsion theory for the category of right R-modules. Then a right R-module M is an ES-module if and only if $M = r(M) \oplus M'$ for some submodule M' of M and both $r(M)$ and M' are ES-modules.*

Proof. Suppose M is an ES-module. Then $r(M)$ is a direct summand of M so that $M = r(M) \oplus M'$ for some submodule M' of M. By Lemma 4, $r(M)$ and M' are ES-modules. Conversely, suppose that $M = r(M) \oplus M'$ for some submodule M' of M. Let K be an exact submodule of M. Then $r(M) \leq K$, and hence $K = r(M) \oplus (K \cap M')$. Now $M/K \cong M'/(K \cap M')$ so that $K \cap M'$ is an exact submodule of M'. Thus $M' = (K \cap M') \oplus K'$ for some submodule K'. Hence $M = K \oplus K'$. It follows that M is an ES-module.

We need to have r is a left exact radical in Theorem 9. We should give the following easy example.

Example 10. Let R be a commutative ring and let M be a torsion module over R which has a unique composition series of length 2. Now, let r denote the socle. Hence M is the only exact submodule of M. It follows that M is an ES-module. However $r(M)$ is essential in M.

We shall be concerned CS-modules. First recall that a module M is called *CS* (or *extending*) if every submodule of M is essential in a direct summand

of M. It is well-known that a direct sum of CS-modules need not to be a CS-module in general (see [1], [4] and references therein). Some conditions are given in [6] which make direct sum of two CS-modules is CS. Now we shall prove some similar results to [6, Proposition 20 and Corollary 21] in general case, namely with left exact preradicals for the category of right R-modules. We begin by mentioning two definitions. Let M_1 and M_2 be modules. Then M_1 is *essentially M_2-injective* if every homomorphism $\alpha : A \longrightarrow M_1$ where A is a submodule of M_2 and $ker\alpha$ is essential in A, can be extended to a homomorphism $\theta : M_2 \longrightarrow M_1$ (see [1] or [6]). A module M is said to have the *finite exchange property* if, for every finite index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and $A_i, i \in I$, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for submodules B_i of A_i, $i \in I$ (see [1]).

The next result generalizes [5, Lemma 11]. Compare it also with [7, Lemma 5].

Proposition 11. *Let R be a ring and let r be a left exact preradical in the category of right R-modules and let M be a right CS-module. Then $M = M_1 \oplus M_2$ be a direct sum of CS-modules M_1, M_2 such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$. In this case M_1 is M_2-injective.*

Proof. Since $r(M) \leq M$ there exists a direct summand M_1 of M such that $r(M)$ is essential in M_1. Then $M = M_1 \oplus M_2$ for some submodule M_2 of M. Note that both M_1 and M_2 are CS-modules (see [1, Lemma 7.1]). Since r is left exact it follows that $r(M) \cap M_2 = r(M_2) = 0$. Hence the first part is proved. For the second part, let N be any submodule of M_2 and let $\varphi : N \longrightarrow M_1$ be a homomorphism. Let

$$L = \{x - \varphi(x) : x \in N\}.$$

Then L is a submodule of M and $L \cap M_1 = 0$. There exists submodules K, K' of M such that $M = K \oplus K'$ and L is an essential submodule of K. Note that $r(K) = K \cap M_1 = 0$, so that $M_1 = r(M) \subseteq K'$. Thus $K' = M_1 \oplus (K' \cap M_2)$ and $M = K \oplus M_1 \oplus (K' \cap M_2)$. Let $\pi : M \longrightarrow M_1$ denote the projection with kernel $K \oplus (K' \cap M_2)$. Let $\theta = \pi \mid_{M_2}$. Then $\theta : M_2 \longrightarrow M_1$ and $\theta(x) = \varphi(x)$ for all $x \in N$. It follows that M_1 is M_2-injective.

Theorem 12. *Let R be a ring and let r be a left exact preradical in the category of right R-modules and let M_1 be a module with finite exchange property and $r(M_1)$ is essential in M_1, M_2 be a module with $r(M_2) = 0$. Then $M_1 \oplus M_2$ is a CS-module if and only if M_1 and M_2 are CS, M_2 is essentially M_1-injective and M_1 is M_2-injective.*

Proof. The sufficiency follows from [6, Theorem 8]. Conversely, let $M = M_1 \oplus M_2$ be a CS-module. Clearly, M_1 and M_2 are CS-modules and M_2 is essentially M_1-injective, by [6, Proposition 12]. Hence the result follows by Proposition 11.

Corollary 13. *Let R be a ring and let r be a left exact preradical in the category of right R-modules and let M_1 be a module with finite exchange property and $r(M_1)$ is essential in M_1, M_2 be any module. If $M_1 \oplus M_2$ is CS then the following conditions are equivalent.*

(i) M_1 is essentially M_2-injective,

(ii) M_1 is $(M_2/r(M_2))$-injective.

Proof. Obviously (ii) implies (i). Suppose that $M = M_1 \oplus M_2$ is CS and M_1 is essentially M_2-injective. Since M_2 is CS then $M_2 = M_{21} \oplus M_{22}$ where $r(M_2)$ is essential in M_{22} and $r(M_{21}) = 0$, by Proposition 11. Thus M_1 is $(M_{22}/r(M_{22}))$-injective, because it is essentially M_{22}-injective. Also, by Theorem 12, M_1 is M_{21}-injective. Thus M_1 is $(M_2/r(M_2))$-injective.

References

[1] N.V. Dung, D. V. Huynh, P.F. Smith and R. Wisbauer, Extending modules, Longman, Harlow 1994.

[2] O. Goldman, *Rings and modules of quotients*, J. Algebra, **13**(1969), 10-47.

[3] K. R. Goodearl, Ring theory, Marcel-Dekker, New York 1976.

[4] A. Harmanci and P.F. Smith, *Finite direct sums of CS-modules*, Houston J. Math. **19**(1993), 523-532.

[5] A. Harmanci, P.F. Smith, A. Tercan and Y. Tiras, *The Bass-Papp theorem and some related results*, Vietnam J. Math., **25**(1997), 33-39.

[6] C. Santa-Clara, *Extending modules with injective or semisimple summands*, J. of Pure and Applied Algebra, **127**(1998), 193-203.

[7] P.F. Smith and A. Tercan, *Direct summands of modules which satisfy* (C_{11}), preprint.

[8] B. Stenstrom, Rings of quotients, Springer-Verlag, New York 1975.

[9] A. Tercan, *On CLS-modules*, Rocky Mount. J. Math., **25**(1995), 1557-1564.

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DECOMPOSITION NUMBERS FOR SOME THREE-PART PARTITIONS

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Abstract

This is the substance of a paper delivered at the Euroconference on „Rings, Modules and Representations" held at the Ovidius University in Constanta, Romania from 14-18 August, 2000.

1. Introduction

Let p be prime, k be a field, and assume throughout this paper that $\lambda = (\lambda_1, \lambda_2, ..., \lambda_s)$ is a p-regular partition of the positive integer n with s parts. We have $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_s > 0$ and $\Sigma_{i=1}^s \lambda_i = n$ and sometimes write $\lambda \vdash n$ to denote that λ is a partition of n. λ is p-regular means that no p parts of λ are equal.

If k has characteristic *zero*, then the irreducible representations of the symmetric group σ_n are characterized by the $k\sigma_n$ Specht modules S^λ. If k has characteristic p then the modular irreducible representations are the quotients D^λ of S^λ modulo its radical $S^\lambda \cap S^{\lambda\perp}$. Let $d_{\lambda\mu} = [S^\lambda : D^\mu]$ denote the composition multiplicity of D^μ in S^λ, when μ is p-regular. For the symmetric group the problem of finding $d_{\lambda\mu}$ given λ (not necesarily p-regular) and μ (for μ p-regular) is still open. It is equivalent to finding the dimension dim D^μ.

The bulk of this paper record some joint work with Gordon James [1]. We give the theorems which enable one to obtain, at least in principle, the composition multiplicity of D^μ in S^λ when $s = 3$ and $\lambda_3 \leq p - 1$. The last

theorem [*cf. Theorem B, 2*] expresses these decomposition numbers in terms of multiplicities of two-part factors and that of the trivial module.

In order to state our theorems we first record some well known facts and definitions.

- $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\mu \trianglerighteq \lambda$ with respect to the dominance order for partitions. In particular, if $\mu_{s+1} > 0$, then $d_{\lambda\mu} = 0$. (Recall that $\mu \trianglerighteq \lambda$ if and only if $\Sigma_{j=1}^i \mu_j \geq \Sigma_{j=1}^i \lambda_j$ for all i).

- If $\mu_s \neq C(B'0$ and $\mu_{s+1} = 0$, then

$$d_{\lambda\mu} = \left[S^{(\lambda_1-1,\lambda_2-1,...,\lambda_s-1)} : D^{(\mu_1-1,\mu_2-1,...,\mu_s-1)} \right].$$

- Identify $\lambda \vdash n$ with the set of nodes $\{(i,j) : 1 \leq j \leq \lambda_i \text{ and } 1 \leq i \leq s\}$. Let A be the node (i,j) and define the *p−residue* of A by $res A = j - i$ mod p and the it $p−$content of λ to be the p-tuple $(c_0, c_1, ..., c_{p-1})$ where c_i is the number of nodes in λ with p-residue i.

- A node $A(i, \lambda_i)$ is called *removable* from a partition λ if $\lambda\backslash\{A\}$ is a partition, and a node $B(i, \lambda_i + 1)$ is *addable* for λ if $\lambda \cup \{B\}$ is again a partition.

- The node A is called *normal* for μ if A is removable and for every addable node B for μ strictly between A and B with $res B = res A$ there exists a removable node $C(B)$ of μ with $res\ C(B) = res A$, and $B \neq B'$ implies that $C(B) \neq C(B')$. Such a node is called good if it is the lowest normal node with this residue.

- The *r-restriction (r-induction)* of a module M for $0 \leq r \leq p - 1$ is the restriction (induced-up module) of a $k\mathfrak{S}_n$ module to $\mathfrak{S}_{n-1}$ (to $\mathfrak{S}_{n+1}$) and to the block with p-content $(c_0, c_1, ..., c_r - 1, ..., c_{p-1})$, (respectively, $(c_0, c_1, ..., c_r + 1, ..., c_{p-1})$).

- Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_s)$ and μ be partitions of n. For $1 \leq i \leq s$, let $\alpha_i = \lambda_i + s - i$ and $\beta_i = \mu_i + s - i$. The sets $\{\alpha_1, ..., \alpha_s\}, \{\beta_1, ..., \beta_s\}$ are called *β-numbers* for the partitions λ and μ, respectively. Note that

$res(i, \lambda_i) \equiv \alpha_i - s \mod p$. Nakayama's Theorem expressed in terms of β-numbers states that S^λ is in the same block as D^μ if and only if λ and μ have the same p-content, that is if and only if $\{\alpha_1, ..., \alpha_s\} \equiv \{\beta_1, ..., \beta_s\}$ as multi-sets.

2. Branching Theorems

As well as the classical branching theorem [cf. 2.2, **2**], crucial to obtaining the decomposition numbers for certain three-part partition are the well known which give bounds for $d_{\lambda\mu}$ in terms of the composition factors of $D^\mu \downarrow$ and $S^\lambda \downarrow$, and a branching theorem of Kleschev [**3**] for D^μ.

2.1. **Proposition** (Bounds for $d_{\lambda\mu}$). *Let $\lambda, \mu \vdash n, \sigma \vdash n - 1, \tau \vdash n + 1$ and μ, σ and τ be p-regular. Then $d_{\lambda\mu} [D^\mu \downarrow_r: D^\sigma] \leq [S^\lambda \downarrow_r: D^\sigma]$ with equality if D^μ is the only composition factor of S^λ which contains D^σ on r-restriction. Similarly, $d_{\lambda\mu} [D^\mu \uparrow^r: D^\tau] \leq [S^\lambda \uparrow^r: D^\tau]$ with equality if D^μ is the only compositions factor of S^λ which contains D^τ on r-inducing up.*

2.2. **Theorem** (Kleschev [**3**]). *Assume that μ is p-regular and has exactly m normal nodes with p-residue congruent to r modulo p and that B is the lowest of these. Then*

(i) $[D^\mu \downarrow_r: D^{\mu\backslash\{B\}}] = m$; in particular if $m = 1$, then $D^\mu \downarrow_r= D^{\mu\backslash\{B\}}$, and if μ has no normal node with $p-$residue congruent to r modulo p, then $D^\mu \downarrow_r= 0$;

(ii) if B is a good node of μ whose p-residue is congruent to r modulo p, then $[D^{\mu\backslash\{B\}} \uparrow^r: D^\mu] > 0$.

3. Node removal theorems

There now follow several results which depend on the β-numbers for μ. We shall assume throughout this section that S^λ and D^μ are in the same block. The first of these is a theorem of Erdmann which applies when all the β-numbers for μ are congruent to *zero* modulo p.

3.1. **Theorem** [cf.2.3, **4**]. *If $\mu = (p - 1)(s - 1, s - 2, ..., 1, 0) + \mu^{(0)}$ and $\mu^{(0)}$ is p-regular then $d_{\lambda\mu} = [S^{\lambda^{(0)}} : D^{\mu^{(0)}}]$.*

The next result may be applied when at least two β-numbers for μ are unequal.

3.2. Theorem [cf.3.6, 1]. *Assume that if $1 \leq i \leq s$ and $y \neq i$, then $\beta_y \not\equiv \beta_i \mod p$ and $B = (y, \mu_y)$ is normal for μ. Let x be the subscript such that $\alpha_x \equiv \beta_y \mod p$ and $A = (x, \lambda_x)$.*

(i) If A is not removable, then $d_{\lambda\mu} = 0$;

(ii) If A is removable then $d_{\lambda\mu} = \left[S^{\lambda\backslash\{A\}} : D^{\mu\backslash\{B\}} \right].$

When some β number for μ is not congruent to zero modulo p and is not congruent modulo p to any $\beta_i + 1$ we have

3.3. Theorem [cf. 3.8, 1] *Assume that there exists y such that $\beta_y \not\equiv 0$ and $\beta_y \not\equiv \beta_i + 1 \mod p$ for all i. Let $r \equiv \beta_y - s \mod p$, B be the lowest node of μ with p-residue r and let A_l be any removable node of λ with p-residue r. Then $d_{\lambda\mu} = \left[S^{\lambda\backslash\{A_l\}} : D^{\mu\backslash\{B\}} \right].$*

When some β-number say β_i is congruent to 1 modulo p and no β-number with subscript smaller than 1 is congruent to 0 modulo p we have

3.4. Theorem [cf. 3.13, 1] *Assume that the following hold.*

(i) For some i with $1 \leq i \leq s$ we have $\beta_i \equiv \mod p$.

(ii) There do not exists i and j with $1 \leq i < j \leq s$ and $\beta_i \equiv 0$ and $\beta_i \equiv 1 \mod p$.

(iii) For every i with $1 \leq i \leq s$ and $\beta_i \equiv 1 \mod p$ the node (i, μ_i) is a removable node of μ.

Let B be the lowest node (i, μ_i) of μ such that $\beta_i \equiv 1 \mod p$.

If λ has no removable node with p-residue $1 - s$, then $d_{\lambda\mu} = 0$.

If λ has a removable node of p-residue $1 - s$, then

$$\left[S^\lambda : D^\mu \right] = \left[S^{\lambda\backslash\{A\}} : D^{\mu\backslash\{B\}} \right]$$

4. Three-part Partition

The last result of this kind is specific to three-part partition and applies when $\beta_1 \equiv 0$ and $\beta_2 \equiv 1$ modulo p.

4.1. **Theorem** [cf. 4.5, 1] *Assume that* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *and* $\lambda_3 > 0$ *and* $\mu = (\mu_1, \mu_2, 0)$ *with the conditions that* $\mu_2 > 0$ *and* $\mu_1 - \mu_2 \neq p - 2$. *Assume also* $\beta_1 \equiv 0$ *and* $\beta_2 \equiv 1 \mod p$ *(so that, in particular, the partition* $(\mu_1 - 1, \mu_2)$ *is p-regular). Then* $\left[S^\lambda : D^\mu \right] = \left[S^\lambda \downarrow_{-3} : D^{(\mu_1 - 1, \mu_2)} \right]$.

With these results to hand, an induction and some intricate combinatorics give the decomposition numbers for $S^{(\lambda_1, \lambda_2, \lambda_3)}$ when $\lambda_3 < p$ and these are expressed in terms of two parameters f_p and θ_p which are defined as follows.

- $f_p(n, m) = 1$ or 0, respectively depending on whether or not $n + 1$ contains m to base p.

- $\theta_p(n, m) = \left[S^{(n-m-1, m, 1)} : D^{(n)} \right]$.

The inductive hypothesis is the case $\lambda_3 = 1$ and the results, obtained by To Law [5], are given in an Appendix of [1].

4.2. **Theorem** [cf. Theorem B, 2] *Assume that* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *and* $\mu = (\mu_1, \mu_2, \mu_3)$ *are partitions of* n, *with* μ *p-regular and* $1 \leq \lambda_3 \leq p - 1$.

(i) *If* $\mu_2 \leq \lambda_3 - 2$ *and* $\mu_3 = 0$, *then*

    a) $d_{\lambda\mu} = \theta_p(\mu_1 - \lambda_3 + 1, \lambda_2)$ *if* $\mu_1 - \lambda_3 + 2 \equiv 0$ *and* $\lambda_2 + 1 \equiv 0 \mod p$;

    b) $d_{\lambda\mu} = \theta_p(\mu_1 - \lambda_3 + 1, \lambda_2 - \mu_2)$ *if* $\mu_1 - \lambda_3 + 2 \equiv 0$ *and* $\lambda_2 \equiv \mu_2 \mod p$

    c) $d_{\lambda\mu} = 0$, *otherwise.*

(ii) *If* $\mu_2 = \lambda_3 - 1$ *and* $\mu_3 = 0$ *and* $1 \leq r \leq p - 1$, *then*

    a) $d_{\lambda\mu} = \theta_p(\mu_1, \lambda_2)$ *if* $\lambda_2 + 1 \equiv 0$ *or* $\lambda_1 + 2 \equiv 0$, *and*

        $n \equiv t \in \{2\lambda_3 - 3, 2\lambda_3 - 2, ..., \lambda_3 - 3 + p\} \mod p$

    b) $d_{\lambda\mu} = \theta_p(\mu_1 - r, \lambda_2)$ *if* $\lambda_2 + 1 \equiv 0$ *and* $\lambda_1 + 2 \equiv r \not\equiv$ *and*

        $n \equiv t \in \{\lambda_3 - 2, \lambda_3 - 1, ..., 2\lambda_3 - 4\} \mod p$

    c) $d_{\lambda\mu} = \theta_p(\mu_1 - r, \lambda_2 - r)$ *if* $\lambda_1 + 2 \equiv 0$ *and* $\lambda_2 + 1 \equiv r \not\equiv 0$ *and*

        $n \equiv t \in \{\lambda_3 - 2, \lambda_3 - 1, ..., 2\lambda_3 - 4\} \mod p$

    d) $d_{\lambda\mu} = 0$, *otherwise.*

(iii) *If* $\lambda_3 - 1 < \mu_2$ *and* $\mu_3 = 0$, *and* $\lambda_1 + 2 \equiv 0$ *and* $\lambda_2 + 1 \not\equiv 0 \mod p$, *then*

    a) $d_{\lambda\mu} = f_p(\mu_1 - \mu_2, \lambda_2 - \mu_2)$ *if* $\mu_2 \not\equiv \lambda_3 - 1 \mod p$;

    b) $d_{\lambda\mu} = f_p(\mu_1 - \mu_2, \lambda_2 - \mu_2) + \Sigma_{k=1}^{\lambda_3 - 1} f_p(\mu_1 - \mu_2, \lambda_2 + k - \mu_2)$ *if* $\mu_2 \equiv \lambda_3 - 1$ mod p;

c) $d_{\lambda\mu} = 0$ *otherwise.*

(iv) If $\lambda_1 + 2 \not\equiv 0$, $\lambda_2 + 1 \equiv 0 \mod p$, $0 < \mu_2$ *and* $\mu_3 = 0$ *then*

  a) $d_{\lambda\mu} = f_p(\mu_1 - \mu_2, \lambda_2 + \lambda_3 - \mu_2)$ *if* $\mu_2 \not\equiv \lambda_3 - 1$ *and* $\lambda_2 + \lambda_3 \leq \lambda_1$;

  a) $d_{\lambda\mu} = f_p(\mu_1 - \mu_2, \lambda_2 + \lambda_3 - \mu_2) + \Sigma_{k=1}^{\lambda_3 - 1} f_p(\mu_1 - \mu_2, \lambda_2 + k - \mu_2)$ *if*
$\mu_2 \equiv \lambda_3 - 1$ *and* $\lambda_2 + \lambda_3 \leq \lambda_1$;

  d) $d_{\lambda\mu} = 0$, *if* $\lambda_2 + \lambda_3 > \lambda_1$.

(v) If $\lambda_1 + 2 \equiv \lambda_2 + 1 \equiv 0 \mod p$, $0 < \mu_2$ *and* $\mu_3 = 0$ *then*

  a) $d_{\lambda\mu} = f_p(\mu_1 - \mu_2, \lambda_2 - \mu_2) = f_p(\mu_1 - \mu_2, \lambda_2 + \lambda_3 - \mu_2)$ *if* $\mu_2 \not\equiv \lambda_3 - 1$;

  b) $d_{\lambda\mu} = f_p(\mu_1 - \mu_2, \lambda_2 - \mu_2) + f_p(\mu_1 - \mu_2, \lambda_2 + \lambda_3 - \mu_2)$ *if* $\mu_2 \equiv \lambda_3 - 1$.

(vi) If $\mu_3 \neq 0$ *then* $d_{\lambda\mu} = \left[S^{(\lambda_1 - \mu_3, \lambda_2 - \mu_3, \lambda_3 - \mu_3)} : D^{(\mu_1 - \mu_3, \mu_2 - \mu_3)} \right]$.

(vii) $d_{\lambda\mu} = 0$, *otherwise.*

References

[1] G. D. James & A. L. Williams, *Decomposition numbers of symetric groups by induction,* J. Algebra **228**, (2000), 119-142.

[2] Al. L. Williams, *Decomposition numbers for some three-part partitions,* J. Algebra (submitted).

[3] A. S. Kleschev, *Branching rules for symmetric groups and apllications,* in ,,*Algebraic Groups and their Applications*" (R.W. Carter and J. Saxl, Eds.) pp. 103-130, Kluwer, Dordrecht/Boston/London, 1998.

[4] K. Erdmann, *Decomposition numbers for symmetric groups and composition factors of Weyl modules,* J. Algebra **180** (1996), 316-320.

[5] K.W. To Law, *Results on Decomposition Matrices for Symmetric Groups,* Ph. D. thesis, University of Cambridge, 1983.

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