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ABOUT THE GROUPS ALL OF WHOSE CHARACTERS ARE RATIONAL VALUED ON THE 2'-ELEMENTS

ION ARMEANU

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ABSTRACT. In this note we shall study the structure of the finite groups all whose irreducible characters are rational valued on the conjugacy classes of odd order elements.

1. ELEMENTARY RESULTS

The notations and terminology are standard (see for example [3] and [5]). All groups will be finite.

DEFINITION. *A $2'$-r group is a group all whose irreducible Characters are rational valued on the conjugacy classes of $2'$-elements.*

PROPOSITION 1. *The following statements are equivalent:*
i) G is a $2'$-r group.
ii) For every $2'$-element g of G the generators of the cyclic group $< g >$ are conjugate in G.
iii) $N_G(< g >)/C_G(g) \simeq Aut(< g >)$ for every $2'-$element $g \in G$.

Proof. Let $g \in G$ and $\chi \in Irr(G)$. Let ω be a primitive n-root of the unity where $n = |G|$ and $G(\omega) = Gal(\mathcal{Q}(\omega), \mathcal{Q})$. Then $G(\omega)$ acts naturally on $Irr(G)$. For every $\sigma \in G(\omega)$ there is a $n_\sigma \in \mathcal{Z}$ such that $\sigma(\omega) = \sigma^{n_\sigma}$ and $(n, n_\sigma) = 1$. $G(\omega)$ acts on G by $g \longmapsto g^{n_\sigma}$. The generators of $< g >$ are g^{n_σ} when $\sigma \in G(\omega)$.

Suppose G is a $2'$-r group and let $g, h \in G$, $2'$-elements such that $< g >=< h >$ but g is not conjugate in G to h. Since $h = g^{n_\sigma}$ for some $\sigma \in G(\omega)$ there exists $\chi \in Irr(G)$ such that $\chi(g) \neq \chi(h)$. Thus $\chi(h) = \chi(g^{n_\sigma}) = \sigma(\chi(g)) = \chi(g)$, contradiction.

Reciprocally, let $g \in G$ be a $2'$-element and $\chi \in Irr(G)$ and suppose that g is conjugate in G with the generators of $< g >$. Then, for every $\sigma \in G(\omega)$, $\sigma(\chi(g)) = \chi(g^{n_\sigma}) = \chi(g)$ hence $\chi(g)$ is rational.

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DEFINITION. *Let G be a group. An element g of G is p-central for a rational prime p if there exists a Sylow p-subgroup S of G such that $S \subset C_G(g)$.*

PROPOSITION 2. *Let G be a $2'$-r group. Then:*
i) There is no 2-central $2'-$elements in G.
ii) $Z(S)$ is an abelian 2-group.
iii) If G is abelian then G is a 2-group.
iv) G/G' is an abelian 2-group.
v) $O^2(G) = O^2(G')$.
vi) Let p be an odd prime and P be a Sylow p-subgroup of G. Then $P < [P, G]$.

Proof. i) Let $g \neq 1$ be a 2-central $2'$-element of G. Then for some $S \in Syl^2(G)$ $S \leq C(x) \leq N(< x >)$ hence 2 does not divide $| Aut(< x >) |$, contradiction.

vi) The focal subgroup of P in G, $Foc(P)$, is generated by the commutators $[g, h]$, where g is in P and h is in G which lie in P. Let $f : G \longrightarrow P/Foc(P)$ the transfer map. Then f is onto and therefore $P/Foc(P)$ is an abelian $2'-$group. Since $p \neq 2$, it follows that $P = Foc(P)$. Hence $P = Foc(P) < P \cap [P, G] < P$ thus $P < [P, G]$.

COROLLARY 3. *Let G be a $2'-r$ group and S be a Sylow 2-subgroup of G. Then $C_G(S) = Z(S)$.*

Proof. The elements of $C(S)$ are 2-central.

DEFINITION. *An element $g \in G$ is real if there is an element $t \in G$ such that $x^t = x^{-1}$.*

LEMMA 4. *Let G be a solvable group and S be a Sylow 2-subgroup of G. Then the odd order elements of $N_G(S)$ are non-real.*

Proof. Induction on the order of G. Let H be a minimal subgroup of G. Since G is solvable, H is an elementary abelian p-group, for a rational prime p. If $x \notin H$ the image of x in G/H is non-real by induction hence x is non-real. If $x \in H$ we have that $[x, S] \subseteq S \cap H = 1$. Since $C_G(x)$ contains a Sylow 2-subgroup of G, we have that the order of $N_G(< x >)/C_G(x)$ is odd hence x is non-real.

By Lemma 4 follows immediately:

PROPOSITION 5. *Let G be a solvable $2'-r$ group and $S \in Syl_2(G)$. Then $N_G(S) = S$.*

COROLLARY 6. *Let G be a $2'-r$ group with $S \in Syl_2(G)$ and let $S < W < G$. Then $N(W) = W$.*

Proof. Let $g \in N(W)$. Then S and S^g are Sylow 2-subgroups of W and hence there is a $w \in W$ such that $S^g = S^w$. Then $w^{-1}g \in N(S) = S < W$ hence $g \in W$.

COROLLARY 7. *Let G be a $2' - r$ group and $P \in Syl_p(G)$, with p an odd prime. Then $N(P) \neq P$.*

Proof. Let x be an element of order p in $Z(P)$. Then $P < C(x) < N(x)$. From Frattini argument, $N_G(x) = C(x)N_{N(x)}(P)$. If $N_G(P) = P$, then $N_G(x) = C_G(x)P = C_G(x)$ which contradicts $p - 1 \neq 1$.

2. Wreath products

DEFINITION *Let H be a permutation group on the set W and let x be in H. The cyclic group $< x >$ acts on W. Denote by $O(x,w)$ the orbit of w. We shall say that H is $2'r$-transversal if for every $2'-$element $x \in H$, and m an integer relatively prime to $| x |$ there exist some element $h \in H$ such that $x^h = x^m$ and $hO(x,w) = O(x,w)$ for every $w \in W$.*

Using techniques of [4] we shall prove the next two statements.

PROPOSITION 1. *Suppose $G \wr H$ is a $2' - r$ group. Then both G and H are $2' - r$ groups.*

Proof. By the definition of the wreath product (see [4]), H is a factor group of $G \wr H$, hence H is a $2' - r$ group.

Let $g \in G$ be a $2'$-element. Define $\pi : W \longrightarrow G$ by setting $\pi(w) = g$ for every $w \in W$. Then $1^*(\pi)(w) = \pi(w) = g$, therefore $1^*(\pi) = \pi$. Hence $| (\pi; 1) |=| g |$ and $(\pi; 1)$ is an $2'-$element in $G \wr H$. Then for every positive integer m relatively prime to $| g |$ there exists $(u; h)$ in $G \wr H$ such that $(u; h)(\pi; 1)(u; h)^{-1} = (\pi; 1)^m$. Hence $u\pi_h u^{-1} = \pi^m$. Since $\pi_h = \pi$ it follows that $u(w)gu(w)^{-1} = \pi(w)^m = g^m$ for every w in W. Hence g is conjugate to g^m.

REMARK. *Hence to construct new $2' - r$ groups by wreathing groups we must consider only $2' - r$ groups.*

In general, it is not true that the wreath product of two $2' - r$ groups is a $2' - r$ group. For example let S_3^{reg} be the left regular permutation representation of the symmetric group S_3. Then $\mathcal{Z}_2 \wr S_3^{reg}$ is not a $2' - r$ group.

THEOREM 2. *Let G be a $2' - r$ group and (H, W) be a $2'r$-transversal group. Then $G \wr (H, W)$ is a $2' - r$ group.*

Proof. Let $(f; x)$ in $G \wr H$ be a $2'$-element and let m be a positive integer relatively prime to $| (f; x) |$. We have to show that $(f; x)^m$ is conjugate to $(f; x)$. Clearly $(f; x)^m = (ff_x...f_{x^{m-1}}; x^m)$. Denote $g = ff_x...f_{x^{m-1}}$. Since H is $2'r$-transversal, there is an element h in H such that $x^h = x^m$ and $hO(x,w) = O(x,w)$ for every w in W. Then $(1; h)(f; x)^m(1; h)^{-1} = (g_h; x)$.

We shall prove now that (g_h) is conjugate to $(f; x)$. It is straightforward to prove that $x^*(g_h)(w) = (x^*(f))(h^{-1}(w)))^m$ for every w in W. Then $h^m(w) \in O(x,w)$ and therefore $(x^*(f)(h^{-1}(w)))^m$ is conjugate to $x^*(f)(w)$. Hence $x^*(g_h)(w)$ is conjugate to $(x^*(f)(w))^m$ and since G is a $2' - r$ group and $x^*(f)(w) \in G$ it follows that $x^*(g_h(w)$ is conjugate to $x^*(f)(w)$.

We shall construct now a map $\mu : W \longrightarrow G$ such that

$$(\mu, 1)(g_h, x)(\mu, 1)^{-1} = (f, x).$$

Let $W = O(x, w_1) \cup ... \cup O(x, w_q)$ be the pairwise disjoint factors decomposition. Let $| O(x, w_i) | = s_i$. By the previous, there exists $\mu(w_i) \in G$ such that

$$\mu(w_i)x^*(g_h)(w_i)\mu(w_i)^{-1} = x^*(f)(w_i)$$

for $i = 1, \dots q$. We define μ on all W by setting

$$\mu(x^{-k}(w_i)) = \{f(w_i) ... f_{k-1}(w_i)\}^{-1}\mu(w_i)\{g_h(w_i) ... g_h(x^{-(k-1)}(w_i)\}$$

for every $1 \leq k \leq s_i - 1$.

It remains to verify that $(\mu, 1)(g_h, x)(\mu, 1)^{-1} = (f, x)$. This follows if we prove that $\mu(w)g_h(w)\mu(x^{-1}(w))^{-1} = f(w)$ for every w in W. For $w = w_i$ this is obvious. In general, write $w = x^{-k}(w_i)$ and straightforward follows the statement.

COROLLARY 3. *Let G be a $2' - r$ group. Then $G \wr S_n$ is a $2' - r$ group.*

THEOREM 4. [1] *Every group G can be embedded in a symmetric group S such that if $x, y \in G$ are conjugate in S then $< x >$ and $< y >$ are conjugate in G.*

COROLLARY 5. *A group G can be embedded in a symmetric group S such that the $2'-$elements of G do not fusion in S iff G is a $2' - r$ group.*

Proof. Let G be a $2' - r$ group embedded in a symmetric group S as in Theorem 4. Then $x^q \sim_G y$ for some positive integer q. Since $x \sim_G x^q$, the $2'-$elements of G do not fusion in S.

Reciprocally, let G be embedded in S such that the $2'-$elements of G do not fusion in in S. Let $\chi \in Irr(G)$. For every $2'-$element x of G we have $\chi^S(x) = e\chi(x)$ with e a positive integer. Then $\chi(x)$ is rational and the statement follows.

COROLLARY 6. *Let G be a group embedded in a symmetric group as in Theorem 4. Then there is a bijection between the conjugacy classes C of odd order elements of S with $C \cap G \neq \emptyset$ and the conjugacy classes of cyclic $2'-$subgroups of G.*

3. SCHUR INDEX FOR $2' - r$ GROUPS

DEFINITION. *Let G be a group.*

i) An element g of G is 2-regular if the order of g is odd.

ii) A conjugacy class of G is 2-regular if its elements are 2-regular.

We shall denote by n_2 the number of conjugacy classes of $2'-$cyclic subgroups of G. Clearly n_2 can be compute from the character table of G.

Let $C_1 = 1, C_2, \dots, C_k$ the conjugacy classes of G and select $g_i \in C_i$. Let $K_i = \mathcal{Q}\ (\chi_1(g_i), ..., \chi_k(g_i))$ where $\chi_j \in Irr(G)$ and $s_i = [K_i : \mathcal{Q}]$. Let $a_i = [G :< g_i >]$. Then $K_i \subset \mathcal{Q}\ (\omega_{a_i})$ and $L_i = Gal(\mathcal{Q}\ (\omega_{a_i}) : K_i)$ is isomorphic to the subgroup of the classes $\overline{z}$ of $\mathcal{Z}^*_{a_i}$ such that $g_i^z \sim g_i$. Clearly, $s_i = [\mathcal{Z}^*_{n_i}; L_i]$. If R_i are representatives modulo L_i, then $card(R_i) = s_i$.

From now on we shall keep these notations.

PROPOSITION 1. *Let G be embedded in a symmetric group S such that the 2-regular elements of G do not fusion in S and let A be a 2-regular conjugacy class of S such that $A \cap G \neq \emptyset$ and $g_i \in C_i \subset A \cap G$. Then $A \cap G$ splits in G in s_i conjugacy classes.*

Proof. Let $x \in A \cap G$. Since $x \sim_S g_i$, we have $< x > \sim_G < g_i >$ hence $x \sim_G g_i^z$ for some $z \in \mathcal{Z}$. Thus $g_i \sim_G g_i^z$ for every z such that $\overline{z} \in G_i$. Hence we can choose a w such that $\overline{w} \in R_i$ and $x \sim_G g_i^w$. Therefore $A \cap G$ splits in at most $card(R_i)$ conjugacy classes in G.

Let y, z be positive integers relatively prime to the order of g_i such that $\overline{y} \not\equiv \overline{z} (mod G_i)$. Then g_i^y and g_i^z are conjugate in S but not in G. Thus $A \cap G$ splits in exactly s_i conjugacy classes in G.

COROLLARY 2. *They are $\sum_j (1/t_j)$ 2-regular conjugacy classes A of S such that $A \cap G \neq \emptyset$, where $t_j = s_i$ for g_i 2-regular classes.*

COROLLARY 3. $n_2 = \sum_j (1/t_j)$.

THEOREM 4. *Let G be a $2' - r$ group. Then G has at least n_2 irreducible characters of odd absolute Schur index.*

Proof. Let G be embedded in a symmetric group S as in Theorem 2.3. Let A_i, $i = 1, ..., n_2$ the 2-regular conjugacy classes of S which intersects G and choose $g_i \in A_i \cap G$. Let $\mu_j \in Irr(S)$. Then the 2-rank of the matrix $(\mu_j(g_i))$ is n_2 and hence they are n_2 irreducible characters $\mu_1, ...\mu_{n_2}$ of S such that $det(\mu_j(x_i)) \not\equiv 0 (mod 2)$. If $\mu_{j|G}(g_i) = \sum_{q=1}^{k} b_{jq}\chi_q(g_i)$, where $\chi_q \in Irr(G)$, let $U = (\mu_j(g_i))$, $V = (b_{jq})$, $W = (\chi_q(g_i))$. Then $U = VW$ and since $det U \not\equiv 0 (mod 2)$ there are n_2 values of q such that $det(b_{jq}) \not\equiv 0 (mod 2)$ and $det(\chi_q(g_i)) \not\equiv 0 (mod 2)$. We can suppose that $q = 1, ..., n_2$ are these values. For every q there is a j such that b_{jq} is odd. Then χ_q appears with odd multiplicity in a $\mathcal{Q}$-representation of G and the absolute Schur index $m_{\mathcal{Q}}(\chi_q)$ is odd.

Using methods of Gow [2] we can improve this result.

THEOREM 5. *Let G be a $2' - r$ group. Then G has at least $n_2 + 1$ irreducible characters of odd absolute Schur index.*

Proof. Suppose G has exactly n_2 irreducible characters $\chi_1, ..., \chi_{n_2}$ of odd absolute Schur index. Let $P \in Syl_2(G)$ and $\nu \in Irr(P)$ be a non-trivial linear character such that $\nu^2 = 1$. Let $\tau = \nu^G$ and $\mu = 1_P^G$. We write

$$\tau = \sum a_i\chi_i + \sum b_j\varphi_j$$

$$\mu = \sum c_i\chi_i + \sum d_J\varphi_j$$

where $\varphi_j \in Irr(G)$ have even absolute Schur index. Since τ and μ are characters of $\mathcal{Q}$-representations it follows that b_j and d_j must be even. Let g_j a 2-regular element of G. Since the conjugacy class of g_j does not intersect P we have

$\tau(g_j) = \mu(g_j) = 0$. Hence

$$\sum_{i=1}^{n_2}(a_i - c_i)\chi_i(g_j) \not\equiv 0(mod2).$$

Thus $a_i \not\equiv c_i(mod2)$ and by Frobenius reciprocity theorem $a_1 = 0$ and $c_1 = 1$, contradiction.

PROPOSITION 6. *Let G be a* $2' - r$ *group. Then* $\mathcal{Q}\ (\chi_1, ..., \chi_k) \subset \mathcal{Q}\ (\omega_{2^a})$.

Proof. Obvious.

THEOREM 7. *Let G be a* $2' - r$ *group having s 2-regular conjugacy classes with* $g_1, ..., g_s$ *the corresponding representatives. Then there exist s rational valued irreducible characters of G which satisfy* $det\chi_i(g_i) \neq 0$, $1 \leq i, j \leq s$.

Proof. By Proposition 6 all characters of G take values in $\mathcal{Q}\ (\omega_{2_n})$. Suppose the statement is not true. Then we can find $a_1, ..., a_s \in \mathcal{Z}$ not all zero, such that for every $\chi \in Irr(G)$, we have $\sum_{i=1}^{s} a_i\chi(g_i) = 0$. Thus for every $\mathcal{Z}$-linear combination of irreducible characters τ of G, $\sum_{i=1}^{s} a_i\tau(g_i) = 0$. Let $g = g_t$, $X =< g >$ and $q =| X |$. Let $P \in Syl_2(C(g))$ and $W = XP$. X has q linear characters $\pi_1, ..., \pi_q$ which extends to linear characters $\mu_1, ..., \mu_q$ of W. Let $\eta = \sum_{i=1}^{q} \pi_i(g)\mu_i^G$. Then $\eta(g_j) = 0$ if $g_j \neq g$ and $\eta(g) \neq 0$. Let $H = Gal(\mathcal{Q}$ $(\omega_{2^n}); \mathcal{Q})$. Clearly η is invariant to H and H acts naturally on $Irr(G)$. The number of characters in an orbit is a power of 2 and by preceding proposition the rational valued irreducible characters are exactly the fix points of this action. Let β_r be the sum of the characters in the r orbit. Then $\eta = \sum b_r\beta_r$. It is clear that $\beta_r(g_i) = 0$ unless the r orbit contains a single character. Hence $b - r \neq 0$ only for the orbits of rational valued characters. Thus $\sum_1^s a_i\eta(g_i) = a_t\eta(g) \neq 0$. Since η is a $\mathcal{Z}$-linear combination of irreducible characters of G, this contradicts the assumption.

COROLLARY 8. *A* $2' - r$ *group G possesses at least* $n_2 + 1$ *rational valued irreducible characters of absolute Schur index 1.*

Proof. By Brauer-Speiser Theorem (see [3]) the rational valued irreducible characters have Schur index at most 2. From preceding theorem and repeating the arguments of Theorems 4 and 5 follows the statement.

REMARK. *Since* S_3 *possesses exactly* $n_2 + 1$ *rational valued irreducible characters of absolute Schur index 1,* $n_2 + 1$ *is the best lower rank.*

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MODULE STRUCTURE OF THE TENSOR PRODUCT OF SIMPLE ALGEBRAS OF KRULL DIMENSION 1

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Throughout let K be a (fixed) field, "module" means a left one, the tensor product $\otimes = \otimes_K$ is over K.

The **Weyl** algebra $A_n = A_n(K)$ of degree n over K is the associative K-algebra with identity generated by the $2n$ indeterminates $X_1, ..., X_n, \partial_1, ..., \partial_n$, subject to the relations:

$$[X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0 \text{ for } i \neq j,\ [\partial_i, X_i] = 1 \text{ for all } i,$$

where $[x, y] = xy - yx$.

If K has the characteristic zero, the Weyl algebra A_n is a simple Noetherian domain.

The next theorem is the key result of [St], the other results of [St] are easy corollaries of this one.

Theorem 0.1. *Any right ideal of $A_n(K)$, char $K = 0$, can be generated by two elements. Moreover, if a right ideal $I = aA_n + bA_n + cA_n$ and $d \neq 0 \in A_n$, then there exist f and $g \in A_n$ such that $I = (a + cfd)A_n + (b + cgd)A_n$.*

For the first Weyl algebra A_1 the theorem was proved in [Di]. The n'th Weyl algebra $A_n = A_1 \otimes \cdots \otimes A_1$, ($n$ times), is the tensor product of the first Weyl algebras and A_1 is a simple Noetherian domain with restricted minimum condition (so of Krull dimension $K.dim\, A_1 = 1$). A ring R has the **restricted minimum condition** if R is not Artinian but every proper left or right factor module of R is Artinian.

In this paper we try to give a positive answer to the following question and realize the following idea of Stafford ([St], Question (3)):

> *"find a more general class of rings for which the results of section 3 ([St]) still hold. Given a simple Noetherian ring that is "build up" from simple*

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rings " with restricted minimum condition (see p. 430, [St]), "it ought to be possible to repeat the arguments of this paper".

On this way, the concepts of strongly simple and special algebras naturally arise.

Definitions. We say that a K-algebra A is **strongly simple** if for any K-linearly independent elements $a_1, \dots, a_r$:

$$A(a_1, \dots, a_r)A = A^{(r)},$$

where $A^{(r)} = A \oplus \cdots \oplus A$, r times. It is clear that a strongly simple algebra is simple.

An algebra A is called **special** if for any two finite dimensional subspaces V and U of A such that $VU \neq 0$ there exist bases $\{a_i\}$ and $\{b_i\}$ of V and U respectively such that the rank of the set of vectors $\{a_i b_j\}$ is greater than the rank of $\{a_i b_j\} \backslash \{a_1 b_1\}$.

As Lemma 2.2 shows, many "good" algebras are special (for example, the Weyl algebra A_n, the universal enveloping algebra of a Lie algebra, but the matrix algebras $M_n(K)$, $n \geq 2$, are not (Proposition 2.3)).

Let R be a domain, the full quotient ring (if exists) is denoted by $\mathcal{D}(R)$ and the *opposite* ring by R^o.

Let S be a subring of R. In order to simplify the notation the localization of R at $S \backslash \{0\}$ (if it exists) is denoted by $S^{-1}R$ rather than the more correct $(S \backslash \{0\})^{-1}R$.

The next theorem gives (partly) an answer to the Stafford's question, its proof is given in Section 3.

Theorem 0.2. *Let A_i $(i = 1, \dots, n)$ be a central special simple algebra with restricted minimum condition. Let $A = A_1 \otimes \cdots \otimes A_n$. Suppose that there exists a subalgebra C of A which is the tensor product $C = C_1 \otimes \cdots \otimes C_n$ of subalgebras C_i of A_i such that*

1. *for $i = 1, \dots, n-1$ there exists the localization $S_i := T_i^{-1} T_{i-1}$ of the $T_{i-1} = C_1 \otimes \cdots \otimes C_{i-1} \otimes A_i \otimes \cdots \otimes A_n$ at T_i which has the restricted minimum condition and there exists $R_i = T_i^{-1} A$;*
2. *for $i = 2, \dots, n$ the quotient division rings $E_i := \mathcal{D}(C_1 \otimes \cdots \otimes C_{i-1})$ and $F_i := \mathcal{D}(C_1 \otimes \cdots \otimes C_{i-1} \otimes A_{i+1} \otimes \cdots \otimes A_n)$ exist and the tensor product of rings $\mathcal{D}(C_1 \otimes \cdots \otimes C_{i-1}) \otimes A_i$ has the restricted minimum condition.*

Let $I = aA + bA + cA$ be a right ideal of A and let d_1, d_2 be arbitrary nonzero elements of A. There exist f and g in A such that

$$I = (a + cfd_1)A + (b + cgd_2)A.$$

So, any finitely generated right ideal of A has no more than two generators. If, in addition, A is Noetherian, then it is true for an arbitrary right or left ideal.

Let D be a ring, $\sigma \in \mathrm{Aut}(D)$ be an automorphism of D and $a \in Z(D)$ an element of the centre of D. The **generalized Weyl algebra** (GWA) $A = D(\sigma, a)$ of degree 1 is the ring generated by D and two indeterminates X and Y subject to the relations:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \ \forall \alpha \in D, \ YX = a \text{ and } XY = \sigma(a).$$

Let the algebras $\mathcal{A}_i$, $i = 1, \dots n$, belongs to one of the two classes of generalized Weyl algebras ($K[H]$ is a polynomial ring in one variable):

1. $K[H](\sigma, a)$, $\sigma(H) = H - \mu$, $\mu \neq 0 \in K$, $char\, K = 0$;
2. $K[H, H^{-1}](\sigma, a)$, $\sigma(H) = \lambda H$, $0 \neq \lambda \in K$ is not a root of 1.

   In both cases $a \neq 0$ and for any two different irreducible multiples of a, say p and q, there is not a nonzero integer $i \in \mathbf{Z}$ such that the maximal ideals of $K[H]$ and $K[H, H^{-1}]$ respectively to case 1 or 2 generated by p and $\sigma^i(q)$ coincide.

It is shown in Section 3 that an algebra

$$A = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \tag{0.1}$$

satisfies the conditions of Theorem 0.2 (Proposition 3.2). If all $\mathcal{A}_i = K[H](\sigma, a = H)$, $\sigma(H) = H - 1$, the algebra A is isomorphic to the Weyl algebra A_n. In [Bav 1] many results of [St] are carried over to the algebra A with all $\mathcal{A}_i$ from the first class and all $\mu = 1$ (only sketches of proofs are given).

Corollaries of Theorem 0.2 are gathered in Section 4, for many of them the proofs are omited since they are the same as in [St]. Name some of them: *let an algebra A be as in Theorem 0.2 and M be a finitely generated A-module.*

- *Then $M \simeq M' \oplus A^{(s)}$, where M' is a module with rank $M' \leq 1$. If M is torsionfree, then M' is isomorphic to a left ideal of A.*
- *Suppose that the rank $M \geq 2$ and $M \oplus A \simeq N \oplus A$ for some module N. Then $M \simeq N$.*

In order to make this paper more accessible (for the reader who read [St]) and to clarify the similarity and difference of this paper and of [St], the author try to keep the notations as well as the way of thinking of [St]. So some parts of proofs of [St] are included practically without changes.

1. Preliminaries

A submodule N of M is called **essential** if it intersects non-trivially any nonzero submodule of M. If a right (or left) ideal I is an essential right (or left) submodule of R it is called an **essential right (or left) ideal.** A ring R is **essential** if any nonzero left or right ideal is essential.

Lemma 1.1. *Let A be a ring every nonzero right ideal of which is essential. Then the set $J_r(A) := \{x \in A \mid ann_r(x) \neq 0\}$ is an ideal of A where $ann_r(x) = \{a \in A \mid xa = 0\}$ is the annihilator right ideal of x.*

Proof. Set $J = J_r$ for short. It is clear that $AJ \subseteq J$. If $u, v \in J$, then by assumption $ann_r(u) \cap ann_r(v) \neq 0$, so $u + v \in J$ and $J + J \subseteq J$, i.e. J is a left ideal of A. So it remains to show that $JA \subseteq J$. Let $x \in J$ and $y \in A$ such that $xy \neq 0$. There are two possibilities: the first $y\, ann_r(x) = 0$. Since $ann_r(x) \neq 0$, we conclude that $y \in J$, thus $xy \in AJ \subseteq J$. The second, $y\, ann_r(x) \neq 0$. All right nonzero ideals of A are essential, so $y\, ann_r(x) \cap ann_r(x) \neq 0$, and there exists $z \neq 0 \in ann_r(x)$ such that $yz \neq 0 \in ann_r(x)$. It follows from $xyz = 0$ that $JA \subseteq J$. ∎

Corollary 1.2. *Let A be a simple ring every right and left nonzero ideal of which is essential. Then A is a domain.*

Proof. It follows from Lemma 1.1 and the simplicity of A. ∎

Corollary 1.3. *Let A be a simple ring with restricted minimum condition. Then each nonzero left or right ideal of A is essential and (by Corollary 1.2) A is a domain.*

Proof. Let J be a nonzero left ideal of A. If J intersects trivially a nonzero left ideal I, $J \cap I = 0$, then $J \oplus I \subseteq A$ and I can be seen as a submodule of the factor module A/J of finite length (A with restricted minimum condition). Thus the length $l_A(I) < \infty$ and $l_A(A) = l_A(A/I) + l_A(I) < \infty$ which contradicts non-artinianity of A. So, J is essential. By symmetry any nonzero right ideal is essential. ∎

The next two results are folklore.

Proposition 1.4. *Let A be a simple algebra, I be a left maximal ideal of A and J be a left ideal such that $I \cap J \neq 0$. Suppose that a sequence of modules $0 \to A/I \to M \to A/J \to 0$ is exact. Then the module M is cyclic.*

Proof. As a vector space M can be decomposed into a direct sum $M = A/I \oplus A/J$ where an element $a \in A$ acts on M as $a_M = \left(\begin{pmatrix} a & \rho(a) \\ 0 & a \end{pmatrix}\right)$ for some map $\rho : A \to Hom_K(A/J, A/I)$. Set $j = I \cap J$, $u = 1 + I \in A/I$ and $v = 1 + J \in A/J$.

If $\rho(j)v \neq 0$, then $\omega = u + v$ is a generator of M. In fact, $j\omega = \rho(j)v \neq 0 \subseteq A/I$, thus $A\omega \supseteq A/I$ and finally $A\omega = M$.

Suppose that $\rho(j)v = 0$. Since $j \neq 0$ and A is simple, $jA = A$, so there exists a nonzero element $b \in A$ such that $jb \not\subseteq I$. Then $\omega = bu + v$ is a generator of M. In fact, $j\omega = jbu \neq 0 \subseteq A/I$, thus $A\omega \supseteq A/I$ and finally $A\omega = M$. ∎

Corollary 1.5. *Let A be a simple algebra which is not a left Artinian module, i.e. $l_A(A) = \infty$. Then any A-module of finite length is cyclic.*

Proof. Let M be an A-module of finite length $l = l_A(M) < \infty$. We use the induction on l. The case $l = 1$ is clear. Suppose that $l > 1$ and it is true for all M such that $l_A(M) < l$. Choose N to be a simple submodule of M, since $l_A(M/N) = l - 1$, by induction, M/N is cyclic. There exist left ideals I and J of A such that a sequence

$$0 \to N = A/I \to M \to M/N = A/J \to 0$$

is exact (I is maximal). Then $I \cap J \neq 0$, otherwise J can be seen as a submodule of A/I, a contradiction ($\infty = l_A(J) \leq l_A(A/I) < \infty$). Now the result follows from Proposition 1.4. ▪

A module M over a K-algebra A is called **Schurian** if $End_A(M) = K$.

A module M is **strongly simple** if for any K-linearly independent elements $m_1, \dots, m_r \in M$:

$$A(m_1, \dots, m_r) = M^{(r)}.$$

Let an abelian group M be both an R- and an S-module. The module ${}_RM$ is **dense** in ${}_SM$ if for any finitely many elements $m_1, \dots, m_s \in M$ and any $s \in S$ there exists $r \in R$ such that

$$sm_i = rm_i \text{ for all } i = 1, \dots, s.$$

Let M be a left R-module. Set R' for the endomorphism ring $R' = End({}_RM)$ of the module M. Then ${}_RM_{R'}$ is an $R - R'$-bimodule:

$$(rm)r' = r(mr') \text{ for all } r \in R, m \in M, r' \in R'.$$

Set R" for the endomorphism ring $R" = End(M_{R'})$ of the right R'-module M, then ${}_{R"}M_{R'}$ is an $R" - R'$-bimodule. The map

$$R \to R", \; a \to (a_M : m \to am), \; m \in M,$$

is a ring monomorphism and we shall identify R with its image in R".

(The density theorem) *Each semisimple module ${}_RM$ is dense in ${}_{R"}M$.*

Theorem 1.6. 1. *A simple Schurian module is strongly simple.*
2. *A simple central algebra is strongly simple.*

Proof. **1** It follows immediately from the density theorem.
2 Set Λ for $A \otimes A^o$. It is clear A is a simple Schurian Λ-module:

$$End_\Lambda(A) \simeq Z(A) = K \;\; (A \text{ is central}),$$

so A is a strongly simple Λ-module, i.e. A is strongly simple. ▪

2. Special and strongly simple modules and algebras

Let $E = \{e_i\}$ be a set of vectors. Remind that the dimension of the vector space $< E >$ generated by E is called the *rank* of E.

A set $E = \{e_j\}$ is called **special** if there exists an element $e_i \in E$ such that the rank of $E_i := E \backslash \{e_i\}$ is smaller than the rank of E (i.e. $\dim < E_i >= \dim E - 1$). Then the element e_i is called **prime** as well as any basis $e_i, e_{i_2}, \dots, e_{i_s}, \dots$ of $< E >$. Any set of linearly independent elements is special and each element of it is prime.

An algebra A is called **special** if for any two finite dimensional subspaces V and U of A such that $VU \neq 0$ there exist bases $\{a_i\}$ and $\{b_i\}$ of V and U respectively such that the set $\{a_i b_j\}$ is special.

Lemma 2.1. *Let $m_1, \dots, m_r$ be a special set of vectors. Then $m_1, \dots, m_s$ are prime elements if and only if $\dim L_1 \cap \dots \cap L_i = n - i$ for all $i = 1, \dots, s$, where n is the dimension of the vector space V generated by all m_j and L_i is the vector space generated by all m_j except m_i.*

Proof. ($\Rightarrow$) Set $L^i = L_1 \cap \dots \cap L_i$ for $i = 1, \dots, s$, then $L^i \supseteq \{m_{i+1}, \dots, m_r\}$, thus $\dim L^i \geq n - i$.

Clear, $L^i \neq L^{i+1}$, otherwise, $m_{i+1} \in L^i = L^{i+1}$, a contradiction, so $\dim L^i = n - i$.

($\Leftarrow$) If we suppose that m_i is the first non-prime in the row $m_1, \dots, m_s$, then $L^i = L^{i-1}$, thus $L^i = L^{i-1} \cap L_i$, a contradiction. ∎

The next lemma shows that special algebras are common.

Lemma 2.2. 1. *Each subalgebra of a special algebra is special too.*

2. *Let $B = S^{-1}A$ be a localization of a special algebra A at a multiplicatively closed subset S of A consisting of regular elements. Then B is special.*
3. *Let G be a well-ordered monoid. Then the monoidal algebra KG is special. In particular, the polynomial ring $K[x_1, \dots, x_n]$ is special.*
4. *If $A = \cup_{i \geq 0}^{\infty} A_i$ $A_0 = K$, is a filtered algebra such that the associated graded algebra $\mathrm{gr}\, A$ is a special domain, then A is special too. In particular, the universal enveloping algebra $U(\mathcal{G})$ of a finite-dimensional Lie algebra $\mathcal{G}$ and the Weyl algebra A_n are special.*
5. *Let A be a special domain. Then the skew polynomial ring $A[X; \sigma, \partial]$ and the skew Laurent polynomial ring $A[X, X^{-1}; \sigma]$ are special (σ is a K-algebra isomorphism of A and ∂ is a σ-derivation of A).*

Proof. 2 The set S consists of regular elements, so the map $A \to B$, $a \to a/1$, is monic. We identify A with its image in B. Let P and Q be finite-dimensional vector subspaces of B such that $PQ \neq 0$. Then there exist s, $t \in S$ such that the vector spaces $V = sP$ and $U = Qt$ are in A. The algebra A is special, choose bases $\{a_i\}$ and $\{b_i\}$ of V and U respectively such that $\{a_i b_j\}$ is special.

Then the bases $\{s^{-1}a_i\}$ and $\{b_jt^{-1}\}$ of P and Q are such that $\{s^{-1}a_ib_jt^{-1}\}$ is special.

5 Let P be a finite-dimensional vector subspace of $R = A[X;\sigma,\partial]$. Then there exists a basis $e_1,\dots,e_r,\dots,e_k$ of P such that

1. $e_1,\dots,e_r$ are polynomials of the same degree, say p, in X and $\deg e_j < p$ for all $j > r$;
2. let $a_i \in A$ be the highest coefficient of e_i, $i = 1,\dots,r$:
$$e_i = a_iX^p + \cdots .$$
The elements $a_1,\dots,a_r$ of A are linearly independent.

For short we say that the basis $\{e_i\}$ is *good* of *degree* p and *order* s.

Let Q be a finite-dimensional subspace of R with a good basis $f_1,\dots,f_t,\dots,f_m$ of degree q and order t, i.e.
$$f_j = b_jX^q + \cdots,\ j = 1,\dots,t,$$
and $\{b_j\}$ are linearly independent in A. Let V and U be the subspaces generated by $\{a_i\}$ and $\{b_j\}$ respectively. Then $\sigma^p(V)$ has the basis $\{\sigma^p(b_j)\}$. The algebra A is a special domain, let $\{a'_i\}$ and $\{\sigma^p(b'_j)\}$ be bases of V and $\sigma^p(U)$ such that $\{a'_i\sigma^p(b'_j)\}$ is special. Without loss of generality we may suppose that all $a'_i = a_i$ and $b'_j = b_j$. Then $\{e_if_j\}$ are special too, thus R is special.

Since the skew Laurent polynomial ring $L = A[X,X^{-1};\sigma]$ is the localization of $A[X;\sigma,\partial = 1]$ at the set $S = \{1,X,X^2,\dots\}$ where X is regular in $A[X;\sigma,\partial = 1]$. So L is special by the statement **2**. ∎

A module M over a ring A is **faithful** if $aM = 0$, a in A, is possible if and only if $a = 0$.

Proposition 2.3. *Suppose that a finite-dimensional algebra A has a left faithful ideal U such that $\dim A > (\dim U)^2 - \dim U + 1$. Then A is not special. In particular, the $n \times n$ matrix ring $M_n(K)$ is not special for $n \geq 2$.*

Proof. The left ideal U is faithful, so the algebra homomorphism
$$A \to End(U),\ a \to (a_U : u \to au), \tag{2.1}$$
is monic. Choose $e_1,\dots,e_n$, $n = \dim U$ and $a_1,\dots,a_m$, $m = \dim A$ to be bases of U and A respectively, then $End(U)$ can be identified with the matrix ring $M_n(K)$ over a field K and by (2.1) A with a matrix subalgebra of $M_n(K)$, i.e. a_i is a matrix $a_i = (a_{i,\alpha\beta}) \in M_n(K)$. Denote by $a_{i,j}$ the j'th column of the matrix a_i. It is enough to show that the set $\{a_ie_j\}$ is not special (in $AU = U$). Suppose the contrary and a_1e_1 is prime. Then the rank of the set $F = \{a_ie_j\}\backslash\{a_1e_1\}$ is $n-1$, thus the rank of all columns of $a_2,\dots,a_m$, is less or equal to $n-1$ (since $a_{i,j} = a_ie_j$). Using this fact we shall show that the vectors $a_2,\dots,a_m$ are linearly dependent, a contradiction. For we need to find m^2-1 scalars $x_2,\dots,x_{m^2}$, not all of them are zero such that

$$\sum_{i=2}^{m^2} x_i a_i = 0.$$

Considering this equality as the matrix one, it is equivalent to the system of n equations:

$$\begin{cases} \sum x_i a_i e_1 = \sum x_i a_{i,1} = 0 \\ \dots \\ \sum x_i a_i e_n = \sum x_i a_{i,n} = 0. \end{cases}$$

Since $rk\, F \leq n-1$, an equation $\sum x_i a_{i,j} = 0$ is equivalent to $\sum y_k a_{k,j} = 0$ where k runs through a set of linearly independent columns of $\{a_{i,j} \mid i = 2, \dots, m^2\}$ and y_k denotes a linear combination of x_i. Thus we have no more than $n(n-1)$ linear homogeneous equations with $m^2 - 1$ indeterminates which have a nonzero solution since by assumption $n(n-1) < m^2 - 1$.

In the case of the matrix ring $A = M_n(K)$, the left ideal $U = M_n(K)E_{11} \simeq K^n$ is faithful and

$$\dim M_n(K) = n^2 > n(n-1) + 1 = \dim U(\dim U - 1) + 1 \text{ for } n \geq 2,$$

where E_{11} is the matrix unit. So, $M_n(K)$, $n \geq 2$, is not special. ∎

We say that a K-algebra A is **strongly simple** if for any K-linearly independent elements $a_1, \dots, a_r$:

$$A(a_1, \dots, a_r)A = A^{(r)}.$$

It is clear that a strongly simple algebra is simple.

Lemma 2.4. *Let A be a strongly simple algebra and B be simple. Then their tensor product $C = A \otimes B$ is a simple algebra too.*

Proof. Let us to show that for any nonzero $c \in C : CcC = C$. For write $c = \sum_1^r a_i \otimes b_i$ as a sum of linearly independent elements $\{a_i\}$ in A and $\{b_i\}$ in B. Then as an A-bimodule

$$AcA = Aa_1 A \otimes b_1 \oplus \dots \oplus Aa_n A \otimes b_n \simeq A(a_1, \dots, a_r)A = A^{(r)},$$

since A is strongly simple and $\{a_i\}$ is linearly independent. Thus

$$CcC \supseteq B(1 \otimes b_1)B = 1 \otimes Bb_1 B \ni 1 \otimes 1 = 1,$$

i.e. $CcC = C$. ∎

Lemma 2.5. *Let algebras A and B be domains and A be special strongly simple. Then*

1. *$A \otimes B$ is a domain.*

2. *If B is a division ring and $S = T^{-1}(A \otimes B)$ is a localization of $A \otimes B$ at a multiplicatively closed subset T of $A \otimes B$, then S is a simple domain and for any linearly independent elements $a_1, \ldots, a_r \in A$ and any $0 \neq t \in S$:*

$$A(a_1, \ldots, a_r)tS = S^{(r)}.$$

Proof. **1** Let u and v be nonzero elements of $C := A \otimes B$. They can be written as sums

$$u = \sum a_i \otimes b_i \text{ and } v = \sum \alpha_j \otimes \beta_j$$

with $\{a_i\}, \ldots, \{\beta_j\}$ linearly independent. The A is special, then for the vector spaces V and U generated by all $\{a_i\}$ and $\{\alpha_j\}$ respectively exist bases $\{a_i'\}$ and $\{\alpha_j'\}$ such that $\{a_i'\alpha_j'\}$ is special. Changing in a proper way b_i'th and β_j'th we may suppose that all $a_i' = a_i$ and all $\alpha_j' = \alpha_j$. Let $a_1\alpha_1, a_{i_2}\alpha_{i_2}, \ldots, a_{i_s}\alpha_{i_s}$ be a prime basis of $\{a_i\alpha_j\}$ and with $a_1\alpha_1$ prime. Since A is strongly simple there exists $\omega = \sum c_i \otimes d_i \in A \otimes A^o$ such that

$$\omega a_1\alpha_1 = \sum c_i a_1 \alpha_1 d_i = 1 \text{ and } \omega a_{i_\nu}\alpha_{i_\nu} = 0 \text{ for all } 2 \leq \nu \leq s.$$

Since $a_1\alpha_1$ is prime, it is easy to see that $\omega a_i\alpha_j = 0$ for all $(i,j) \neq (1,1)$. Then $\omega uv = 1 \otimes b_1\beta_1 \neq 0$, thus $uv \neq 0$, i.e. C is a domain.

2 It follows from **1** and Lemma 2.4 that S is a simple domain. It is enough to prove the above equality in the case $S = A \otimes B$. Write $t = \sum \alpha_j \otimes \beta_j$ with $\{\alpha_j\}$ and $\{\beta_j\}$ linearly independent. Note that any K-linear nondegenerate transformation in $S^{(r)}$ (i.e. the basis $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ is changing to $e_i' = \sum \lambda_{ji} e_j$, where $\Lambda = (\lambda_{ij}) \in GL_r(K)$) will not affect either the hypotheses or validity of the lemma. Using the same argument as above for the vector spaces V and U generated by $\{a_i\}$ and $\{\alpha_j\}$ respectively we can find an element $\omega \in A \otimes A^o$ as above, i.e. (modulo the above arguments)

$$\omega a_1\alpha_1 = 1 \text{ and } \omega a_i\alpha_j = 1 \text{ for all } (i,j) \neq (1,1).$$

Then $\omega(a_1, \ldots, a_r)t = (1, 0, \ldots, 0) \otimes \beta = (\beta, 0, \ldots, 0) \in A(a_1, \ldots, a_r)tS$ where $0 \neq \beta \in B$ is a unit. Induction now completes the argument. ∎

The next lemma is an extension of sublemma 2.4, [St], on general situation (for the reason of completeness a proof is given, it is almost the same as in [St]).

Lemma 2.6. *Let S, t and $a_1, \ldots, a_r$ be as in Lemma 2.5.(2) such that S is not Artinian. Suppose that N is a right submodule of $S^{(r)}$ of finite colength. Then there exists $f \in A$ such that*

$$N + f(a_1, \ldots, a_r)tS = S^{(r)}.$$

Proof. We use the induction on the length l of $S^{(r)}/N$. The case $l = 0$ is clear. So suppose that the lemma holds for any right submodule N' of $S^{(r)}$ with length $S^{(r)}/N' < l$ and $l(S^{(r)}/N) = l$. By Lemma 2.5 choose $g \in A$ such that

$$g(a_1, \ldots, a_r)tS \not\subseteq N.$$

Since S is not Artinian but $S^{(r)}/N$ is, choose $0 \neq v \in S$ such that $g(a_1, \dots, a_r)tv \in N$. Denote by M a right submodule of $N + g(a_1, \dots, a_r)tS$ containing N and M/N is simple. By the induction there exists $f \in A$ such that

$$M + f(a_1, \dots, a_r)tvS = S^{(r)}.$$

If $N + f(a_1, \dots, a_r)tS = S^{(r)}$ the result follows. Otherwise,

$$N + f(a_1, \dots, a_r)tvS = N + f(a_1, \dots, a_r)tS.$$

Set $J = (g + f)(a_1, \dots, a_r)tS + N$. Then

$$J \supseteq (g+f)(a_1, \dots, a_r)tvS + N = f(a_1, \dots, a_r)tvS + N = f(a_1, \dots, a_r)tS + N.$$

So $J + M = S^{(r)}$. But also $J \supseteq g(a_1, \dots, a_r)tS + N \supseteq M$. Thus $J = S^{(r)}$, as required. ∎

Lemma 2.7. *Let S, t and $a_1, \dots, a_r$ be as in Lemma 2.6 such that S with restricted minimum condition and let $0 \neq p \in S$. Then there exists $f \in A$ such that*

$$S^{(r+1)} = (pS)^{(r+1)} + (e_1 + f(a_1, \dots, a_r)t)S,$$

where $e_1 = (1, 0, \dots, 0) \in S^{(r+1)}$.

Proof. The algebra S has the restricted minimum condition, thus the right submodule $N = (pS)^{(r+1)}$ of $S^{(r+1)}$ is of finite colength. Applying Lemma 2.6 we find $f \in A$ such that

$$N + f(a_1, \dots, a_r)tpS = S^{(r)}.$$

Set $I = N + (e_1 + f(0, a_1, \dots, a_r)t)S$, then

$$I \supseteq N + (e_1 + f(0, a_1, \dots, a_r)t)pS = N + f(0, a_1, \dots, a_r)tpS =$$

$$N + (0, S, \dots, S) \ni f(0, a_1, \dots, a_r).$$

Thus I contains also e_1 and so $I = S^{(r+1)}$. ∎

3. Proof of Theorem 0.2

Theorem 3.1. *Let an algebra A be a special strongly simple domain and an algebra B be a simple domain. Let $R = T^{-1}C$ be a localization of $C = A \otimes B$ at a multiplicatively closed subset T of C such that the subalgebra S of R generated by T, T^{-1} and A equals to a localization $S = F^{-1}(A \otimes D)$ where D is the full ring of fractions of some subalgebra B_1 of B, $F \subseteq A \otimes B$ is a multiplicatively closed subset. Suppose that S has the restricted minimum condition. Let $U = S \cap C$. Choose $0 \neq p \in U$ and a, $b \in C$ and let $I = pR + aR + bR$ be the right ideal of R. Let $0 \neq d \in C$. Then there exists $f \in C$ such that $I = pR + (a + bfd)R$.*

Proof. The algebra C is the tensor product $A \otimes B$ of algebras, thus the element d can be written as

$$d = \sum_{i=1}^{m} d_i \otimes c_i$$

with $\{d_i\}$ and $\{c_i\}$ linearly independent in A and B respectively.

Consider the right ideal $J = pR + aR + \sum bd_iR \subseteq I$ and the right module $M = J/pR$. Since A is special strongly simple, by Lemma 2.5.(2) the algebra S is a simple Noetherian domain. Thus there exist nonzero elements $q_0, q_1, \dots, q_m \in S$ such that aq_0 and each $b_id_iq_i \in pS$. Each $q_i = p_it_i^{-1}$ for some $p_i \in C$ and $t_i \in T$ (in fact, all $p_i \in C \cap S = U$). Hence, ap_0 and each $b_id_ip_i \in pC \subseteq pR$. Let $x \in S$ be a common multiple in S of the p_i, i.e. $x = p_ix_i$ for some $x_i \in S$. Choose $t \in T$ such that all $q_i := x_it \in C$. Then $q := xt \in S \cap C = U$ and $q \in \cap p_iU$.

By Lemma 2.5.(1) C is a domain, thus $R = T^{-1}C$ is too as well. By the choice of q the module M is a homomorphic image of

$$N = aR/aqR \oplus bd_1R/bd_1qR \oplus \cdots \oplus bd_mR/bd_mqR$$

and, hence, (R is a domain) of $D := (R/qR)^{(m+1)}$. By Lemma 2.6 the right S-module $Q := (S/qS)^{(m+1)}$ is cyclic, moreover, there exists $f \in A$ such that the element

$$(1, fc_1, \dots, fc_m) + (qS)^{(m+1)}$$

is a generator. Consider the map

$$\varphi : P \to Q,\ x + (qS)^{(m+1)} \to x + (qR)^{(m+1)}.$$

Using φ we can see that the R-homomorphism

$$Q \otimes_S R \to P,\ q \otimes r \to \varphi(q)r,$$

is epic. Thus M is cyclic too, moreover,

$$J = pR + (a + \sum bd_ifc_i)R = pR + (a + bf \sum d_ic_i)R = pR + (a + bfd)R.$$

If $J = I$, it is nothing to prove. Otherwise, since by Lemma 2.4 C is a simple ring, for each $i : Cd_iC = C$, or there exist elements $s_{ij},\ t_{ij} \in C$ such that $\sum_j s_{ij}d_it_{ij} = 1$, hence, $\sum s_{ij}dC = C$. Let $s_1, \dots, s_t$ be the elements of the set $\{s_{ij}\}$, then $\sum_{k,i} s_kd_iC = C$. So,

$$I = pR + aR + \sum bs_1d_iR + \cdots + \sum bs_td_iR.$$

Doing as above t times with B replaced by bs_k on the k'th occasion, we obtain $f_1, \dots, f_k \in A$ such that $I = pR + (a + bgd)R$ where $g = \sum s_kf_k$. ∎

3.1. **Proof of Theorem 0.2.**

Remark. By Corollary 1.3 all A_i are *domains*, thus by Theorem 1.6 all A_i are *strongly simple*, by Lemmas 2.4 and 2.5 A is a *simple domain* and by (1) the localization $T_1^{-1}A = T_1^{-1}T_0$ is a Noetherian ring, so the quotient division ring $\mathcal{D}(A)$ of A exists.

Proof. If one of the a, b, c is zero it is nothing to prove. So we assume that each of them is nonzero. According to [St] replacing a by $a + cfd_1$ and b by $b + cgd_2$ for some f and $g \in A$ will be called a **refinement** of a and b (it does not change the hypotheses of the theorem). As we have seen above the ring A has the quotient division ring $\mathcal{D}(A)$, thus any nonzero left or right ideal of A is essential. Thus,

$$cq \in aA + bA \text{ for some } q \neq 0 \in A \tag{3.1}$$

and to finish the proof it is sufficient to show that there exists $q \neq 0 \in K$ such that (3.1) holds. It will be done by two steps. The first step is to show, by using the localizations S_i, that there exist refinements of a and b such that (3.1) holds for some $q \in C$. The second, by the hypothesis 2, there exists a refinement of a and b such that (3.1) holds for some $q \neq 0 \in C_1$ and then for $q \neq 0 \in K$. Then the results follows from Theorem 3.1.

Step 1. So suppose that, after possibly refining a and b, (3.1) holds for some $q \neq 0 \in T_{i-1}$. We aim to show that there exists a refinement of a and b such that q can be chosen in T_i.

It follows from (3.1) that there exist $x_1, x_2 \in A$ such that $cq = ax_1 + bx_2$. Since A is essential we may choose x_i'th to be nonzero, then $Ad_1x_1 \cap Ax_2d_2 \neq 0$ (A is an essential domain), so there exist nonzero $y_1, y_2 \in A$ such that

$$y_1d_1x_1 = -y_2d_2x_2. \tag{3.2}$$

By the same argument there exist nonzero s and t in A such that $bt = cs$. It is clear that $R_i = qR_i + sR_i + 1R_i$. Now applying Theorem 3.1 for $C = A$, $A = A_i$, $B = \otimes_{j\neq i} A_j$, $R = R_i$, $S = S_i$, $F = T_i\backslash\{0\}$, $D = F_i$ and $d = y_2d_2t$ we find $f \in A$ such that

$$R_i = qR_i + (s + fy_2d_2t)R_i. \tag{3.3}$$

It follows from (3.2) that $cq \in J_i := (a + cfy_1d_1)R_i + (b + cfy_2d_2)R_i$. In fact,

$$(a + cfy_1d_1)x_1 + (b + cfy_2d_2)x_2 = ax_1 + bx_2 = cq.$$

It follows from (3.3) and $bt = cs$, that

$$J_i \supseteq cqR_i + (b + cfy_2d_2)tR_i = c(qR_i + (s + fy_2d_2t)R_i) = cR_i \ni c.$$

Since R_i is the localization of A at $T_i\backslash\{0\}$, there exists $0 \neq q \in T_i$ such that

$$cq \in (a + cfy_1d_1)A + (b + cfy_2d_2)A.$$

Thus, by induction, a and b can be refined so that (3.1) holds for $q \in T_n = C$.

Step 2. So suppose that, after possibly refining a and b, (3.1) holds for some $q \in C_{(i)} := C_1 \otimes \cdots \otimes C_i$, set $C_{(0)} = K$. We will show that there exists a refinement of a and b such that q can be chosen from $C_{(i-1)}$. We use induction on i. The case $i = n$ is proved. Fix $i < n$ and suppose that for all $n \geq j \geq i$ it is true. An exact duplicate of the argument of Step 1 with applying Theorem 3.1 in the following situation : $C = A$, $A = A_i$, $B = \otimes_{j \neq i} A_j$, $F = T = C_{(i)} \backslash \{0\}$, $D = K$, $S = \mathcal{D}(C_{(i-1)}) \otimes A_i$, $R = T^{-1}A$, shows that there exist f and g in A such that

$$cR \subseteq (a + cfd_1)R + (b + cgd_2)R.$$

Thus there exists $0 \neq q \in C_{(i-1)}$ such that

$$cq \in (a + cfd_1)A + (b + cgd_2)A.$$

By induction it is true for $i = 0$, but $C_{(0)} = K$ and the result follows. ∎

A criterion of simplicity of a generalized Weyl algebra $D(\sigma, a)$ of degree 1 was established in [Jor] for commutative D and in [Bav 3] for arbitrary D.

Proposition 3.2. *Let an algebra $A = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ be as in (0.1). Then A satisfies the condition of Theorem 0.2.*

Proof. Let $\mathcal{A}$ be one of the $\mathcal{A}_i$. The simplicity of $\mathcal{A}$ follows from [Bav 4]. By [Bav 2, 4] $\mathcal{A}$ has the restricted minimum condition. Let $\Lambda = D(\sigma, a)$ be a generalized Weyl algebra of degree 1. Then $\Lambda = \oplus_{n \in \mathbf{Z}} \Lambda_n$ is a $\mathbf{Z}$-graded ring where

$$\Lambda_n = \begin{cases} DX^n, & \text{if } n > 0; \\ D, & \text{if } n = 0; \\ DY^{-n}, & \text{if } n < 0. \end{cases}$$

If D is Noetherian (resp. a domain and $a \neq 0$) then by [Bav 4] a generalized Weyl algebra $D(\sigma, a)$ is Noetherian (resp. a domain). So $\mathcal{A}$ is a Noetherian domain, thus there exists a quotient division ring of $\mathcal{A}$.

Suppose that z belongs to the centre of $\mathcal{A}$. Since $\mathcal{A}$ is a simple $\mathbf{Z}$-graded domain, $z \in D$ is a unit, i.e. $z \in K$ and $\mathcal{A}$ is central.

Set $D = K[H]$ or $K[H, H^{-1}]$. It is easy to verify that $Q := D\backslash\{0\}$ is a multiplicatively closed subset of $\mathcal{A}$ (which satisfies the Ore condition) and the localization $Q^{-1}\mathcal{A}$ is the skew Laurent polynomial ring $T[X, X^{-1}; \sigma]$ with coefficients in the field of rational functions $T = K(H)$. By Lemma 2.2 $T[X, X^{-1}; \sigma]$ is special, so $\mathcal{A}$ is special too.

Suppose that $\mathcal{A}_i = D_i[\sigma_i, a_i]$ where $D_i = K[H_i]$ or $K[H_i, H_i^{-1}]$. It is clear that the algebra $C = C_1 \otimes \cdots \otimes C_n$ with $C_i = D_i$ satisfies the condition of Theorem 0.2. ∎

4. The module structure of the tensor product of simple algebras with restricted minimum condition

The next results and Corollaries of Theorem 0.2 and their proofs are the **same** as for the Weyl algebra A_n over a field K of characteristic zero. For more details the reader is referred to [St].

Let R be a domain with quotient ring Q (which is supposed to exist) and let M be a left R-module. The **rank** of M is the dimension of the left vector space $Q \otimes_R M$, i.e.

$$rk\, M = \dim_Q Q \otimes_R M.$$

An element of M is called *torsion* if it is annihilated by some nonzero element of R. The set $t(M)$ of all torsion elements of M is called the *torsion submodule* of M. It is the kernel of the R-homomorphism $M \to Q \otimes_R M$, $m \to 1 \otimes m$. A module M is called *torsion*, if $t(M) = M$, and *torsionfree*, if $t(M) = 0$.An element $m \in M$ is not torsion if and only if there exists $\phi \in Hom_R(M, R)$ such that $\phi(m) \neq 0$. Define

$$O(m) = \{f(m) : f \in Hom_R(M, R)\}.$$

So, $O(m)$ is a right ideal of R. An element $m \in M$ is called *unimodular* if $O(m) = R$, equivalently, m generates a free direct summand of M.

Theorem 4.1. *Let A be as in Theorem 0.2 . Let N be a finitely generated left A-module and M be a submodule of N of rank $rk\, M \geq 2$. Then M contains an element m which is unimodular as an element of N. That is $M = Am \oplus M' \subseteq Am \oplus N' = N$ where N' is a submodule of N and $M' = M \cap N'$.*

Proof is the same as in (Theorem 3.4, [St]). ∎

The next theorem follows from Theorem 4.1 by induction on the rank of M.

Theorem 4.2. *Let A be as in Theorem 0.2 . Let M be a finitely generated left A-module. Then $M \simeq M' \oplus A^{(s)}$, where M' is a module with rank $M' \leq 1$. If M is torsionfree, then M' is isomorphic to a left ideal of A.* ∎

Lemma 4.3. *Let A be as in Theorem 0.2 . Let M be a finitely generated left A-module with rank $M = r \geq 2$. Suppose that $m \oplus t \in M \oplus A$ is unimodular. Then there exists $\phi \in Hom_R(A, M)$ such that $m + \phi(t)$ is unimodular in M.*

Proof is the same as (Lemma 3.5, [St]). ∎

Theorem 4.4. *Let A be as in Theorem 0.2 . Let M be a finitely generated left A-module with rank $M \geq 2$. Suppose that $M \oplus A \simeq N \oplus A$ for some module N. Then $M \simeq N$.*

Proof. The element $1 = 0 \oplus 1 \in N \oplus A$ is unimodular, it can be written as $1 = m \oplus t \in M \oplus A$. Let $\phi \in Hom_A(A, M)$. Then the map

$$M \oplus A \to M \oplus A, \ (m, a) \to (m + \phi(a), a)$$

is a module isomorphism. Using this fact and Lemma 4.3 we may suppose that m is unimodular, hence, $M = Am \oplus M'$, $Am \simeq A$, for some submodule M' of M. Suppose that $t \neq 0$. The isomorphism $Am \to At$, $am \to at$, can be extended to the homomorphism $M \to At \subseteq A$ putting $\phi(M') = 0$. Using the isomorphism

$$M \oplus A \to M \oplus A, \ (m, a) \to (m, a - \varphi(m)),$$

we may suppose $t = 0$, i.e. $A1 = Am$ and $M \simeq A \oplus M'$. Now

$$N \simeq (N \oplus A)/A \simeq (M \oplus A)/Am \simeq M' \oplus A \simeq M. \blacksquare$$

A left module M over a ring R is called **stably n-generated** if, given any $r \geq n$ and $m_1, \dots, m_{r+1} \in M$ such that $M = \sum_1^{r+1} Rm_i$, then there exist $f_i \in R$ such that $M = \sum_1^r R(m_i + f_i m_{r+1})$.

Theorem 4.5. *Let A be as in Theorem 0.2 and Noetherian.*

1. *If I is a left (or right) ideal of A, then I is stably two-generated;*
2. *let M be a finitely generated torsion left A-module. Then M is a homomorphic image of a projective left ideal of A. Thus M is stably two-generated.*

Proof. **1** It follows from Theorem 0.2.

2 The module M can be written as $M = A^{(s+1)}/N$ with $rk\, N = s+1$. Applying Theorem 4.1 s times we obtain unimodular in $A^{(s+1)}$ elements $n_1, \dots, n_s \in N$, thus

$$M = A^{(s+1)}/N = (An_1 \oplus \cdots \oplus An_s \oplus P)/(An_1 \oplus \cdots \oplus An_s \oplus N') \simeq P/N'$$

where P is isomorphic to a left ideal of A and $N' = N \cap P$. $\blacksquare$

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ON THE GROUP OF UNITS IN MODULAR GROUP ALGEBRAS

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ABSTRACT. We give a summary of known and new results on unitary elements, generating systems, conjugacy classes, nilpotency class and solvable length in the group of units of a modular group algebra.

Let $\mathbb{F}G$ be the group algebra of a group G over a field $\mathbb{F}$ of characteristic $p > 0$ and $V(\mathbb{F}G)$ the group of normalized units (that is, of the units with augmentation 1) in $\mathbb{F}G$.

1. Unitary units

The anti-automorphism $g \mapsto g^{-1}$ extends linearly to an anti-automorphism

$$a = \sum_{g \in G} \alpha_g g \mapsto a^* = \sum_{g \in G} \alpha_g g^{-1}$$

of $\mathbb{F}_p G$, where $\mathbb{F}_p$ is the field of p elements; this extension leaves $V(\mathbb{F}_p G)$ setwise invariant. The elements v of $V(\mathbb{F}_p G)$ satisfying $v^{-1} = v^*$ are called *unitary normalized units* of $\mathbb{F}_p G$; these form a subgroup which we denote by $V_*(\mathbb{F}_p G)$.

The interest in unitary units arose from algebraic topology. S. Novikov had raised the problem of studying the group $V_*(\mathbb{F}G)$ and of determining the invariants and a basis of $V_*(\mathbb{F}G)$ when G is a finite abelian p-group. We describe the basis of $V_*(\mathbb{F}G)$.

Let $G = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$ be a direct decomposition into cyclic factors of a finite abelian p-group G. Let $L(G)$ be the set of all t-tuples of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ with coordinates satisfying the conditions $0 \leq \alpha_i < o(a_i)$ and at least one of the coordinates is not divisible by p. By Sandling's Theorem

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[San; 84b], the subset

$$B(V) = \{u_\alpha = 1 + (a_1 - 1)^{\alpha_1}(a_2 - 1)^{\alpha_2} \cdots (a_t - 1)^{\alpha_t} \mid \alpha \in L(G)\}$$

is a basis for the group $V(\mathbb{F}_p G)$.

Bovdi-Szakács [Bo-Sz; 89] described all invariants of $V_*(\mathbb{F}_p G)$ and, using Sandling's basis, they obtained the following result.

Theorem 1.1. *If G is a finite abelian p-group and $p > 2$, then* $\{z_\alpha = u_\alpha^* u_\alpha^{-1} \mid \alpha \in L(G)$ *and* $\alpha_1 + \alpha_2 + \cdots + \alpha_t$ *is odd*$\}$ *is a basis for* $V_*(\mathbb{F}_p G)$.

In case $p = 2$ we gave [Bo-Sz; 95] an algorithm to construct a basis for $V_*(\mathbb{F}_2 G)$. It would be interesting to give a basis in an explicit form.

We know only the following result about the unitary subgroup $V_*(\mathbb{F}_2 G)$ for a nonabelian group G.

Theorem 1.2. (V. Bovdi-Rozgonyi [B-R; 92]). *Let G be a finite 2-group which contains an abelian normal subgroup A of index 2. Suppose that there exists an element $b \in G \setminus A$ of order 4 such that $b^{-1}ab = a^{-1}$ for all $a \in A$. Then the unitary subgroup $V_*(\mathbb{F}_2 G)$ is the semidirect product of G and a normal subgroup H. The subgroup H is the semidirect product of the normal elementary abelian 2-group $W = \{1 + (1 + b^2)zb \mid z \in \mathbb{F}_2 A\}$ and the abelian subgroup L, where $V_*(\mathbb{F}_2 A) = A \times L$. The abelian group W is the direct product of $\frac{1}{2}|A|$ copies of the additive group of the field $\mathbb{F}_2$.*

Definition 1.3. *Let M be an ideal of the group ring $\mathbb{F}_p G$. Then the subgroup $M^+ = \{u \in V(\mathbb{F}_p G) \mid u - 1 \in M\}$ is called a* **congruence subgroup** *of $V(\mathbb{F}_p G)$ and the subgroup $M_*^+ = \{u \in V_*(\mathbb{F}_p G) \mid u - 1 \in M\}$ is called a* **unitary congruence subgroup** *of $V(\mathbb{F}_p G)$.*

Problem 1.4. Is the unitary subgroup $V_*(\mathbb{F}_p G)$ a semidirect product of G and a normal subgroup? Moreover, is there any unitary congruence subgroup M_*^+ such that $V_*(\mathbb{F}_p G) = M_*^+ \rtimes G$?

Since G is a subgroup of $V_*(\mathbb{F}_p G)$, it is clear that if $V(\mathbb{F}_p G)$ is the semidirect product of G and a normal subgroup (or a congruence subgroup M^+) then $V_*(\mathbb{F}_p G)$ is also the semidirect product of G and a normal subgroup (or a unitary congruence subgroup M_*^+). A normal complement exists in the following cases:

1. (Moran-Tench [M-T; 77], Bovdi [Bo; 77, Bo; 82]). If G is a nilpotent p-group of class 2 and exponent p, then $V(\mathbb{F}_p G) = M^+ \rtimes G$;
2. (Bovdi [Bo; 77, Bo; 82, Bo; 96]). If G is a nilpotent group of class 2 and exponent 4 then $V(\mathbb{F}_2 G) = M^+ \rtimes G$;
3. (Bovdi [Bo; 82], Sandling [San; 74a]). If G is the circle group of a radical ring of characteristic p then $V(\mathbb{F}_p G) = M^+ \rtimes G$;

4. (Sandling [San; 89c]). If G' is a central-elementary-by-abelian p-group then $V(\mathbb{F}_pG) = N \rtimes G$.

Recall [Iv; 80] that for the 2-group G of maximal class of order greater than 8 (that is the dihedral, semidihedral or generalized quaternion 2-group) there is no normal complement to G in $V(\mathbb{F}_2G)$. Note that in this situation we do not know whether a normal complement exists for the unitary subgroup.

Bovdi-Erdei [Bo-E; 96] described the unitary subgroup in the group algebras $\mathbb{F}_2G$ of nonabelian 2-groups of order 16. In each of these cases the subgroup G has a normal complement in $V_*(\mathbb{F}_2G)$.

Example 1.5. $V_*(\mathbb{F}_2D_4)$ is a direct product of the dihedral group D_4 and three groups of order two.

Example 1.6. $V_*(\mathbb{F}_2D_8) = D_8 \ltimes N$, where N is a direct product of a dihedral group of order 8 and five groups of order two.

Example 1.7. Let $G = \langle a, b \mid a^8 = 1,\ b^2 = 1,\ b^{-1}ab = a^5 \rangle$ and

$$\begin{aligned} v_1 &= 1 + (1 + a^2 + a^4 + a^6)a & v_4 &= 1 + (1 + a^2 + a^4 + a^6)ab \\ v_2 &= 1 + (1 + a^2 + a^4 + a^6) & v_5 &= 1 + (1 + a^4)a^2b \\ v_3 &= 1 + (1 + a^4)b & u &= 1 + a^3 + a^5 + (a^5 + a^7)b. \end{aligned}$$

Then $V_*(\mathbb{F}_2G) = (G \ltimes L_1) \times L_2$, where

1. L_1, L_2 are elementary 2-groups of order 8;
2. $L_1 = \langle u \rangle \times \langle v_1 \rangle \times \langle v_4 \rangle$, $u^a = uv_4$, $u^b = uv_1v_4$, $v_1^a = v_1$, $v_1^b = v_1$, $v_4^a = v_4$, $v_4^b = v_4$ and $L_2 = \langle v_2 \rangle \times \langle v_3 \rangle \times \langle v_5 \rangle$.

Example 1.8. Let $G = \langle a, b \mid a^8 = 1,\ b^2 = 1,\ b^{-1}ab = a^3 \rangle$ and

$$\begin{aligned} v_1 &= 1 + (1 + a^2 + a^4 + a^6) & v_4 &= 1 + (1 + a^4)ab \\ v_2 &= 1 + (1 + a^2 + a^4 + a^6)a & v_5 &= 1 + (1 + a^4)a^2b \\ v_3 &= 1 + (1 + a^4)b & v_6 &= 1 + (1 + a^4)a^3b \end{aligned}$$

$$u = 1 + a^3 + a^5 + (a^4 + a^6)b.$$

Then $V_*(\mathbb{F}_2G) = G \ltimes L$ where

1. L is an elementary abelian group of order 2^7;
2. $L_1 = \langle u \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \times \langle v_4 \rangle \times \langle v_5 \rangle \times \langle v_6 \rangle \times \langle v_3 \rangle$ subject to the relations

$$u^a = uv_1v_4v_5v_6\,,\ u^b = uv_1v_2v_4v_5v_6\,,\ v_1^a = v_1\,,\ v_1^b = v_1\,,\ v_2^a = v_2\,,$$
$$v_2^b = v_2\,,\ v_3^a = v_5\,,\ v_3^b = v_3\,,\ v_4^a = v_6\,,\ v_4^b = v_6\,,\ v_5^b = v_5\,.$$

Let G be a locally finite p-group. For each element g in G, let $\hat{g}$ denote the sum (in $\mathbb{F}G$) of the distinct powers of g. The elements $1+(g-1)h\hat{g}$ with $g, h \in G$ are the *bicyclic* units of $\mathbb{F}G$, introduced by Ritter and Sehgal. V. Bovdi and

Kovács [B-K; 94] described those $\mathbb{F}$ and G for which all bicyclic units in $V(\mathbb{F}G)$ are unitary. This is an answer to the difficult question of Sehgal and the first step to determine the generators of the unitary subgroup $V_*(\mathbb{F}_pG)$. If the map $^f : KG \to KG,\ x \mapsto x^f$ of the group ring KG, where K is a commutative ring with unity, has the properties

1. $(x+y)^f = x^f + y^f$;
2. $(xy)^f = y^f x^f$;
3. $(x^f)^f = x$

then it is called an *involution* of KG. We considered above the classic involution of $\mathbb{F}G$, which is an extension of the anti-automorphism $g \to g^{-1}$ of the group G. We give another example of involution. Let $\zeta(G)$ be the center of a finite 2-group G, and suppose that the central quotient $G/\zeta(G)$ is a direct product of two groups of order two and that the commutator subgroup $G' = \langle e \mid e^2 = 1\rangle$ is of order two. Then the mapping $^\odot : G \to G$, defined by

$$g^\odot = \begin{cases} \text{g}, & \text{if } g \in \zeta(G), \\ \text{ge}, & \text{if } g \notin \zeta(G) \end{cases}$$

is an anti-automorphism of order two. If $x = \sum_{g\in G} \alpha_g g \in KG$, then $u \longmapsto u^\odot = \sum_{g\in G} \alpha_g g^\odot$ is an involution.

Problem 1.9. Let G be a finite p-group. Find all the involutions of $V(\mathbb{F}_pG)$ (for instance, for an abelian G). Study the unitary subgroups related with these involutions.

V. Bovdi and Rozgonyi described the unitary subgroup $V_\odot(\mathbb{F}_2G)$ in [B-R; 92].

There is an extensive literature on the unitary subgroup in integral group rings, see [Art-Bo; 89], [Bo; 87], [Seh; 93].

2. Generators of the unit group

Let G be a finite p-group. We know that the augmentation ideal $A(\mathbb{F}_pG)$ is nilpotent and the dimension subgroups $D_n = \{g \in G \mid g-1 \in A^n(\mathbb{F}_pG)\}$ form a central series such that D_i/D_{i+1} is an elementary abelian p-group of order p^{d_i}.

Let $D_i/D_{i+1} = \prod_{j=1}^{d_i} \langle u_{ij}D_{i+1}\rangle$ be a decomposition into a direct product of cyclic groups of order p. Then any element $g \in G$ can be written uniquely in the form

$$g = u_{11}^{z_{11}} u_{12}^{z_{12}} \dots u_{1d_1}^{z_{1d_1}} u_{21}^{z_{21}} \dots u_{2d_2}^{z_{2d_2}} \dots u_{s1}^{z_{s1}} \dots u_{sd_s}^{z_{sd_s}} \ (0 \le z_{ij} < p).$$

The set $\{u_{ij} \mid j = 1, \ldots, d_i,\ i = 1, \ldots, s\}$ will be called a dimension basis of G. If an element

$$w = \prod_{k=1}^{s} \prod_{j=1}^{d_k} (u_{kj} - 1)^{y_{kj}} \quad (0 \leq y_{kj} < p)$$

has indices of its factors in the lexicographic order then we call it regular. Clearly, $\nu(w) = \sum_{k=1}^{s} \sum_{j=1}^{d_k} k y_{kj}$ is the weight of this element. From Jennings' theory it is well known that all elements of the form $1 + w$ generate the group $V(\mathbb{F}G)$. But this generating system is a very big one. Jointly with Sehgal we constructed a new generating system (yet unpublished), which is a part of this system, and if G is abelian then this is just Sandling's basis and a minimal system. Using commutator calculations we exclude some elements from this generating system. This rather complicated algorithm determines which elements $1 + v$ of the generating system may be replaced by the commutator $(1 + w, g)$, where $g \in G$. Clearly, these elements can be excluded from the generating system. This algorithm is very effective for the groups G satisfying one of the following conditions: G is a 2-group of maximal class, $|G| = p^3$ or $|G| = 16$. For example, we obtained:

Example 2.1. Let G be one of the following groups:

1. $D_{2^n} = \langle\, a, b \mid a^{2^n} = 1,\ b^2 = 1,\ b^{-1}ab = a^{-1} \,\rangle,\ n \geq 2;$
2. $Q_{2^n} = \langle\, a, b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ b^{-1}ab = a^{-1} \,\rangle,\ n \geq 2;$
3. $D_{2^n} = \langle\, a, b \mid a^{2^n} = 1,\ b^2 = 1,\ b^{-1}ab = a^{2^{n-1}-1} \,\rangle,\ n \geq 3.$

Then $V(\mathbb{F}_2G)$ is generated by a, b and $\{1 + (a+1)^{4k+1}(b+1) \mid 0 \leq k \leq 2^{n-2} - 1\}$.

Example 2.2. Let $G = \langle\, a, b \mid a^4 = 1,\ b^4 = 1,\ b^{-1}ab = a^3 \,\rangle$. Then $V(\mathbb{F}_2G)$ is generated by the units a, b, $1 + (a+1)(b+1)$, $1 + (a+1)(b+1)^2$, $1 + (b+1)^3$, $1 + (a+1)(b+1)^3$, and $1 + (a+1)^3(b+1)^3$.

Example 2.3. Let $G = \langle a, b, c \mid a^4 = 1, b^2 = 1, c^2 = 1, b^{-1}ab = ac, c^{-1}ac = a, c^{-1}bc = b\rangle$. Then $V(\mathbb{F}_2G)$ is generated by the units a, b, $1 + (a+1)(b+1)$, $1 + (a+1)^3$, $1 + (a+1)^2(b+1)$, $1 + (a+1)^3(b+1)$ and $1 + (a+1)^3(b+1)(c+1)$.

Example 2.4. Let $G = \langle\, a, b \mid a^{2^n} = 1,\ b^2 = 1,\ b^{-1}ab = a^{2^{n-1}+1},\ n \geq 3 \,\rangle$. Then $V(\mathbb{F}_2G)$ is generated by a, b, $\{1 + (a+1)^{4k+3+2^{n-1}}(b+1) \mid 0 \leq k < 2^{n-3}\}$ and

$$\{1 + (a+1)^{2k+1} \mid 0 \leq k < 2^{n-2}\} \cup \{1 + (a+1)^k(b+1) \mid 0 \leq k < 2^{n-1}\}.$$

Until now in the nonabelian case the generating system was not studied with the exception of the groups $V(\mathbb{F}_2G)$ for all groups G of order $|G| \leq 32$, which was done by Sandling [San; 92d] and Rao [R; 93].

3. Conjugacy classes of the unit group

We know very little of conjugacy classes in the group of units. The first result was obtained by Coleman [C; 64]:

Theorem 3.1. *Let G be a finite p-group and C_u a conjugacy class in $V(\mathbb{F}_pG)$. If C_u contains an element from G, then $C_u \cap G$ is a conjugacy class in G.*

The next theorem was obtained by Rao-Sandling [R-San; 95b].

Theorem 3.2. *Let G be a finite p-group and $\mathbb{F}$ a field of characteristic p. If u is a noncentral element in $\mathbb{F}G$ then the conjugacy subset $C_u = \{x^{-1}ux \mid x \in V(\mathbb{F}G)\}$ has the following properties:*

1. *if $\mathbb{F}$ is an infinite field then C_u is infinite;*
2. *if $p^r = max\{p^2, |\mathbb{F}|\}$ then p^r divides $|C_u|$;*
3. *if $|C_u|$ is finite then the sum of all elements of C_u is zero.*

Theorem 3.3. *If G is a finite p-group and $\mathbb{F}$ is a field of characteristic p, then there exists in $V(\mathbb{F}G)$ a conjugacy class C_u which does not contain elements from G.*

Theorem 3.4. *Let $\mathbb{F}$ be a field of characteristic p and suppose that the finite group G satisfies the following conditions: there exists a proper normal subgroup H of G such that $G = \langle g, H\rangle$, the subset $W = \{(g,h) \mid h \in H\}$ of commutators is a normal subgroup of G and $W' \subseteq C_G(g)$. If $\widehat{W} = \sum_{h\in W} h$ and $z \in (\mathbb{F}\zeta(G) \cap \mathbb{F}H)\widehat{W}$ (where $\zeta(G)$ is the centre of G), then $L = \langle g+z, H\rangle$ is a group basis for $\mathbb{F}G$ over $\mathbb{F}$ and the subgroups G and L are not conjugate in $V(\mathbb{F}G)$.*

Clearly, every finite p-group of nilpotency class 2 satisfies the conditions of Theorem 3.4.

4. The nilpotency class of the unit group

Let $\mathbb{F}G$ be a modular group algebra. By Khripta's [Khr; 72] result $V(\mathbb{F}G)$ is nilpotent if and only if G is nilpotent and the derived subgroup G' is a finite p-group. Denote by $cl(V), cl(G)$ respectively the nilpotency class of $V(\mathbb{F}G)$ and G.

Problem 4.1. Which values are taken by the function $f(G) = cl(V) - cl(G)$, and determine all groups G for which $f(G) = n$.

Denote by $Syl_p(G)$ the Sylow p-subgroup of G.

Theorem 4.2. *(Khripta [Krh; 72]). $f(G) = 0$ if and only if G satisfies one of the following conditions:*

1. *G is an abelian group;*
2. $G' = Syl_p(G)$ *and* $|G'| = 3$*;*
3. $cl(G) = 2$, $G' = Syl_p(G) = \langle a \mid a^2 = 1\rangle \times \langle b \mid b^2 = 1\rangle$*;*
4. $cl(G) = 3$, $G' = Syl_p(G)$ *and* $|G'| = 4$.

Let G be a finite noncommutative p-group. Coleman-Passman [C-P; 70] showed that $V(\mathbb{F}_pG)$ involves a wreath product $C(p) \wr C(p)$ of two groups of order p, and, as a consequence, $cl(V) \geq p$. The values of the function are determined in the following cases:

1. (Baginski [Bag; 87]) $cl(V) = p$ if and only if $|G'| = p$ and $f(G) = p - 2$.
2. (Shalev [Sh; 93]) If $cl(V) > p$ and $p > 3$, then $cl(V) \geq 2p - 1$. Equality holds if G' is elementary abelian of order p^3 and $cl(G) = 2$ and hence $f(G) = 2p - 3$.
3. (Shalev [Sh; 93]). If $cl(V) > 2p - 1$ and also $p > 3$, then we have the relation $cl(V) \geq 3p - 2$. Equality holds if G' is elementary abelian and one of the following conditions holds:
   (a) $|G'| = p^3$ and $cl(G) = 2$; thus $f(G) = 3p - 4$;
   (b) $|G'| = p^2$ and $cl(G) = 3$; thus $f(G) = 3p - 5$.
4. (Shalev [Sh; 93]). If G' is a central elementary subgroup of order $p^n, p > 3$, then $f(G) = n(p - 1) + 1 - cl(G)$.
5. (Konovalov [K; 95]). If G is a 2-group of maximal class, then $f(G) = |G'| - cl(G)$.

It is easy to see that in the next theorem $f(G) = 1$ or 0, and the question of determining when $f(G) = 1$ is still open. Theorem 4.3 for finite p-groups was also proved by Rao-Sandling [R-San; 95a].

Theorem 4.3. *Let G be any nilpotent group with a nontrivial Sylow p-subgroup and $\mathbb{F}$ a field of prime characteristic p. Then $cl(V) = 3$ if and only if one of the following conditions is satisfied:*

1. $p = 3$, $cl(G) = 2$, $|G'| = 3$ *and* $Syl_p(G) \neq G'$*;*
2. $p = 2$, $cl(G) = 2$, $Syl_p(G) = G'$ *is cyclic of order 4;*
3. $p = 2$, $cl(G) = 3$, $Syl_p(G) = G'$ *is cyclic of order 4;*
4. $p = 2$, $cl(G) = 3$, $Syl_p(G') = G' = \langle a \mid a^2 = 1\rangle \times \langle b \mid b^2 = 1\rangle$
5. $p = 2$, $cl(G) = 2$, $Syl_p(G) \neq G'$ *and* $G' = \langle a \mid a^2 = 1\rangle \times \langle b \mid b^2 = 1\rangle$*;*
6. $p = 2$, $cl(G) = 2$, $G' = \langle a \mid a^2 = 1\rangle \times \langle b \mid b^2 = 1\rangle \times \langle c \mid c^2 = 1\rangle$ *and* $Syl_p(G)$ *is central of order at most* 16.

5. Solvable length of the unit group

Problem 5.1. Describe the derived length of the group $V(\mathbb{F}_pG)$.

Although the problem of its solvability has formerly been solved, few facts are known of the derived length of the group of units $V(\mathbb{F}G)$ of the group algebra $\mathbb{F}G$ of either a torsion [Bo-Khr; 77] or a nilpotent group G [Bo; 92]. The first result was obtained by Shalev [Sh; 91] for finite groups: $V(\mathbb{F}_pG)$ is metabelian if and only if G is abelian provided $p > 3$; for $p = 3$, $V(\mathbb{F}_pG)$ is metabelian if and only if G is either abelian or nilpotent with a derived subgroup of order 3. Kurdics [Kur; 96] and also Coleman-Sandling [C-San; 94] treated the case $p = 2$ and obtained the following result:

Theorem 5.2. *Let G be a finite group and $\mathbb{F}$ a field of characteristic 2. The group of units $V(\mathbb{F}G)$ is metabelian if and only if one of the following conditions holds:*

1. *G is abelian;*
2. *G is nilpotent of class 2 and has an elementary abelian derived subgroup of order 2 or 4;*
3. *$\mathbb{F} = \mathbb{F}_2$, the field of two elements, and G is an extension of an elementary abelian 3-group H by the group $\langle b\rangle$ of order 2 with $b^{-1}ab = a^{-1}$ for every $a \in H$.*

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MODULES OVER GROUP ALGEBRAS

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0. Introduction

This paper is a report of a series of three lectures which I presented at a conference on Representation Theory of Groups, Algebras and Orders at Ovidius University in Constanta, Romania. The aim of my lectures was to give a sketch of a new approach to the module theory of group algebras that has been developed over the last few years. The new methods involve the study of infinitely generated modules and various subcategories and quotient categories of the module category. Some surprising new theorem have already been discovered and it seems likely that many other interesting results will come out of these developments.
Throughout the notes G denotes a finite group and k denotes an algebraically closed field of prime characteristic $p > 0$. As general references on representation theory we refer the reader to the book [CR] or to volume 1 of [B1]. For background on the cohomology theory see Volume 2 of [B1] or [E1]. An introduction to a lot of the material presented here is contained in [C5].

1. Rank varieties

Let $E = \langle x_1, \ldots, x_n \rangle$ be an elementary abelian group of order p^n. That is, $E \cong (\mathbb{Z}/p)^n$ so that $x_i^p = 0$ and $x_i x_j = x_j x_i$ for all i and j. We say that n is the rank of E, and in general, for any finite group G, the p-rank of G is largest of the ranks of the elementary abelian p-subgroups of G. The group algebra kE is a local ring with maximal ideal generated by the elements $x_i - 1$ for $i = 1, \ldots, n$. Note that for any $x, y \in E$, $(x-1)^p = x^p - 1 = 0$ and $xy - 1 = (x-1) + (y-1) + (x-1)(y-1)$. So the ideal generated by $\{x_i - 1\}$ is the same as the augmentation ideal of kE.
More generally the group algebra of any p-group is a local ring. This implies that any projective module must be free and hence the dimension of any projective

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module over a p-group must be divisible by the order of the group. Another important fact is that the group algebra of any finite group is a self-injective ring. That is, injective modules are projective and vice versa.

Suppose that M is a finitely generated kE-module. We define the rank variety of M to be the set

$$V_E^r(M) = \{\alpha \in k^n | M_{\langle u_\alpha \rangle} \text{ is not a free module}\} \cup \{0\}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$ and $u_\alpha = 1 + \Sigma \alpha_i(x_i - 1)$. Notice that $u_\alpha \in kE$ is a unit of order p. By $M \downarrow_{\langle u_\alpha \rangle}$ we mean the restriction of M to a $k\langle u_\alpha \rangle$-module where $\langle u_\alpha \rangle$ is the subgroup of the group of units of kE generated by u_α. With the zero element $0 \in k^n$ thrown in, the rank variety $V_E^r(M)$ is a closed homogeneous subvariety of affine k-space, k^n, given the Zariski topology (see [C1]). Note that if p does not divide the dimension of M then $M_{\langle u_\alpha \rangle}$ can not be free (projective) if $\alpha \neq 0$. So in such a case $V_E^r(M) = k^n$. In particular $V_E^r(k) = k^n$ where k denotes the one-dimensional trivial kE module.

Several properties of the rank variety are useful in the module theory. For example, if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of kE-modules and if V is a closed subset of $V_E^r(k) = k^n$ with two of $V_E^r(L)$, $V_E^r(M)$ and $V_E^r(N)$ in V, then the third of these rank varieties is also in V [C1]. This implies, for example, that the subcategory of all kE-modules with varieties contained the fixed closed set $V \subset k^n$ is a "thick" subcategory of the stable category of finitely generated kG-modules modules modulo projectives (see [BCR2, BCR3, R]). We will explain this more fully later. Two of the most useful facts about rank varieties are contained in the following theorem.

Theorem 1.1. *(a)* (Dade's Lemma [D]) $V_E^r(M) = \{0\}$ *if and only if M is a projective kE-module.*

(b) (Tensor Product Theorem [C1]). *If M and N are finitely generated kG-modules then*

$$V_E^r(M \otimes N) = V_E^r(M) \cap V_E^r(N).$$

By $M \otimes N$ we mean the space $M \otimes_k N$ given the diagonal action of E ($g(m \otimes n) = gm \otimes gn$ for $g \in E$, $m \in M$ and $n \in N$).

For an example consider the case in where $p = 2$ and $E = \langle x, y \rangle$ is an elementary abelian group of order 4. Let M be a k-vector space with basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ and define an action of E on M by

$$(x - 1)a_i = b_i, (y - 1)a_i = b_{i+1}$$

and

$$(x - 1)b_i = (y - 1)b_i = 0 \text{ for } i = 1, \ldots, n.$$

Thus the matrices for the actions of x and y with respect to the given basis have the forms

$$x \longleftrightarrow \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \qquad y \longleftrightarrow \begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$$

where I is the $n \times n$ identity matrix and

$$A = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & 1 & 0 \end{bmatrix}.$$

We claim that $V_E^r(M) = \{(0, \alpha_2) | \alpha_2 \in k\} \subset k^2$.
To understand this we must look at the free module for a cyclic group of order 2. If $U = \langle u \rangle$ is such a group then kU has a basis consisting of 1 and $u - 1$. With respect to this basis the matrix of u has the form

$$u \longleftrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

So a kU-module is projective (free) if and only if it is a sum of such modules. It can be seen that the following holds.

Lemma 1.2. *A kU-module N is free if and only if the rank of the matrix of $u - 1$ on N is exactly half of the dimension of N.*

Note that because $(u-1)^2 = 0$, the rank can't be more than half of the dimension of N. The following is another way of saying the same thing. The advantage of this formulation is that it doesn't depend on N being finite dimensional.

Lemma 1.3. *A kU-module N is free if and only if*

$$(u-1)N = \{m \in N | (u-1)m = 0\}.$$

Now let's examine the module M in the example. Suppose that $\alpha = (\alpha_1, \alpha_2) \in k^2$ and $u = u_\alpha = 1 + \alpha_1(x-1) + \alpha_2(y-1)$. Suppose that $\alpha_1 \neq 0$, then notice that the matrix of $u - 1$ is

$$\begin{bmatrix} 0 & 0 \\ \alpha_1 I + \alpha_2 A & 0 \end{bmatrix}$$

and has rank equal to half the dimension of M. So M_U is a free kU-module. Likewise

$$(u-1)M = \{m \in M | (u-1)m = 0\}$$

is the subspace spanned by $b_1, \ldots, b_n$. So we have that $V_E^r(M) = \{(0, \alpha_2) | \alpha_2 \in k\}$ as claimed.

The second example is an infinite dimensional example defined in much the same way. Let M be the space with basis $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$. Define the action of $E = \langle x, y \rangle$ ($|E| = 4, p = 2$ as before) by

$$(x-1)a_i = b_i, (y-1)a_i = b_{i+1}$$

and

$$(x-1)b_i = 0 = (y-1)b_i$$

for $i = 1, 2, \ldots$.

Suppose we try to measure the rank variety in the same way. This time, we claim that

$$V_E(M) = k^2 \backslash \{(\alpha_1, 0) | \alpha_1 \in k\}.$$

Clearly if $\alpha = (\alpha_1, 0)$ then u_α acts freely on M by Lemma 1.3. However, if $\alpha = (\alpha_1, \alpha_2), \alpha_2 \neq 0$ then it is impossible to express b_1, which is in $\{m \in M | (u_\alpha - 1)m = 0\}$, as an element of $(u_\alpha - 1)M$. That is, we would need to write b_1 as a <u>finite</u> linear combination of the elements

$$(u_\alpha - 1)a_i = \alpha_1 b_i + \alpha_2 b_{i+1}.$$

This is impossible if $\alpha_2 \neq 0$.

Therefore in the second example $V_E^r(M)$ is an open set in k^2 and not a closed set. The situation is actually much worse. It is possible to construct an infinitely generated module M with the property that $V_E^r(M) = \{0\}$, but M is not projective. Thus there are counter-examples to Dade's lemma which are infinitely generated. It is also possible to find infinite dimensional counter-examples to the Tensor Product Theorem.

It has long been clear that there are problems with infinitely generated modules. However recently it has been shown that a change in the definition can recover some of these important properties [BCR1, BCR2]. The trick is to enlarge the field. Suppose that K is a large algebraically closed transcendental extension of k. The transcendence degree of K over k should be at least the rank of E. Then we define a new rank variety ($E = \langle x_1, \ldots, x_n \rangle$) as

$$\mathcal{V}_E^r(M) = \{\alpha \in K^n \mid (K \otimes M)_{\langle u_\alpha \rangle} \text{ is not free } \} \cup \{0\}.$$

That is $\mathcal{V}_E^r(M) = V_E^r(K \otimes M)$.

Notice that the new rank variety, $\mathcal{V}_E^r(-)$, is still not a variety in that it may not be a closed set in K^n. But we do have the following [BCR2].

Theorem 1.4. *(a) (Dade's Lemma) $\mathcal{V}_E^r(M) = \{0\}$ if and only if M is projective.*

(b) (Tensor Product Theorem) $\mathcal{V}_E^r(M \otimes N) = \mathcal{V}_E^r(M) \cap \mathcal{V}_E^r(N)$.

There is another view of the new variety that comes from looking at generic points. Recall that a point $\alpha \in K^n$ is generic for an irreducible subvariety V of k^n provided the collection of k-rational polynomials satisfied by α generate the ideal of V. For example suppose that $\gamma_1, \gamma_2 \in K$ are algebraically independent over k. Let $\alpha = (\gamma_1, \gamma_1\gamma_2, \gamma_2, \gamma_2)$. Then α satisfies the k-rational polynomials $X_1X_3 - X_2$ and $X_3 - X_4$. So α is generic for the irreducible subvariety V which is the zero set in k^4 of the polynomials $X_1X_3 - X_2$ and $X_3 - X_4$. Every point in K^n is generic for some irreducible subvariety of k^n and conversely every irreducible subvariety of k^n has a generic point in K^n as long as the transcendence degree of K over k is at least n.

The connection of generic points with varieties of modules comes from a theorem which says that if $\alpha, \beta \in K^n$ are generic for the same irreducible subvariety of k^n and if M is a kE-module then $\alpha \in \mathcal{V}_E^r(M)$ if and only if $\beta \in \mathcal{V}_E^r(M)$. Hence it is possible to think of $\mathcal{V}_E^r(M)$ as a collection of closed irreducible subvariety of k^n.

2. Cohomology and representations

In the last section we concentrated on representations over elementary abelian p-groups. We begin this section by outlining some features of the module theory for general groups. Most of this will not be needed for the rest of the paper, but it shows why the elementary abelian case is so important. Throughout, G denotes a finite group and k is an algebraically closed field of prime characteristic p.

An old theorem of Evens [E2] and Venkov [V] shows that the cohomology ring $H^*(G,k) \cong \text{Ext}^*_{kG}(k,k)$ is a finitely generated k-algebra. A major implication of the finite generation is that the set of all maximal ideals of $H^*(G,k)$ is an affine variety, called the maximal ideal spectrum of $H^*(G,k)$. We denote this by $V_G(k)$. In the case that $G = E$ is an elementary abelian group of order p^n, then $H^*(G,k)/\text{Rad}H^*(G,k) \cong k[x_1, \dots, x_n]$ is a polynomial ring in n-variables. Here $\text{Rad}H^*(G,k)$ is the Jacobson radical. So the maximal ideal spectrum $V_G(k) \cong k^n$. That is, every $\alpha \in k^n$ is associated to the maximal ideal which is the kernel of the map $H^*(G,k) \to k$ given by evaluating any $f = f(X_1, \dots, X_n)$ at $\alpha \in k^n$.

If M and N are finitely generated kG-modules then $\text{Ext}^*_{kG}(M,N)$ and $\text{Ext}^*_{kG}(N,M)$ are finitely generated modules over $H^*(G,k)$ ([E1] or [V] again). So let $J(M)$ be the annihilator in $H^*(G,k)$ of $\text{Ext}^*_{kG}(M,-)$ and $\text{Ext}^*_{kG}(-,M)$. Actually $J(M)$ is equal to the annihilator of the identity homomorphism $\text{Id}_M \in \text{Ext}^0_{kG}(M,M) \subset \text{Ext}^*_{kG}(M,M)$. Then the variety of M is $V_G(M) = V_G(J(M))$ is the set of all maximal ideals in $V_G(k)$ that contain $J(M)$. $V_G(M)$ is sometimes called the cohomological variety of M or the support variety of M.

If H is a subgroup of G then there is a restriction map on cohomology

$$\text{res}_{G,H} : H^*(G,k) \to H^*(H,k).$$

This induces a map on maximal ideal spectra

$$\text{res}^*_{G,H} : V_H(k) \to V_G(k),$$

which sends $\mathfrak{m} \in V_H(k)$ to $\{\alpha \in H^*(G,k) | \text{res}_{G,H}(\gamma) \in \mathfrak{m}\} \in V_G(k)$.
The following result was a fundamental contribution of Quillen [Q] in the case that $M = k$. It was extended to general finite dimensional modules by Alperin and Evens [AE] and independently by Avrunin [A].

Theorem 2.1. *For any kG-module M, $V_G(M) = \bigcup res^*_{G,E}(V_E(M_E))$ where the union is taken over all elementary abelian p-subgroups of G.*

Hence we see that the support varieties can be measured at the level of the elementary abelian p-subgroups. In the case that $G = E \cong (\mathbb{Z}/p)^n$ is elementary abelian, the support variety defined above is related to the rank variety of Section 1 by the following.

Theorem 2.2.[C1, AS]. *There is a one-to-one onto map of varieties (isogeny) $V_E(M) \to V^r_E(M)$.*

For $p > 2$, the map in the theorem involves applying the Frobenius automorphism to k^n. Hence it is an isogeny not an isomorphism because its inverse is not a polynomial map.
The first part of this section was written in order to emphasize the role of the elementary abelian p-subgroups in the module theory of kG-modules. We will not really use much of it in the rest of the paper. Most of what we do will involve the cohomological variety $V_G(M)$ rather than the rank variety $V^r_E(M)$. However the reader should keep Theorem 2.2 in mind.
Suppose that

$$(P_*, \epsilon) \qquad \cdots \longrightarrow P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \xrightarrow{\epsilon} k \longrightarrow 0$$

is a minimal projective kG-resolution of the trivial module k. By minimal we mean that the kernels of the boundary maps, $\Omega^n(k) = \text{Ker}\partial_{n-1} = \text{Im}\partial_n$, have no projective submodules. One property of projective modules over kG is that if P is projective and M is any kG-module then $P \otimes M = P \otimes_k M$ is also projective. Therefore $(P_*, \epsilon) \otimes M$ is a projective resolution (though not necessarily minimal) of $k \otimes M \cong M$. It follows that $\Omega^n(k) \otimes M \cong \Omega^n(M) \oplus \text{proj}$ where by "$\oplus$proj" we mean the direct sum with some projective module.
Now $H^*(G,k) = \text{Ext}^*_{kG}(k,k)$ is the cohomology of the complex $\text{Hom}_{kG}(P_*, k)$. So an element $\zeta \in H^n(G,k)$ is represented by a cocycle $\hat{\zeta} : P_n \to k$. Of course

"cocycle" means that the composition $\hat{\zeta}\partial_{n+1} : P_{n+1} \to P_n \to k$ is zero. So $\hat{\zeta}(\partial_{n+1}(P_{n+1})) = 0$ and we have an induced homomorphism

$$\bar{\zeta} : \Omega^n(k) \to k$$

since $\Omega^n(k) \cong P_n/\partial_{n+1}(P_{n+1})$. In fact, it can be shown that $\bar{\zeta}$ represents the cohomology class $\zeta \in H^n(G,k)$ uniquely. If $\zeta \neq 0$ then $\hat{\zeta} \neq 0$ and hence it is surjective. It follows that we have an exact sequence

$$0 \to L_\zeta \to \Omega^n(k) \xrightarrow{\bar{\zeta}} k \to 0 \tag{1}$$

where L_ζ is the kernel of $\bar{\zeta}$. The following can be proved using rank varieties. [C2].

Lemma 2.3. $V_G(L_\zeta) = V_G(\zeta) \subset V_G(k)$. *Here $V_G(\zeta)$ is the collection of all maximal ideals in $H^*(G,k)$ that contain ζ.*

Another way of saying this is that the ideal $J(L_\zeta)$ has the same radical in $H^*(G,k)$ as the ideal generated by ζ. This implies that some power of ζ annihilates the cohomology of L_ζ. In fact it can be shown that $\zeta^2 \in J(L_\zeta)$ (see [B1]).

To get another view of this consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_\zeta & = & L_\zeta & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^n(k) & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots \\
 & & \bar{\zeta}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & k & \xrightarrow{\sigma} & \Omega^{-1}(L_\zeta) & \longrightarrow & P_{n-2} & \xrightarrow{\partial_{n-2}} & \cdots \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here the left hand column is the sequence (1). The lower left hand square is a pushout diagram. $\Omega^{-1}(L_\zeta)$ is by definition the cokernel of the embedding of L_ζ into its injective hull, which is P_{n-1}. Because the kernel of ∂_{n-2} is $\Omega^{n-1}(k)$ we have an exact sequence

$$E_\zeta : 0 \longrightarrow k \xrightarrow{\sigma_\zeta} \Omega^{-1}(L_\zeta) \longrightarrow \Omega^{n-1}(k) \longrightarrow 0.$$

Now the class of E_ζ in $\mathrm{Ext}^1_{kG}(\Omega^{n-1}(k), k) \cong \mathrm{Ext}^n_{kG}(k,k) \cong H^n(G,k)$ is ζ. This all follows by standard homological algebra. Another way of saying it is the following.

Lemma 2.4. *The sequence E_ζ represents the class ζ in the sense that ζ annihilates the cohomology of M (i.e. $\zeta \in J(M)$) if and only if $E_\zeta \otimes_k M$ splits.*

The point of this is that the class of $E_\zeta \otimes M$ in

$$\operatorname{Ext}^1_{kG}(\Omega^{n-1}(k) \otimes M, M) \cong \operatorname{Ext}^1_{kG}(\Omega^{n-1}(M), M) \cong \operatorname{Ext}^n_{kG}(M, M)$$

is precisely the cup product $\zeta \mathrm{Id}_M$.

Notice that the lemma is more a statement about the ideal generated by ζ than it is about the element ζ. Viewed this way it has a generalization. Suppose that $I = (\zeta_1, \ldots, \zeta_t)$ is an ideal in $H^*(G,k)$ generated by homogeneous elements $\zeta_1, \ldots, \zeta_t$. Let $\sigma_i : k \longrightarrow \Omega^{-1}(L_{\zeta_i})$ be the map $\sigma_i = \sigma_{\zeta_i}$ in the sequence E_{ζ_i} as above. So we have a map

$$\sigma_1 \otimes \cdots \otimes \sigma_t : k \otimes \cdots \otimes k \longrightarrow \Omega^{-1}(L_{\zeta_1}) \otimes \cdots \otimes \Omega^{-1}(L_{\zeta_t}).$$

Suppose that $\hat{U}$ is the cokernel of this map. Notice that $k \otimes \cdots \otimes k \cong k$, and let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_t$. Then we have an exact sequence

$$E(I) : 0 \longrightarrow k \xrightarrow{\sigma} \bigotimes_{i=1}^{t} \Omega^{-1}(L_{\zeta_i}) \longrightarrow \hat{U} \longrightarrow 0. \tag{2}$$

Proposition 2.5. ([CW2]) *The sequence $E(I)$ represents the ideal $I \subset H^*(G,k)$ in the sense that, for any finitely generated kG-module M, $I \subset J(M)$ if and only if $E(I) \otimes M$ splits.*

The sequence $E(I)$ and some of the terms in the sequence have several other interesting properties which are explained in some detail in [CP].

3. Idempotent modules and applications

In this section we present a brief description of one of the surprises of dealing with infinitely generated modules. One of the main motivations for the investigation of infinitely generated modules arose in the study of quotient categories of kG-modules in [CDW]. There it was shown that certain quotient categories have no Krull-Schmidt Theorem, no unique decomposition of objects as direct sums of indecomposables. Even the trivial module, which is indecomposable, has the property that a direct sum of certain number of copies of the trivial module is also expressible as a sum of modules induced from proper subgroups [C3]. This also happens for other modules [CW1].

It was Rickard who observed that the lack of a Krull-Schmidt theorem could be repaired if the category allowed certain infinite direct sums. However then it became necessary to extend the notions of complexity and varieties to the infinitely generated modules. This was accomplished in [BCR1] and [BCR2].

Recall from the last section that if $\zeta \in H^n(G,k)$, $\zeta \neq 0$, then we have an exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^n(k) \xrightarrow{\hat{\zeta}} k \longrightarrow 0$$

where $\bar{\zeta}$ is a cocycle representing ζ. For M a kG-module let $M^* = \mathrm{Hom}_k(M, k)$ be the k-dual with G-action given by $(gf)(m) = f(g^{-1}m)$. Then taking duals is an exact contravariant functor and from the previous exact sequence we get a new one:

$$0 \longrightarrow k \xrightarrow{\hat{\zeta}} \Omega^{-n}(k) \longrightarrow L_\zeta^* \longrightarrow 0$$

Here $(\Omega^n(k))^* \cong \Omega^{-n}(k)$ because the dual of a projective resolution of k is an injective resolution. Of course, we should remember that injective modules are also projective and in fact $(kG)^* \cong kG$ as kG-modules.
Now suppose that $P_1 \xrightarrow{\rho} \Omega^{-n}(k)$ is a projective cover of $\Omega^{-n}(k)$. Then we get an exact sequence

$$0 \longrightarrow \Omega(L_\zeta^*) \longrightarrow k \oplus P_1 \xrightarrow{\begin{pmatrix} \hat{\zeta} \\ \rho \end{pmatrix}} \Omega^{-n}(k) \longrightarrow 0\,. \tag{3}$$

It is an exercise to prove that the kernel of $\begin{pmatrix} \hat{\zeta} \\ \rho \end{pmatrix}$ is isomorphic to $\Omega(L_\zeta^*)$. Now recall also that $\Omega^n(k) \otimes \Omega^n(k) \cong \Omega^{2n}(k) \oplus \mathrm{proj}$, where "$\oplus$ proj" indicates the direct sum of some projective module. So we have a commutative diagram

$$\begin{array}{ccccc} \Omega^{2n}(k) & \xrightarrow{i} & \Omega^n(k) \otimes \Omega^n(k) & \xrightarrow{\bar{\zeta}\otimes\bar{\zeta}} & k \\ & & {\scriptstyle 1\otimes\bar{\zeta}} \downarrow & & \| \\ & & \Omega^n(k) & \xrightarrow{\bar{\zeta}} & k \end{array}$$

where i is the split inclusion. We denote the composition $(1 \otimes \bar{\zeta})i$ by $\Omega^n(\bar{\zeta})$. Taking duals we get a system of commutative diagrams

$$\begin{array}{ccc} k & \xrightarrow{\hat{\zeta}} & \Omega^{-n}(k) \\ \| & & \downarrow {\scriptstyle \Omega^{-n}(\hat{\zeta})} \\ k & \xrightarrow{\hat{\zeta}^2} & \Omega^{-2n}(k)) \\ \| & & \downarrow {\scriptstyle \Omega^{-2n}(\hat{\zeta})} \\ k & \xrightarrow{\hat{\zeta}^3} & \Omega^{-3n}(k) \\ \vdots & & \vdots \end{array}$$

Now adding projectives to the middle terms we can get a directed system of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(L_\zeta^*) & \longrightarrow & k\oplus P_1 & \longrightarrow & \Omega^{-n}(k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega(L_{\zeta^2}^*) & \longrightarrow & k\oplus P_2 & \longrightarrow & \Omega^{-2n}(k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega(L_{\zeta^3}^*) & \longrightarrow & k\oplus P_3 & \longrightarrow & \Omega^{-3n}(k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{4}$$

where the middle column is a sequence of identity maps on k plus projectives. Taking direct limits is an exact functor so we get a sequence of the direct limits:

$$0 \longrightarrow E(\zeta) \longrightarrow k \oplus (\text{proj}) \longrightarrow F(\zeta) \longrightarrow 0\,. \tag{5}$$

Now suppose that M is a finitely generated module and that $\zeta^m \in J(M)$ for some m. Then we can tensor the system (4) with the module M and notice that in the right hand column

$$\Omega^{-n}(k)\otimes M \longrightarrow \Omega^{-2n}(k)\otimes M \longrightarrow \dots$$

any composition of m of these maps factors through a projective. It can be proved that this implies that the limit of the system, $F(\zeta)\otimes M$, is a projective module. Hence by tensoring sequence (3) with M, we get a sequence

$$0 \longrightarrow E(\zeta)\otimes M \longrightarrow M \oplus \text{proj} \longrightarrow \text{proj} \longrightarrow 0$$

which splits. Consequently $E(\zeta)\otimes M \cong M \oplus (\text{proj})$. In addition, $E(\zeta)$ is itself a direct limit of such modules, and we must have that $E(\zeta)\otimes F(\zeta)$ is projective.

Lemma 3.1. *For $\zeta \in H^n(G,k)$, $\zeta \neq 0$, we have that*
- i $E(\zeta)\otimes F(\zeta) \cong$ *proj,*
- ii $E(\zeta)\otimes E(\zeta) \cong E(\zeta)\oplus$ *proj,*
- iii $F(\zeta)\otimes F(\zeta) \cong F(\zeta)\oplus$ *proj.*

The last two statements follow from (i) and the sequences obtained by tensoring sequence (3) with $E(\zeta)$ and $F(\zeta)$ respectively. Because of the results of the lemma we say that $E(\zeta)$ and $F(\zeta)$ are (orthogonal) idempotent modules.
Everything that we have said so far in this section is a special case of a more general construction of Rickard [R]. To get a broader viewpoint we should note first that the condition that $\zeta^m \in J(M)$ for some m is equivalent to the statement that $V_G(M) \subset V_G(\zeta)$. In fact, the modules $E(\zeta)$ and $F(\zeta)$ do not depend so much on ζ as they do on the variety $V = V_G(\zeta)$. Moreover we can substitute $V = V_G(\zeta_1,\dots,\zeta_t)$ for $V_G(\zeta)$ by performing a similar construction beginning with sequences (2) of the last section [CW2].

Rickard's best result is even more general that what we have stated below. To state the main theorem of [R] in full generality requires some explanation of thick subcategories of the stable category of kG-modules modulo projectives. Such an explanation is beyond the scope of these notes.

Theorem 3.2. [R] *Let $\mathcal{V} \subset \mathcal{V}_G(k)$ be any collection of closed homogeneous subvarieties of $V_G(k)$. Assume that $\mathcal{V}$ is closed under specialization (i.e. if $U \subset V \in \mathcal{V}$ then $U \in \mathcal{V}$). Let $\mathcal{M}_{\mathcal{V}}$ be the full subcategory of all finitely generated kG-modules M with $V_G(M) \in \mathcal{V}$. There is an exact sequence*

$$0 \longrightarrow E(\mathcal{V}) \longrightarrow k \oplus proj \longrightarrow F(\mathcal{V}) \longrightarrow 0$$

of kG-modules such that

i *$E(\mathcal{V})$ and $F(\mathcal{V})$ are idempotent modules in the sense of Lemma 3.1,*
ii *$E(\mathcal{V})$ is a direct limits of objects in $\mathcal{M}_{\mathcal{V}}$, and*
iii *$F(\mathcal{V})$ is $\mathcal{M}_{\mathcal{V}}$ local (i.e. if $X \in \mathcal{M}_{\mathcal{V}}$ and $\alpha : X \longrightarrow F(\mathcal{V})$ then α factors through a projective module).*

In fact properties (ii) and (iii) characterize the sequence up to direct sums with projective modules and maps which factor through projectives.
We end this paper by mentioning two applications. First the idempotents modules allows us to define a cohomological variety for all kG-modules. The definition goes as follows. Let V be a homogeneous closed irreducible subset of $V_G(k)$. Let

$$a(V) = \{W \subset V_G(k) | W \text{ closed, homogeneous and } W \subset V\},$$

$$b(V) = \{W \in a(V) | W \neq V\}.$$

Let $\kappa(V) = E(a(V)) \otimes F(b(V))$. Then the cohomological variety of a kG-module M is

$$\mathcal{V}_G(M) = \{V \in V_G(k) | V \text{ closed, irreducible and } \kappa(V) \otimes M \text{ not projective}\}.$$

This "variety" again is not really a variety. However it does satisfy Dade's Lemma and the Tensor Product Theorem [BCR2]. Also if G is an elementary abelian p-group then it has a natural association with the rank variety of M, as in Theorem 2.2.
Finally the idempotent modules can be used to give necessary and sufficient condition on a closed subvariety $V \subset V_G(k)$ and a subgroup H such that if M is a kG-module with $V_G(M) \subset M$ then M must be stably induced from H in the sense that there is a kH-module L with $L^{\uparrow G} \cong M \oplus \text{proj}$. The condition is very technical and we will not state it here. This type of theorem was first proved by Benson in [B2] in the case that V is a line. It was generalized in [C4].

Other applications of the technology include a characterization of the thick subcategories of the stable category for some groups [BCR3] and a solution to some questions on the vanishing of group cohomology [B2].

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QUASI-HEREDITARY ALGEBRAS REVISITED

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ABSTRACT. This paper is an attempt to complement my series of lectures on quasi-hereditary algebras with brief references to existing literature, reiterating the basic concepts of the existing theory, revisiting the definition of quasi-hereditary algebras and pointing out some of the recent developments. The basic aim of the series was to provide an accessible introduction to the theory — initiated by Cline–Parshall–Scott ([PS], [CPS1]) in their studies of semi-simple complex Lie algebras and algebraic groups — for a general algebraically oriented audience. The presentation was based on a previous exposition [DK], as well as on my recent work with I. Ágoston and E. Lukács; here, a reference to [A] could also be found helpful. Due to time limitations, a selection of topics was necessary and thus in no way it reflected all developments in this rapidly expanding field. The subject was presented as a part of ring theory with only a few particular references to the numerous applications which have clearly governed a number of recent results in the area.

1. NOTATIONS, DEFINITIONS

Let A be a (finite dimensional associative) K-algebra and, without loss of generality, let A be basic (and connected). Thus the right regular representation decomposes into a direct sum $A_A = \bigoplus_{i=1}^{n} P(i)$ of pairwise non-isomorphic (right) indecomposable projective A-modules. Write

$$P(i) = e_i A \ \text{ with } \ \sum_{i=1}^{n} e_i = 1.$$

Now, we shall always consider an order of the set $\{ P(i) \mid 1 \le i \le n \}$, i. e. we shall consider a (complete) sequence of the primitive orthogonal idempotents $\mathrm{e} = (e_1, e_2, \dots, e_n)$. In addition, we shall consider the related idempotents $\varepsilon_i = e_i + e_{i+1} + \ldots + e_n$ for $1 \le i \le n$ and $\varepsilon_{n+1} = 0$. Equivalently, we consider an

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order $(S(1), S(2), \dots, S(n))$ of the simple A-modules $S(i) = P(i)/\operatorname{rad} P(i) = e_i A/e_i \operatorname{rad} A$.

Definition 1.1. Define (with respect to the fixed order **e**!) the sequence

$$\Delta = (\Delta(1), \Delta(2), \dots, \Delta(n))$$

of the right *standard* A-modules, and the sequence

$$\overline{\Delta} = (\overline{\Delta}(1), \overline{\Delta}(2), \dots, \overline{\Delta}(n))$$

of the right *proper standard* A-modules by

$$\Delta(i) = e_i A/e_i \operatorname{rad} A\varepsilon_{i+1} A, \quad 1 \le i \le n,$$

and

$$\overline{\Delta}(i) = e_i A/e_i \operatorname{rad} A\varepsilon_i A, \quad 1 \le i \le n,$$

respectively.

Remark. Denote by $P^\circ(i)$, $S^\circ(i)$, $\Delta^\circ(i)$ and $\overline{\Delta}^\circ(i)$ corresponding (i.e. indecomposable projective, simple, standard and proper standard) left A-modules:

$$P^\circ(i) = Ae_i, \quad S^\circ(i) = Ae_i/\operatorname{rad} Ae_i, \quad \Delta^\circ(i) = Ae_i/A\varepsilon_{i+1} \operatorname{rad} Ae_i \text{ and}$$
$$\overline{\Delta}^\circ(i) = Ae_i/A\varepsilon_i \operatorname{rad} Ae_i\,.$$

Observe that for any K-algebra A and any order **e**, $\Delta(n)$ and $\Delta^\circ(n)$ are projective and $\overline{\Delta}(1)$ and $\overline{\Delta}^\circ(1)$ are simple A-modules. Clearly,

$$\operatorname{End}_A \overline{\Delta}(i) = \operatorname{End}_A \overline{\Delta}^\circ(i) = e_i Ae_i/e_i \operatorname{rad} Ae_i \big(= \operatorname{End}_A S(i) = \operatorname{End}_A S^\circ(i)\big) = D_i$$

for all $1 \le i \le n$. In fact, one can see immediately that $\Delta(i) = \overline{\Delta}(i)$ (and consequently $\Delta^\circ(i) = \overline{\Delta}^\circ(i)$) if and only if

$$\operatorname{End}_A \Delta(i) = e_i Ae_i/e_i A\varepsilon_{i+1} Ae_i = \varepsilon_i A\varepsilon_i/\varepsilon_i A\varepsilon_{i+1} A\varepsilon_i \big(= \operatorname{End}_A \Delta^\circ(i)\big) = D_i\,,$$

i.e. if and only if $e_i A\varepsilon_{i+1} Ae_i = e_i \operatorname{rad} Ae_i$. In this case, the standard module $\Delta(i)$ is said to be *Schurian* [DK]. Consequently, Δ and $\overline{\Delta}$ coincide if and only if all $\Delta(i)$'s (and thus also all $\Delta^\circ(i)$'s) are Schurian.

Definition 1.2. Given a K-algebra $(A, \mathbf{e})$, i.e. A with an order **e** of the simple A-modules, call

$$\mathcal{S} = \mathcal{S}_A = \big(D_i = \operatorname{End}_A S(i),\ {}_iW_j = D \operatorname{Ext}^1_A (S(i), S(j)), 1 \le i, j \le n\big)$$

the *ordered K-species* of A.

Recall that, for any K-species $\mathcal{S}$, there is a canonical tensor algebra

$$T(\mathcal{S}) = \bigoplus_{k\geq 0} W^{\otimes^k}, \text{ where } W^{\otimes^1} = \bigoplus_{i,j} {}_iW_j, \quad W^{\otimes^0} = \prod_{i=1}^{n} D_i = D$$

and the multiplication is induced by $\underset{D}{\otimes}$. Denote by $d_i = [D_i : K]$, $1 \leq i \leq n$, the K-dimension of D_i.

One of the basic concepts of the theory of (A, e) is the concept of the canonical trace filtration of a module X_A afforded by the traces $X\varepsilon_i A$ of the projective A-modules $\varepsilon_i A$.

Definition 1.3. Given an A-module X, define its *trace filtration* (with respect to e) by

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \ldots \supseteq X^{(i)} \supseteq X^{(i+1)} \supseteq \ldots \supseteq X^{(n)} \supseteq X^{(n+1)} = 0,$$

where $X^{(i)} = \langle \varphi(\varepsilon_i A) \mid \varphi \in \operatorname{Hom}(\varepsilon_i A, X) \rangle = X\varepsilon_i A$ for $1 \leq i < n$.

In particular, we have a canonical filtration of the algebra (A, e) by the idempotent (two-sided) ideals $A\varepsilon_i A$, and the respective sequence of K-algebras

$$B_i = A/A\varepsilon_{i+1}A, \quad 1 \leq i \leq n.$$

Given an A-module X_A, denote the B_i-module $X\varepsilon_i A/X\varepsilon_{i+1}A$ by $\Phi_X(i)$. Notice that, for each $1 \leq i \leq n$, $\Delta(i) = \Delta_A(i) = \Delta_{B_i}(i)$ is a projective B_i-module and that there is an epimorphism

$$(*) \qquad \bigoplus_{finite} \Delta(i) \longrightarrow \Phi_X(i).$$

In particular, as A-modules, $B_1 = \Delta(1)$.

The central definition of a quasi-hereditary algebra reflects the fact that for $X = A_A$ and for each $1 \leq i \leq n$, there is an isomorphism $(*)$, and that all $\Delta(i)$'s are Schurian ([PS], [DR1]).

Definition 1.4. The algebra (A, e) is said to be *quasi-hereditary* (with respect to the order e!) if $\Phi(i) = A\varepsilon_i A/A\varepsilon_{i+1}A$ is a projective $A/A\varepsilon_{i+1}A$-module and $e_i \operatorname{rad} A\varepsilon_{i+1} \operatorname{rad} Ae_i = e_i \operatorname{rad} Ae_i$ for all $1 \leq i \leq n$.

In other words, (A, e) is quasi-hereditary if and only if A_A has a Δ-filtration and all $\Delta(i)$'s are Schurian. Furthermore, since for any algebra (A, e), each trace factor $\Phi(i)$ contains a B_i-projective direct summand $\Delta(i)$, (A, e) is quasi-hereditary if and only if $\operatorname{End}_{B_i} \Phi(i)$ is a simple K-algebra for all $1 \leq i \leq n$.

Let us refer to [DK], for a number of other important characterizations of quasi-hereditary algebras. In the following Section 2, we will offer yet another description of these algebras.

2. Δ-FILTERED ALGEBRAS

In this section, we shall characterize a class of K-algebras which contains the class of quasi-hereditary algebras, in terms of their K-dimension. In turn, we shall obtain a new characterizations of quasi-hereditary algebras. Let us start with a simple lemma. Recall that we deal with a K-algebra (A, e) with a fixed order of the simple A-modules, the canonical trace filtration of A, the respective factors $\Phi(i)$ and factor algebras B_i, as well as the standard and proper standard A-modules.

Lemma 2.1. *Always*

$$\dim_K Ae_nA \leq \frac{1}{d_n} dim_K \overline{\Delta}^{\circ}(n) \dim_K \Delta(n).$$

The equality

$$(**) \qquad \dim_K Ae_nA = \frac{1}{d_n} dim_K \overline{\Delta}^{\circ}(n) \dim_K \Delta(n)$$

holds if and only if the right A-module $(Ae_nA)_A$ is projective, i. e. if $(Ae_nA)_A$ is filtered by $\Delta(n)$. In turn, this is equivalent to the fact that the left A-module ${}_A(Ae_nA)$ is filtered by $\overline{\Delta}^{\circ}(n)$. Moreover, the number of factors $\overline{\Delta}^{\circ}(n)$ in Ae_nAe_i is equal to $\frac{1}{d_n}\dim_K e_nAe_i$ for every $1 \leq i \leq n$; in particular, the number of factors $\overline{\Delta}^{\circ}(n)$ in $\Delta^{\circ}(n)$ is equal to $\frac{1}{d_n}\dim_K e_nAe_n$.

Proof. Given an A-module X, we have $[X : S(i)] = \frac{1}{d_i}\dim_K Xe_i$ for every $1 \leq i \leq n$. Thus, in particular,

$$[A/Ae_n \operatorname{rad} Ae_nA : S(n)] = \frac{1}{d_n} dim_K Ae_n/Ae_n \operatorname{rad} Ae_n = \frac{1}{d_n} dim_K \overline{\Delta}^{\circ}(n).$$

Now,

$$\dim_K Ae_nA \leq [A/Ae_n \operatorname{rad} Ae_nA : S(n)] \cdot \dim_K e_nA,$$

and thus

$$\dim_K Ae_nA \leq \frac{1}{d_n} \dim_K \overline{\Delta}^{\circ}(n) \cdot \dim \Delta(n).$$

Obviously, the equality $(**)$ is equivalent to the fact that $(Ae_nA)_A$ is a direct sum of copies of $\Delta(n)$, and thus to the projectivity of $(Ae_nA)_A$. In fact, analyzing the isomorphism

$$(***) \qquad \overline{\Delta}^{\circ}(n) \underset{D_n}{\otimes} \Delta(n) \longrightarrow Ae_nA$$

given by the multiplication map, we obtain a filtration of ${}_A(Ae_nA)$ by $\overline{\Delta}^{\circ}(n)$. On the other hand, the existence of such a filtration implies that the multiplication map $(***)$ is a monomorphism. Furthermore, the number of factors $\overline{\Delta}^{\circ}(n)$ in the direct summand Ae_nAe_i of ${}_A(Ae_nA)$ is clearly equal to $\frac{1}{d_n}\dim_K e_nAe_i$ for every $1 \leq i \leq n$.

Remark. In fact, one can show, for a given $1 \leq i \leq n$, that the right B_i-module $\Phi(i) = A\varepsilon_iA/A\varepsilon_{i+1}A$ is filtered by $\Delta(i)$ if and only if the left B_i-module $\Phi(i)$ is filtered by $\overline{\Delta}^{\circ}(i)$, and express the number of factors $\overline{\Delta}^{\circ}(i)$ in $\Phi(i)e_j$ for each $1 \leq j \leq i$. We hope that the reader will find this to be a useful exercise.

Lemma 2.2. *Every right projective A-module has a Δ-filtration if and only if every left projective A-module has a $\overline{\Delta}^{\circ}$-filtration. The number of factors $\overline{\Delta}^{\circ}(i)$ in $\Delta^{\circ}(i)$ equals $\frac{1}{d_i}\dim_K e_iAe_i/e_iA\varepsilon_{i+1}Ae_i$ for each $1 \leq i \leq n$.*

Proof. Every right projective A-module has a Δ-filtration if and only if A_A has a Δ-filtration. Applying induction to the trace filtration of A_A, Lemma 2.1 yields a $\overline{\Delta}^{\circ}$-filtration of ${}_AA$. Lemma 2.2 follows.

Definition 2.3. The algebra (A, e) is said to be Δ-*filtered* if all right B_i-modules $\Phi(i) = A\varepsilon_iA/A\varepsilon_{i+1}A$ are projective. Equivalently, (A, e) is Δ-filtered if A_A has a Δ-filtration.

Theorem 2.4. *The algebra (A, e) is Δ-filtered if and only if*

$$\dim_K A = \sum_{i=1}^{n} \frac{1}{d_i} \dim_K \overline{\Delta}^{\circ}(i) \cdot \dim_K \Delta(i).$$

Proof. This is an immediate consequence of Lemma 2.1 when applied inductively to the trace filtration of A_A.

Remark. If (A, e) is quasi-hereditary, then both (A, e) and (A^{op}, e) are Δ-filtered. Observe that if (A, e) is Δ-filtered, then (A^{op}, e) does not have to be Δ-filtered:

Consider the path algebra of the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \circlearrowright \gamma$$

modulo the ideal $\langle \beta\alpha, \gamma\beta, \gamma^2 \rangle$.

On the other hand, even if both (A, e) and (A^{op}, e) are Δ-filtered, (A, e) does not have to be quasi-hereditary:

Consider the path algebra of the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

modulo the ideal $\langle \beta\alpha\beta\alpha \rangle$.

However, we have the following simple characterizations of quasi-hereditary algebras.

Theorem 2.5. *The algebra (A, e) is quasi-hereditary if and only if (A, e) is Δ-filtered and, for every $1 \leq i \leq n$, $\overline{\Delta}(i) = \Delta(i)$ (and thus also $\overline{\Delta}^{\circ}(i) = \Delta^{\circ}(i)$).*

Proof. Clearly, $\overline{\Delta}(i) = \Delta(i)$ if and only if $[\Delta(i) : S(i)] = 1$, i.e. if and only if $\Delta(i)$ is Schurian.

Corollary 2.6. (see [D], also [W]). *The algebra (A, e) is quasi-hereditary if and only if (A, e) is Δ-filtered and $gl.dim\, A < \infty$.*

Proof. If (A, e) is Δ-filtered and not quasi-hereditary, then there is i, $1 \leq i \leq n$, such that $\left[\Delta^{\circ}(i) : \overline{\Delta}^{\circ}(i)\right] \geq 2$. But then obviously

$$proj.dim_{B_i} \overline{\Delta}^{\circ}(i) = \infty,$$

and thus $gl.dim\, A \geq gl.dim\, B_i = \infty$ (cf. [DR1]).

Corollary 2.7. *The algebra (A, e) is quasi-hereditary if and only if*

$$\dim_K A = \sum_{i=1}^{n} \frac{1}{d_i} \dim_K \Delta^{\circ}(i) \dim_K \Delta(i).$$

Remark. One can derive the last Corollary 2.7 directly. E.g., the necessity of the equality can be obtained by using the Bernstein–Gelfand–Gelfand reciprocity law

$$\left[P(k) : \Delta(i)\right] = \frac{d_k}{d_i}\left[\Delta^{\circ}(i) : S^{\circ}(k)\right]$$

for quasi-hereditary algebras (see [DK]). We get immediately

$$\begin{aligned} \dim_K A &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} d_j \left[P(k) : \Delta(i)\right] \left[\Delta(i) : S(j)\right] = \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{d_k d_j}{d_i} \left[\Delta^{\circ}(i) : S^{\circ}(k)\right] \left[\Delta(i) : S(j)\right] = \\ &= \sum_{i=1}^{n} \frac{1}{d_i} \dim_K \Delta^{\circ}(i) \dim_K \Delta(i). \end{aligned}$$

3. Basic facts

If $(A, \mathbf{e})$ is quasi-hereditary, then gl.dim$A \leq 2(n-1)$ and Loewy length of $A \leq 2^n - 1$; both bounds are optimal [DK]. Moreover, as already mentioned, the (Bernstein-Gelfand-Gelfand) reciprocity law holds:

$$d_j\,[P(i) : \Delta(j)] = d_i\,[\Delta^\circ(j) : S^\circ(i)] \quad \text{for all } 1 \leq i, j \leq n.$$

Thus, denoting by $C = (c_{ij})$ the $n \times n$ Cartan matrix of A (i.e. $c_{ij} = [P(i) : S(j)]$) and defining the nilpotent lower triangular $n \times n$ matrices $\Delta = (u_{ij})$ and $\Delta^o = (v_{ij})$ by $u_{ij} = [\Delta(i) : S(j)]$ and $v_{ij} = [\Delta^o(i) : S^o(j)]$, respectively, we have $C = D\Delta^{ot}D^{-1}\Delta$, where $D = (d_{ij})$ is the diagonal matrix $d_{ii} = d_i$. Consequently the determinant $\mid C \mid = 1$.

For the sake of further references, denote the kernel of the canonical epimorphism $P(i) \to \Delta(i)$ by $V(i)$, and write $U(i) = \operatorname{rad}\Delta(i)$. Thus, for each $1 \leq i \leq n$, we have a short exact sequence

$$0 \to V(i) \to \operatorname{rad} P(i) \to U(i) \to 0.$$

Of course, there are similar canonical short exact sequences

$$0 \to V^o(i) \to \operatorname{rad} P^o(i) \to U^o(i) \to 0.$$

of left $A-$modules.

The quasi-hereditary algebras which appear in the applications have additional special properties. One of such distinguished property is to be *lean* (see [ADL1]). Recall that X is a *top* submodule of Y if $\operatorname{rad} X = X \cap \operatorname{rad} Y$. A *top filtration* of a module is a filtration whose members are all top submodules.

Definition 3.1 The algebra $(A, \mathbf{e})$ is said to be *lean* (with respect to the order $\mathbf{e}$) if, for each $1 \leq i \leq n$, $V(i)$ is a top submodule of $\operatorname{rad} P(i)$ and all $U(i) = \operatorname{rad}\Delta(i)$ have top $\Delta-$filtrations. Equivalently, $(A, \mathbf{e})$ is lean if

$$e_i \operatorname{rad}^2 Ae_j = e_i \operatorname{rad} A\varepsilon_m \operatorname{rad} Ae_j$$

for all $1 \leq i, j \leq n$ and $m = min\{i, j\}$.

In fact, there are very important homological characterizations of lean algebras (see [ADL2]). The following two classes of lean quasi-hereditary algebras are of fundamental importance.

Definition 3.2 A quasi-hereditary algebra is said to be *shallow*, or *replete* if all $\operatorname{rad} P(i), 1 \leq i \leq n$, have top filtrations with factors $S(j), 1 \leq j \leq i-1$, and $\Delta(j), i+1 \leq j \leq n$, or with factors $\Delta(j), 1 \leq j \leq i-1$, and $P(j), i+1 \leq j \leq n$, respectively.

Let us point out that $\operatorname{rad}^3 A = 0$ for a shallow algebra A and gl.dim$A \leq 2$ for a replete algebra A.

Given an ordered K–species $\mathcal{S} = (D_i, {}_iW_j; 1 \leq i,j \leq n)$, there is a canonical construction of a shallow algebra $S_{\mathcal{S}}$ and a replete algebra $R_{\mathcal{S}}$ over $\mathcal{S}$:

$$S_{\mathcal{S}} = T(\mathcal{S})/ < {}_iW_j \otimes {}_jW_k \mid j < max\{i,k\} >$$

and

$$R_{\mathcal{S}} = T(\mathcal{S})/ < {}_iW_j \otimes {}_jW_k \mid j < min\{i,k\} > .$$

Theorem 3.3 *Denote $\mathcal{S} = \mathcal{S}_A$ the (ordered) K–species of a lean quasi-hereditary algebra A. Then*

$$\dim_K S_{\mathcal{S}} \leq \dim_K A \leq \dim_K R_{\mathcal{S}}.$$

In fact, if $\dim_K S_{\mathcal{S}} = \dim_K A$, then A is shallow and if $\dim_K A = \dim_K R_{\mathcal{S}}$, then A is replete.

As an illustration, consider the "total" ordered species

$$\mathcal{S} = \mathcal{S}_n = (D_i = K,\ {}_iW_j = K \text{for all } 1 \leq i,j \leq n,\ \ i \neq j).$$

Then $\dim_K \mathcal{S}_{\mathcal{S}} = \frac{1}{6}n(n+1)(2n+1)$, $\dim_K R_{\mathcal{S}} = \frac{1}{3}(2^{2n}-1)$, gl.dim$\mathcal{S}_{\mathcal{S}} = 2(n-1)$ and Loewy length of $\mathcal{R}_{\mathcal{S}} = 2n-1$.

Recently, Lakatos [L] characterized the lean, shallow and replete algebras in terms of the construction $A(\gamma)$ [DR3].

4. Homological duality

The Kazhdan-Lusztig theory of Cline, Parshall and Scott [CPS2] leads to quasi-hereditary algebras A whose homological dual $A^* =$ Ext –algebra of A is again a quasi-hereditary algebra. One of the objectives of [ADL3] is to find a natural class of such algebras:

Definition 4.1 The algebra (A, e) is said to be *solid* if the following conditions are satisfied for all $1 \leq i \leq n$:

(1) $\Delta(i)$ is Schurian;
(2) $V(i)$ a top submodule of rad $P(i)$;
(3) $U(i)$ has a top filtration by $S(j)$'s and $\Delta(j)$'s for $j < i$;
(4) $V(i)$ has a top filtration by $\Delta(j)$'s and $P(j)$'s for $j > i$.

Proposition 4.2. ([ADL3]). *A solid algebra (A, e) is a lean quasi-hereditary algebra. Moreover, all $S(i), \Delta(i)$ and $U(i)$ belong to a subcategory $C_A \subset$ mod–A defined as follows: $X_A \in C_A$ if and only if its minimal projective resolution*

$$\cdots \overset{d_{j+1}}{\to} \mathcal{P}_j(X) \overset{d_j}{\to} \cdots \overset{d_2}{\to} \mathcal{P}_1(X) \overset{d_1}{\to} \mathcal{P}_0(X) \overset{d_0}{\to} X \to 0$$

satisfies the condition that all $\operatorname{Ker} d_j$ *are top submodules of* $\operatorname{rad} P_j(X)$.

Recall the concept of the Ext −algebra A^* of an algebra A. It is, by definition, the $K-$algebra whose underlying vector space is

$$\bigoplus_{k\geq 0}\bigoplus_{i,j\in I}\operatorname{Ext}_A^k\left(S(i),S(j)\right)$$

and the multiplication is defined by the Yoneda product of extensions.

One of the main results of [ADL3] is the following

Theorem 4.3. *Let $(A,\mathbf{e})$ be a solid algebra. Then: $\left((A^*)^{op},\mathbf{f}\right)$, where $\mathbf{f}$ denotes the "reverse" order to $\mathbf{e}$, is solid. Moreover, the species $\mathcal{S}(A^*)$ is dual to the species of $\mathcal{S}(A)$ and $\dim_K A^{**}=\dim_K A$.*

Remark. Here, it is worthwhile pointing out that, in general, $A^{**}\simeq A$ does not hold for a solid algebra A (see [ADL3]).

Corollary 4.4. *If the algebra $(A,\mathbf{e})$ is shallow, or replete, then $(A^*,\mathbf{f})$ is replete, or shallow over the dual species, respectively.*

Remark. Let us point out that for the monomial algebras A, the connection between $(A,\mathbf{e})$ and $(A^*,\mathbf{f})$ can be described explicitly and that it results in a relationship between leanness and quasi-heredity. In particular, it turns out that the homological dual of a lean quasi-hereditary monomial algebra is again a lean quasi-hereditary algebra (see [ADL3]).

5. Two constructions

For the sake of reference in the next section, let us formulate the following two theorems.

Theorem 5.1. (Construction 1, see [DR2]). *Let R be an arbitrary finite-dimensional K-algebra. Then the endomorphism algebra $\operatorname{End}_K X$, where $X=\bigoplus_{t=1}^{d}R/(\operatorname{rad}R)^t$ with $(\operatorname{rad}R)^d=0$, is a quasi-hereditary algebra with an idempotent $e=e^2\in A$ such that $eAe\simeq R$.*

Theorem 5.2. (Construction 2, see [DHM]). *Let R be a commutative local selfinjective $K-$algebra over a splitting field K; let $\dim_K R=n$. Let $\{X_1=R,X_2,\dots,X_m\}$ be a sequence of local ideals such that $X_i\subset X_j$ implies $i>j$. Then $A=\operatorname{End}(\bigoplus_{t=1}^{m}X_t)$ is quasi-hereditary (with respect to the order of the summands) if and only if $n=m$ and $\operatorname{rad}X_i=\sum_{X_j\subset X_i}X_j$. In fact, under these conditions, A is lean, each $\Delta(i)$ has a simple socle isomorphic to $S(1)$, all $[\Delta(i):S(j)]\leq 1$ and there is a duality $D:\text{mod}-A\to\text{mod}-A$ such that $DS(i)\simeq S(i)$ for all $1\leq i\leq n$, and thus $\dim_K A=\sum_{i=1}^{n}(dim_K\Delta(i))^2$.*

Let us point out that the Construction 2 has been recently instrumental in a major study of Cline-Parshall-Scott on stratification of endomorphism rings [CPS3].

6. Well-filtered algebras

Yet another additional characteristic of the quasi-hereditary algebras which appear in the applications is the property that their filtrations behave well in the sense of [ADL4]. Let us briefly present some of the results of [ADL4].

Definition 6.1. The quasi-hereditary algebra $(A, \mathbf{e})$ is said to be *right well-filtered* if $\bigoplus_{i=1}^{n} V^o(i)$ is a (two-sided) ideal of A; denote this ideal by I^+ and define $A^+ = A/I^+$.

Remark. Left well-filtered algebras are defined similarly. Observe that shallow algebras are both right and left well-filtered. Clearly, $(A, \mathbf{e})$ is right well-filtered if and only if ${}_AA^+ \simeq \bigoplus_{i=1}^{n} \Delta^o(i)$.

The well-filtered algebras are characterized by the property that the quotient algebra $A^+ = A/I^+$ describes the filtration of A_A by standard modules, i.e. the image of every standard filtration A_A under the canonical map $A \to A^+$ gives a composition series of $A^+_{A^+}$ with composition factors corresponding to the tops of the respective standard modules. Note that the algebras which are both right and left well-filtered are necessary lean.

If A is a replete well-filtered algebra or one of the quasi-hereditary endomorphism algebras described by Construction I or Construction II (these are necessarily right well-filtered), then there is a section $A^+ \to A$ with respect to $0 \to I^+ \to A \to A^+ \to 0$. Moreover, the image of this section coincides with the subalgebra defined earlier by Dyer [Dy] and König [K1].

In fact, if A is a replete well-filtered algebra, then A^+ is hereditary and $(A^+)^* \simeq (A^*)^+$, where A^* denotes, as before, the Ext-algebra of A.

If A is the endomorphism algebra defined by Construction I, then A^+ is a uniserial hereditary subalgebra.

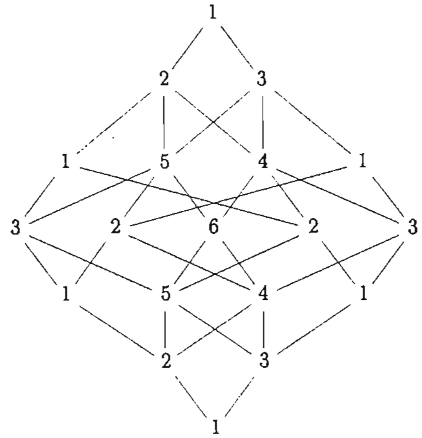
Both Construction I and Construction II were illustrated on typical examples. In particular, the basic algebra to the principal block of $sl(3)$ was presented by Construction II with 6-dimensional selfinjective algebra

$$R = K[x, y]/ < x^2 - y^2, x^3 > .$$

(see also [K2]). In this case,

$$\dim_K A = 1^2 + 2^2 + 2^2 + 4^2 + 4^2 + 6^2 = 77,$$

$P(1)$ contains a copy of every $P(i)$, $1 \leq i \leq n$, and A is homologically selfdual. Explicitly, the structure of $P(1)$ is as follows:



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SOME APPLICATIONS OF INTEGRAL REPRESENTATIONS OF FINITE GROUPS

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The integral representation theory of a finite group G is the study of the representations of G on finitely generated abelian groups, or equivalently, the study of finitely generated $\mathbb{Z}G$-modules. Such representations occur, of course, within abstract group theory, but also in algebraic topology and algebraic number theory. We shall look at an example of such an occurrence in each of these three fields.

A word of warning about the three stories that follow. I decided not to burden the account with chapter and verse for all the unproved statements. These vary from the elementary to the rather deep. The interested reader will find all the relevant details in the references at the end of each section.

1. Group Theory

We work with a given fixed *finite* group G. The natural augmentation $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ (i.e., $\varepsilon(g) = 1$ for all g in G) is a ring homomorphism with kernel the augmentation ideal $\Delta G = \mathfrak{g}$. For any $\mathbb{Z}G$-module A, homological algebra provides a natural isomorphism $H^2(G,A) \simeq \mathrm{Ext}^1_{\mathrm{G}}(\Delta \mathrm{G}, \mathrm{A})$; and we know that $H^2(G,A)$ classifies group extensions of the form $A \rightarrowtail H \twoheadrightarrow G$, while $\mathrm{Ext}^1_{\mathrm{G}}(\Delta \mathrm{G}, \mathrm{A})$ classifies $\mathbb{Z}G$-module extensions $A \rightarrowtail M \twoheadrightarrow \mathfrak{g}$. We begin by explaining a down-to-earth process of obtaining this homological isomorphism.

Start with the group extension

$$\text{(i)} \qquad 1 \longrightarrow A \longrightarrow H \xrightarrow{\pi} G \longrightarrow 1 .$$

This yields a ring homomorphism $\pi : \mathbb{Z}H \to \mathbb{Z}G$ whose kernel is $H\mathfrak{a}$ (the ideal of $\mathbb{Z}H$ generated by $\mathfrak{a} = \Delta A$). Moreover, π induces $\mathfrak{h} \twoheadrightarrow \mathfrak{g}$ and hence we have the exact sequence $0 \to H\mathfrak{a}/\mathfrak{a}\mathfrak{h} \to \mathfrak{h}/\mathfrak{a}\mathfrak{h} \to \mathfrak{g} \to 0$, which is one of (left) $\mathbb{Z}G$-modules. (Observe that for any $\mathbb{Z}H$-module M, $M/\mathfrak{a}M \simeq \mathbb{Z}G \otimes_{\mathbb{Z}H} M$).

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Now $a \mapsto (a-1)+\mathfrak{a}\mathfrak{h}$ defines a $\mathbb{Z}$-homomorphism $A \to H\mathfrak{a}/\mathfrak{a}\mathfrak{h}$ which is easily seen to be a $\mathbb{Z}$-isomorphism. It is even a G-isomorphism: if $\pi(h) = g$, then $ga = hah^{-1}$ and $hah^{-1} - 1 \equiv h(a-1) \bmod \mathfrak{a}\mathfrak{h}$.

Thus from (i) we have constructed the module extension

$$\text{(ii)} \qquad 0 \longrightarrow A \longrightarrow \mathfrak{h}/\mathfrak{a}\mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Let $(-|G)$ be the category of all group extensions like (i) but with varying A and where morphism means a commutative diagram of groups

$$\begin{array}{ccccc} A & \longrightarrow & H & \longrightarrow & G \\ \downarrow & & \downarrow & & \| \\ A' & \longrightarrow & H' & \longrightarrow & G \end{array} .$$

Similarly, let $(-|\mathfrak{g})$ be the category of all module extensions like (ii) with varying A. Then (i) $\to$ (ii) defines a *translation functor* $\Phi : (-|G) \longrightarrow (-|\mathfrak{g})$.

There is also a functor Ψ in the opposite direction. Given $A \rightarrowtail M \overset{\tau}{\twoheadrightarrow} \mathfrak{g}$ in $(-|\mathfrak{g})$, define a multiplication on M by $x.y = \tau(x)y + x + y$. This makes M into a multiplicative semigroup with identity element 0; and A is the subsemigroup whose product is the given addition on A. If $E = \{x \in M \mid \tau(x) = g-1$ some $g \in G\}$, then E is a sub*group* of M $(x^{-1} = -g^{-1}x)$ and $\tau' : E \to G$ via $\tau'(x) = g$ is a surjective group homomorphism with kernel A. So $A \rightarrowtail E \twoheadrightarrow G$ is in $(-|G)$. It is easy to check that $\Phi\Psi$ and $\Psi\Phi$ are equivalent to the appropriate identity functors and so we have an equivalence of categories: $(-|G) \sim (-|\mathfrak{g})$.

Let $R \hookrightarrow F \twoheadrightarrow G$ be a free presentation of G. So F is a free group and we always assume F is finitely generated, say F is free on $x_1, \dots, x_d$. If $\bar{R} = R/[R,R]$, then the conjugation action of F on R induces an action on $\bar{R}$ for which R acts trivially. Hence G acts on $\bar{R}$, making $\bar{R}$ into a $\mathbb{Z}G$-module. This is the *relation module* determined by the given free presentation. Note that $\bar{R}$ is a finitely generated free additive group and thus $\bar{R}$ is a $\mathbb{Z}G$-lattice.

Applying the translation functor gives $0 \to \bar{R} \to \mathfrak{f}/\mathfrak{r}\mathfrak{f} \to \mathfrak{g} \to 0$. Here $\mathfrak{f}/\mathfrak{r}\mathfrak{f}$ is $\mathbb{Z}G$-free on $x_1 - 1, \dots, x_d - 1$. Hook this sequence onto $0 \to \mathfrak{g} \to \mathbb{Z}G \to \mathbb{Z} \to 0$ to obtain

$$\text{(1)} \qquad 0 \to \bar{R} \to (\mathbb{Z}G)^d \to \mathbb{Z}G \to \mathbb{Z} \to 0,$$

thus exhibiting $\bar{R}$ as the second kernel in a projective (even free) $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Some notation: if H is a group, $d(H)$ denotes the minimum number of elements needed to generate H; if M is a $\mathbb{Z}G$-module, $d_G(M)$ is the minimum number of elements needed to generate M as $\mathbb{Z}G$-module.

Group theorists are interested in R and its properties. General questions about R are very difficult to handle. Until the development of integral representation theory, people often had to be content with a very weak form of R, viz., $R/[R,F]$, the largest image on which the action of G is completely lost. Now $R/[R,F] \simeq F/F' \oplus H_2(G,\mathbb{Z})$ (recall that G is finite!). It follows that $d(R/[R,F]) - d(F) = d(H_2(G,\mathbb{Z}))$ and so the left hand side is a constant: it is independent of the choice of free presentation.

Here is a generalization of this fact.

(2) Theorem. *$d_G(\bar{R}) - d(F)$ is constant.*

We sketch the proof, which is a good illustration of the use of integral representation theory.

Let $\mathbb{Z}_{(G)} = \bigcap_{p/|G|} \mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid (b,|G|) = 1\}$. This is a semilocal ring: the only primes are those that divide $|G|$. If A is a $\mathbb{Z}G$-lattice, then $A_{(G)}$ means $\mathbb{Z}_{(G)} \otimes A$. We shall use the following facts.

(3). (i) *$A \vee B$ (A and B are in the same genus) if, and only if, $A_{(G)} \simeq B_{(G)}$;*

(ii) *P is projective if, and only if, $P_{(G)}$ is $\mathbb{Z}_{(G)}G$-free;*

(iii) *$A_{(G)} = U \oplus V$ implies $A = B \oplus C$ with $B_{(G)} \simeq U$, $C_{(G)} \simeq V$; and hence*

(iv) *A has a non-zero projective direct summand if, and only if, $A_{(G)}$ has one.*

If $A = A' \oplus P$, where P is projective and A' has no non-zero projective summand, then A' is called a *core* of A. The decomposition need not be unique, but by (3) and semi-local cancellation, A' and P are both determined to within their genus. Thus if $P_{(G)} \simeq (\mathbb{Z}_{(G)}G)^r$, then r is an invariant of A, called the *projective rank* and written $\mathrm{pr}A$.

(4). *The cores of A lie in a single genus and $\frac{1}{|G|}\dim_{\mathbb{Q}}(\mathbb{Q}\otimes P)$ is the projective rank of A.*

Let $0 \to A \to P \to \mathfrak{g} \to 0$ be a minimal projective presentation of $\mathfrak{g}$. This means that $d_G(P_{(G)}) = d_G(\mathfrak{g}_{(G)})$, or equivalently, that A is core-equal. If another minimal presentation $0 \to A' \to P' \to \mathfrak{g} \to 0$ is given, then Schanuel's Lemma, (3) (ii) and semi-local cancellation show $A_{(G)} \simeq A'_{(G)}$, whence $d_G(A_{(G)})$ is an invariant of G. Schanuel also gives (cf. (1))

$$A \oplus (\mathbb{Z}G)^{d(F)} \simeq \bar{R} \oplus P \tag{5}$$

and so $d_G(\bar{R}_{(G)}) - d(F) = d_G(A_{(G)}) - d_G(\mathfrak{g}_{(G)})$: the right hand side here is obviously an invariant of G and so therefore is the left hand side. But $d_G(\bar{R}_{(G)}) = d_G(\bar{R})$, whence (2) follows. □

Let $P_* = (P, C)$ be a projective resolution of $\mathbb{Z}$ with all terms finitely generated:

$$\cdots \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$
$$P_i \searrow C_i \nearrow P_{i-1}$$

The n-th *partial Euler characteristic* of P_* is defined as

$$\chi_n(P) = \sum_{i=0}^{n} (-1)^{n-i} \operatorname{rank} P_i ,$$

where $\operatorname{rank} P_i = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes P_i)$. We may also view $\chi_n(P)$ as the ordinary Euler characteristic $\chi({}_nP)$ of ${}_nP$, the truncated complex $P_n \to \cdots \to P_0$. The set $\{\chi_n(P) \mid \forall P_*\}$ is bounded below with minimum value written $\chi_n(\mathbb{Z})$.

(6). $\chi_n(\mathbb{Z}) = \chi_n(P)$ *for any minimal projective resolution of* $\mathbb{Z}$.

Thus $\chi_1(\mathbb{Z}) = (d_G(\mathfrak{g}_{(G)}) - 1)|G| = (d_G(\mathfrak{g}) - 1)|G|$ since $d_G(\mathfrak{g}_{(G)}) = d_G(\mathfrak{g})$.

Further, (5) shows that A belongs to the genus of a core of $\bar{R}$ (actually it *is* a core of $\bar{R}$), whence $\operatorname{pr}\bar{R} = d(F) - d_G(\mathfrak{g})$. If $d(F) = d(G)$ (the chosen free presentation is "minimal"), then $\operatorname{pr}\bar{R} = d(G) - d_G(\mathfrak{g})$. We call the right hand side here the *generation gap* of G. It is a remarkable fact that this number can be non-zero.

Finally, (5) and (6) show $\chi_2(\mathbb{Z}) = d(\bar{R}) - d(F) + 1$.

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2. TOPOLOGY

Topologists compare projective resolutions by chain homotopies. If (P, C), (P', C') are projective resolutions of $\mathbb{Z}$ (over a finite group G, as always), then to say the truncated complexes ${}_mP$, ${}_mP'$ are *homotopically equivalent* (write ${}_mP \sim {}_mP'$) means that there exists chain maps (of degree 0) $\alpha\colon {}_mP \to {}_mP'$, $\beta\colon {}_mP' \to {}_mP$ such that $\beta\alpha$ and $\alpha\beta$ are chain homotopically equivalent to the appropriate identity maps. Clearly, ${}_mP \sim {}_mP'$ implies that α restricts to an isomorphism $C_{m+1} \xrightarrow{\sim} C'_{m+1}$ (whose inverse is the restriction of β to C'_{m+1}); and thus $\chi_m(P) = \chi_m(P')$.

Let $[_mP]$ denote the equivalence class of all $_mP' \sim {}_mP$ and write $\mathbf{P}(m;\ell)$ for the set of all $[_mP]$ with $\chi_m(P) = \ell$.

(7). *If* $[_mP]$, $[_mP']$ *are any elements in* $\mathbf{P}(m;\ell)$, *then* $C_{m+1} \vee C'_{m+1}$.

Now let X be a finite, connected 2-dimensional CW-complex. Thus X has finitely many cells and only cells in dimensions 0, 1 and 2. The universal cover $\widetilde{X}$ of X is also a connected 2-dimensional CW-complex and its cellular chain complex $C(\widetilde{X})$ has the form $E_2 \to E_1 \to E_0$, where E_i is a free $\pi_1(X)$-module with basis all i-cells. The homology of $C(\widetilde{X})$ is $\mathbb{Z}$ in dimension 0, 0 in dimension 1 and $\pi_2(X)$ in dimension 2.

The following theorem was proved 45 years ago by Saunders Mac Lane and Henry Whitehead:

(8). *If* Y *is another space like* X, *then* X, Y *are homotopically equivalent if, and only if, there exists a group isomorphism* $\pi_1(X) \xrightarrow{\sim} \pi_1(Y)$ *under which* $C(\widetilde{X})$, $C(\widetilde{Y})$ *are homotopically and equivariantly equivalent as augmented chain complexes.*

This result translates a topological problem into an algebraic one. Progress with the algebraic problem hinges on understanding the relevant integral representation theory of the fundamental groups. Assume now that the fundamental groups are isomorphic to our finite group G. Given $\theta_X : \pi_1(X) \xrightarrow{\sim} G$ and $\theta_Y : \pi_1(Y) \xrightarrow{\sim} G$, we say X, Y are *G-linked homotopically equivalent* spaces if there exists a homotopy equivalence $f: X \to Y$ such that $\theta_Y f_1 = \theta_X$, where f_1 is the homomorphism on fundamental groups induced by f.

Let us say that an element $[_2P]$ in $\mathbf{P}(2;\ell)$ is *free* if there exists $_2P' \sim {}_2P$ such that P'_0, P'_1, P'_2 are G-free.

The essential content of (8) can now be rephrased as

(8′). *The set of all G-linked homotopy classes of 2-dimensional* CW*-complexes with Euler characteristic* ℓ *is bijective with a subset of the free elements in* $\mathbf{P}(2;\ell)$.

It may be that the G-linked homotopy classes are bijective with *all* free elements, but this is not known at present for general (finite) G.

To understand $\mathbf{P}(2;\ell)$ we use the Grothendieck group $K_0(\mathbb{Z}G)$. Let $\{P\}$ denote the isomorphism class of the projective module P (recall again that we only consider finitely generated modules) and form the free abelian group A on all $\{P\}$. Let B be the subgroup generated by all $\{P \oplus Q\} - \{P\} - \{Q\}$ and set $K_0(\mathbb{Z}G) = A/B$, with the image of $\{P\}$ in $K_0(\mathbb{Z}G)$ written $[P]$. Then

$[P] \mapsto \text{rank}P$ is a well-defined homomorphism $K_0(\mathbb{Z}G) \twoheadrightarrow \mathbb{Z}$ and the kernel is $\text{Cl}(\mathbb{Z}G)$, the *projective class group* of $\mathbb{Z}G$.

Given ${}_2P$, we define its *Euler class* to be

$$\varepsilon\,[{}_2P] = [P_2] - [P_1] + [P_0].$$

This is an element in $K_0(\mathbb{Z}G)$. Now ${}_2P \sim {}_2P'$ implies $\varepsilon\,[{}_2P] = \varepsilon\,[{}_2P']$ and so ε is well-defined on $\mathbf{P}(2;\ell)$.

Choose and fix some ${}_2Q$ of Euler characteristic ℓ. Then

(9). (i) *There exists a map* $\mathbf{P}(2;\ell) \to \text{Cl}(\mathbb{Z}G)$ *via* $[{}_2P] \mapsto \varepsilon[{}_2P] - \varepsilon[{}_2Q]$.

(ii) *If* ${}_2Q$ *is free, then the set of all free elements in* $\mathbf{P}(2;\ell)$ *is bijective with the inverse image of* 0 *under the map of part* (i).

(iii) *If* ${}_2Q$ *is free and* ℓ *is non-minimal (i.e.,* $\ell > \chi_2(\mathbb{Z})$*), then* $[{}_2Q]$ *is the only free element.*

(10) Corollary. *Two connected, finite 2-dimensional* CW*-complexes, with isomorphic fundamental groups and equal non-minimal Euler characteristics, are homotopically equivalent spaces.*

References

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3. Number Theory

Let K/k be a finite Galois extension of algebraic number fields. Our friend G now plays the role of the Galois group $\text{Gal}(K/k)$. The ring $\mathfrak{O}_K$ of algebraic integers in K is a $\mathbb{Z}G$-lattice. Its structure has been the subject of intense investigation during the last twenty years, principally by A. Fröhlich and his school. This is *additive theory*. The *multiplicative theory* concerns the $\mathbb{Z}G$-module structure of $U = \mathfrak{O}_K^{\times}$, the group of units in K. Interest in U is actually older than that in $\mathfrak{O}_K$ but far less is known. This and related material is collectively known as *Galois module theory*.

We shall look here only at the local situation, which means that k is a finite extension of $\mathbb{Q}_p$, the field of rational p-adic numbers. This case is considerably easier than the global one. What follows is a report on joint work with Alfred Weiss.[1]

The ring of integers $\mathfrak{O}_K$ in K is here a discrete valuation ring with unique maximal ideal $\mathfrak{P}$ and $U_1 = 1 + \mathfrak{P}$ is a G-invariant subgroup of U, called the group of *principal units.* Now $U = U_1 \times \mu(p')$, where $\mu(p')$ is the subgroup of all elements whose order is prime to p, and the decomposition is one of G-modules. Thus, in order to understand U we only need to understand U_1. Since U_1 is the pro-p-completion of U, U_1 is a $\mathbb{Z}_p$-module (any abelian pro-p-group is, in a natural way, a $\mathbb{Z}_p$-module) and in fact,

(11) *U_1 is a finitely generated $\mathbb{Z}_pG$-module.*

(By contrast, U is *not* finitely generated as $\mathbb{Z}G$-module.)

The proof of (11) uses the p-adic logarithm

$$\log: \qquad 1+x \quad \mapsto \quad \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n, \qquad\qquad x \in \mathfrak{P}.$$

This gives a homomorphism of the multiplicative group U_1 into the additive group K with kernel $\mu(p)$, the group of p-power roots of unity. If $U_n = 1 + \mathfrak{P}^n$, then for n large enough, $\log U_n = \mathfrak{P}^n$, whence $\log U_n$ is a $\mathbb{Z}_pG$-lattice. Now the finiteness of $|U_1 : U_n|$ establishes (11).

The aim is to extract enough algebraic data from the arithmetic situation to enable us to construct a $\mathbb{Z}_pG$-module M that can be shown to be isomorphic to U_1. We already possess one algebraic fact about U_1, its torsion submodule $\mu(p)$. There is a second easy one, its character. For

$$\mathbb{Q}_p \otimes U_1 \simeq \mathbb{Q}_p \otimes \log U_n \simeq \mathbb{Q}_p \otimes \mathfrak{O}_K \simeq K$$

and $K \simeq (\mathbb{Q}_pG)^{[k:\mathbb{Q}_p]}$ by the normal basis theorem. So

(12) $$\mathbb{Q}_p \otimes U_1 \simeq (\mathbb{Q}_pG)^{[k:\mathbb{Q}_p]}.$$

The third algebraic fact involves cohomology. We shall use Tate cohomology throughout. Thus, for all positive dimensions we have the usual cohomology groups; however $H^0(G,A) \simeq A^G/\widehat{G}A$, where $A^G = \{a \in A \mid ga = a \ \forall g \in G\}$ and $\widehat{G} = \Sigma_{g\in G}\, g$; $H^{-1}(G,A) \simeq {}_{\widehat{G}}A/(\Delta G)A$, where ${}_{\widehat{G}}A = \{a \in A \mid \widehat{G}a = 0\}$; and $H^i(G,A) = H_{-i-1}(G,A)$ for all $i \leq -2$.

If v is the normalized valuation on K, then $0 \to U \to K^\times \xrightarrow{v} \mathbb{Z} \to 0$ is an exact sequence of $\mathbb{Z}G$-modules with $v(\Pi) = 1$ where $\Pi\mathfrak{O}_K = \mathfrak{P}$. If $\mathfrak{p} = \mathfrak{P} \cap k = \pi\mathfrak{O}_k$ and $v(\pi) = e$, then e is the *ramification index* of K/k. If $\bar{K} = \mathfrak{O}_K/\mathfrak{P}$, $\bar{k} = \mathfrak{O}_k/\mathfrak{p}$, then $\bar{K}/\bar{k}$ is a finite Galois extension of finite fields, whose Galois group $\bar{G}$ is cyclic of order f, the *inertial degree* of K/k, and

[1] I have learned that Anatoly Yakovlev and Alexandra Yakovleva have quite recently obtained similar results.

$ef = [K : k]$. The unique generator F of $\bar{G}$ given by $F(x) = x^{|\bar{k}|}$ $(x \in \bar{K})$ is called the *Frobenius automorphism* of K/k. The natural homomorphism $G \to \bar{G}$ is surjective and has kernel G_0, the inertial subgroup of G.

There exists a homomorphism (the *Hasse invariant*) $inv_{K/k}: H^2(G, K^\times) \to \mathbb{Q}/\mathbb{Z}$ which is one-one and has image $\frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$. Let $u_{K/k}$ (the *fundamental class*) be the unique element in $H^2(G, K^\times)$ whose invariant image is $\frac{1}{|G|} + \mathbb{Z}$. Use $u_{K/k}$ to construct a group extension G_V and take the push-out along v:

$$\begin{array}{ccccc} K^\times & \rightarrowtail & G_V & \twoheadrightarrow & G \\ {\scriptstyle v}\downarrow & & \downarrow & & \| \\ \mathbb{Z} & \rightarrowtail & G_W & \twoheadrightarrow & G \end{array} .$$

If $\chi = v\, u_{K/k}$, then χ is the cohomology class of the extension G_W and χ may be viewed as a homomorphism $G/[G,G] \to \mathbb{Q}/\mathbb{Z}$ via the isomorphism

$$H^2(G, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(G/[G,G], \mathbb{Q}/\mathbb{Z})$$

induced by integral duality. Taking the cup-product with $u_{K/k}$ produces an isomorphism $H^{-2}(G, \mathbb{Z}) \xrightarrow{\sim} H^0(G, K^\times)$ (by a theorem of Tate) and $H^0(G, K^\times) \simeq k^\times/N\,K^\times$ (where we have abbreviated the norm $N_{K/k}$ as N).

Now $v(k^\times) = e\mathbb{Z}$ and $v(N\,K^\times) = |G|\,\mathbb{Z}$ since $v(gx) = v(x)$ for all $g \in G$. Hence v induces a surjection $k^\times/NK^\times \twoheadrightarrow \frac{1}{f}\mathbb{Z}/\mathbb{Z}$. If $\phi : \bar{G} \to \frac{1}{f}\mathbb{Z}/\mathbb{Z}$ is the isomorphism given by $\phi(F) = \frac{1}{f} + \mathbb{Z}$, then we have the square

$$\begin{array}{ccc} G/[G,G] & \longrightarrow & \bar{G} \\ {\scriptstyle u_{K/k}\cdot -}\downarrow & & \downarrow{\scriptstyle \phi} \\ k^\times/NK^\times & \xrightarrow{v} & \frac{1}{f}\mathbb{Z}/\mathbb{Z} \end{array} ,$$

where both vertical maps are isomorphisms. *Local classfield theory ensures that this square is commutative.* Hence χ is also the composite $G/[G,G] \to \bar{G} \to \frac{1}{f}\mathbb{Z}/\mathbb{Z}$.

Take a minimal free presentation of $\bar{G}$ and pull back along $G \to \bar{G}$:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & H & \longrightarrow & G \\ \| & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & C & \longrightarrow & \bar{G} \end{array}$$

Here $C = \langle c \rangle$ is infinite cyclic, $c \mapsto F$ and so $\mathbb{Z} \to C$ is $1 \mapsto c^f$. The cohomology class of the free extension is $\phi \in \mathrm{Hom}(\bar{G}, \mathbb{Q}/\mathbb{Z})$ and so the pull-back has cohomology class χ. Hence H and G_W are equivalent group extensions, which gives us the commutative diagram

$$\begin{array}{ccccc} K^{\times} & \longrightarrow & G_V & \longrightarrow & G \\ \downarrow & & \downarrow & & \| \\ \mathbb{Z} & \longrightarrow & G_W & \longrightarrow & G \\ \| & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & C & \longrightarrow & \bar{G} \end{array} .$$

Applying the translation functor (cf. the group theory section of this essay) produces the $\mathbb{Z}G$-module diagram

$$\begin{array}{ccccc} U & = & U & & \\ \downarrow & & \downarrow & & \\ K^{\times} & \longrightarrow & V & \longrightarrow & \Delta G \\ v\downarrow & & \downarrow & & \| \\ \mathbb{Z} & \longrightarrow & W & \longrightarrow & \Delta G \\ \| & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}\bar{G} & \longrightarrow & \Delta\bar{G} \end{array}$$

where $\mathbb{Z}\bar{G} \to \Delta\bar{G}$ is $1 \mapsto F - 1$.

The point is that the top two rows determine W arithmetically while the lower two rows determine W in a totally explicit algebraic manner. We call W the *inertial lattice* of K/k. The crucial part of the diagram is the exact sequence

(13) $$0 \to U \to V \to W \to 0.$$

The first bonus for translating is that, while there seemed nothing special about the group G_V, the module V has a remarkable cohomological property: it is cohomologically trivial. This means that $H^r(H, V) = 0$ for all subgroups H of G and all $r \in \mathbb{Z}$. (A cohomologically trivial *torsion-free* module is projective. But V is usually not torsion-free.)

The pro-p-completion of (13) is the exact sequence of finitely generated $\mathbb{Z}_pG$-modules

$$0 \to U_1 \to \widehat{V} \to \widehat{W} \to 0.$$

Here $\widehat{W}$ is a $\mathbb{Z}_pG$-lattice and $\widehat{V}$ is still cohomologically trivial.

Thus we have an explicitly known $\mathbb{Z}_pG$-lattice $\widehat{W}$ and we require a cohomologically trivial presentation of $\widehat{W}$ whose kernel has given torsion (namely $\mu(p)$) and given character (by (12)).

(14) *How do we construct such a presentation and how unique is it?*

Suppose for the moment $\mu(p) = 1$. Thus $\widehat{V}$ is torsion-free, hence $\mathbb{Z}_pG$-projective, and if $0 \to M \to P \to \widehat{W} \to 0$ is any projective presentation then $M \oplus \widehat{V} \simeq U_1 \oplus P$. If $\mathbb{Q}_p \otimes M \simeq \mathbb{Q}_p \otimes U_1$, then $\mathbb{Q}_p \otimes \widehat{V} \simeq \mathbb{Q}_p \otimes P$, from which it follows (by a theorem of Swan) that $\widehat{V} \simeq P$, whence $M \simeq U_1$ (by the Krull-Schmidt Theorem).

It will be useful to work with an abstract copy T of $\mu(p)$ and an embedding $j : T \to U_1$ with $j(T) = \mu(p)$. Our third algebraic invariant is the kernel $\mathcal{U}_1$ of

$$H^1(G, \mathrm{Hom}(\widehat{W}, j)) : H^1(G, \mathrm{Hom}(\widehat{W}, T)) \to H^1(G, \mathrm{Hom}(\widehat{W}, U_1)).$$

(15). *The $\mathbb{Z}_pG$-module U_1 is determined up to isomorphism by $[k : \mathbb{Q}_p]$, T and $\mathcal{U}_1$.*

Thus, if M is a $\mathbb{Z}_pG$-module with

(i) $\mathbb{Q}_p \otimes M \simeq (\mathbb{Q}_pG)^{[k:\mathbb{Q}_p]}$,

(ii) its torsion module isomorphic to T via an embedding j', and

(iii) the kernel of $H^1(G, \mathrm{Hom}(\widehat{W}, j'))$ equal to $\mathcal{U}_1$,

then $M \simeq U_1$.

It looks as though $\mathcal{U}_1$ involves a knowledge of U_1. This is an illusion. One observes that $\mathcal{U}_1$ is (easily seen to be) the kernel of the homomorphism $H^1(G, \mathrm{Hom}(W, T)) \to H^1(G, \mathrm{Hom}(W, U))$ and any cohomology class x in the image of $H^1(G, \mathrm{Hom}(W, T))$ can be represented by an explicit cocycle. So what is needed is a test whether x is zero. There is such a test, because there exists a character t: $H^1(G, \mathrm{Hom}(W, U)) \to \mathbb{Q}/\mathbb{Z}$ whose calculation involves only Hilbert's Theorem 90 and the Hasse invariant; and there is a particular $\mathbb{Z}G$-module endomorphism ω of W with the property that $x = 0$ if, and only if, $t(x) = t(x\omega) = 0$.

If $\mathcal{W}$ denotes the finite ring $H^0(G, \mathrm{Hom}(W, W))$, then clearly $H^1(G, \mathrm{Hom}(W, T))$ is a right $\mathcal{W}$-module. Now t determines a unique (right) $\mathcal{W}$-module homomorphism $H^1(G, \mathrm{Hom}(W, T)) \to \mathcal{W}$ whose kernel is our $\mathcal{U}_1$.

Question (14) is solved by the following purely module-theoretic result.

(16) The Recognition Theorem. *Given a $\mathbb{Z}_pG$-lattice L, a finite $\mathbb{Z}_pG$-module T and the kernel $\mathcal{M}$ of some $\mathcal{L}$-module homomorphism*

$$H^1(G, \mathrm{Hom}(L, T)) \to \mathcal{L}, \ \textit{where } \mathcal{L} = H^0(G, \mathrm{Hom}(L, L)).$$

Then

(i) (*construction*) *there exists an exact sequence $0 \to M \to C \to L \to 0$ with C cohomologically trivial, M a finitely generated $\mathbb{Z}_pG$-module with torsion isomorphic to T via an embedding j and $\mathcal{M}$ is the kernel of $H^1(G, \mathrm{Hom}(L, j))$;*

(ii) (*uniqueness*) *if* $0 \to M' \to C' \to L \to 0$ *is another such sequence, then there exist projective modules* P, P' *so that* $M \oplus P \simeq M' \oplus P'$.

The conclusion in part (ii) becomes $M \simeq M'$ if $\mathbb{Q}_p \otimes M \simeq \mathbb{Q}_p \otimes M'$. To obtain (15) we apply (16) with $L = \widehat{W}$ and $\mathcal{M} = \mathcal{U}_1$.

The Recognition Theorem remains true if $\mathbb{Z}_p$ is replaced by $\mathbb{Z}$. In this form it becomes the first step on the road to understanding the structure of global units.

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PERPENDICULAR CATEGORIES TO EXCEPTIONAL MODULES

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ABSTRACT. We will survey some results on perpendicular categories to exceptional modules over Artin algebras. Since their introduction by Geigle, Lenzing and Schofield, perpendicular categories have become a powerful tool in the representation theory of Artin algebras. They inherit essential properties from $\mathrm{mod}\,\Lambda$, furnish a reduction procedure and open the possibility for proofs by induction. We will illustrate this in several instances. Moreover we pose several problems.

1. EXCEPTIONAL MODULES

Let Λ be an Artin algebra over a commutative Artin ring R. The category of finitely generated Λ-modules will be denoted by $\mathrm{mod}\,\Lambda$. For a Λ-module X we denote by $\mathrm{pd}_\Lambda X$ (resp. $\mathrm{id}_\Lambda X$) the projective (resp. injective) dimension of X. We denote by $\mathrm{gl.dim}\,\Lambda$ the global dimension of Λ. A Λ-module X is called *selforthogonal* if $\mathrm{Ext}^i_\Lambda(X,X) = 0$ for all $i > 0$ and is called *exceptional* if it is selforthogonal and $\mathrm{pd}_\Lambda X < \infty$.

Given any Λ-module X we may decompose it $X \simeq \bigoplus_{i=1}^m X_i^{d_i}$ where X_i is indecomposable and $d_i > 0$ for all i and moreover $X_i \not\simeq X_j$ for $i \neq j$. We call X *multiplicity-free* or *basic* in case $d_i = 1$ for all i. The uniquely determined number m occuring in the direct sum decomposition above will be denoted by $\delta(Z)$. The classical Nakayama conjecture or a version thereof [Na], [AR], (see also [H5]) states that $\delta(X) \leq n$ for a selforthogonal Λ-module X, with $n = \mathrm{rk}\,K_0(\Lambda)$, where $K_0(\Lambda)$ denotes the Grothendieck group of Λ. Equivalently n coincides with the number of isomorphism classes of simple Λ-modules. Unfortunately there is not much known in general at the present stage about the validity of this conjecture. We point out that this applies even to the

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case of Λ being local (i.e. $\delta({}_\Lambda\Lambda) = 1$). Also the question remains open under the very restrictive extra assumption that X is exceptional. The only general result in this direction is a result due to Bongartz [Bo] which states that an exceptional Λ-module X with $\mathrm{pd}_\Lambda X \leq 1$ satisfies $\delta(X) \leq n$. For further details in this direction we refer to [H3].

Of some interest seems to be the following problem. Given an exceptional Λ-module X then study the endomorphism algebra $\mathrm{End}\, X$ of X. We recall that a Λ-module X is called a *brick* if $\mathrm{End}\, X$ is a division ring. The question is under which conditions an indecomposable selforthogonal Λ-module is a brick. Of course this will only hold under very restrictive assumptions. In fact, let Λ be a non-simple local algebra. Then the indecomposable projective is selforthogonal but not a brick. It was shown in [HR] (compare also [H1]), that for a hereditary Artin algebra Λ any indecomposable selforthogonal module is a brick. It also follows that this holds for the more general class of quasitilted algebras [HRS]. Clearly this easily follows from the following result in [HU].

Theorem. *Let Λ be an Artin algebra with* $\mathrm{gl.dim}\,\Lambda \leq 3$. *Let Z be an indecomposable Λ-module with* $\mathrm{pd}_\Lambda Z \leq 1$ *and* $\mathrm{id}_\Lambda Z \leq 2$. *If Z is not a brick and* $\mathrm{Ext}^1(Z, Z) = 0$, *then there exists an indecomposable subfactor V of Z with* $\mathrm{Ext}^2(V, V) \neq 0$.

The problem mentioned above is related to studying exceptional sequences of Λ-modules or exceptional sequences of coherent sheaves on projectives varieties. For details we refer to [CB], [Ri2] and the literature given there.

There are examples of Artin algebras Λ with $\mathrm{gl.dim}\,\Lambda = 2$ where all indecomposable Λ-modules X satisfy $\mathrm{Ext}^2(X, X) = 0$ but there exist indecomposable exceptional modules which are not bricks. For details we refer to [HU]. These algebras are necessarily of infinite representation type (i.e. there exist infinitely many pairwise non-isomorphic indecomposable Λ-modules) as the following result from [HU] shows.

Theorem. *Let Λ be a representation-finite Artin algebra with* $\mathrm{gl.dim}\,\Lambda \leq 3$. *Let Z be an indecomposable Λ-module with* $\mathrm{pd}_\Lambda Z \leq 2$ *and* $\mathrm{id}_\Lambda Z \leq 2$. *If Z is not a brick and* $\mathrm{Ext}^1(Z, Z) = 0$, *then there exists an indecomposable subfactor V of Z with* $\mathrm{Ext}^2(V, V) \neq 0$.

In the next section we will study perpendicular categories to modules of projective dimension at most one. This can be done for arbitrary modules but only in case of exceptional modules we will remain in the class of Artin algebras.

2. Perpendicular categories

For a Λ-module X we denote by $X^{\perp}$ the *right perpendicular category* of X. It is by definition the full subcategory of $\operatorname{mod}\Lambda$ consisting of those Λ-modules Y satisfying $\operatorname{Ext}^i_\Lambda(X,Y) = 0$ for all $i \geq 0$. Of particular interest is the perpendicular category of a Λ-module X which satisfies $\operatorname{pd}_\Lambda X \leq 1$. This has been introduced and studied in [GL] and [S].

It is straightforward to see that in this case $X^{\perp}$ is an abelian category, which is closed under extensions and that the inclusion functor $X^{\perp} \to \operatorname{mod}\Lambda$ is exact.

The next theorem states some fundamental properties of the perpendicular category determined by X. This is a slight generalisation of a result in [GL] and is contained in [H4].

Theorem. *Let $X \in \operatorname{mod}\Lambda$ with $\operatorname{pd}_\Lambda X \leq 1$ and $\operatorname{Ext}^1_\Lambda(X,X) = 0$, then there exists ${}_\Lambda Q \in X^{\perp}$ such that $X^{\perp} \simeq \operatorname{mod}\Lambda_0$, with $\Lambda_0 = \operatorname{End}_\Lambda Q$. If X is indecomposable, then $\operatorname{rk} K_0(\Lambda_0) = \operatorname{rk} K_0(\Lambda) - 1$, where $K_0(\Lambda)$ denotes the Grothendieck group of Λ.*

For the convenience of the reader we will recall the construction of Q.

There exists an exact sequence

$$0 \to {}_\Lambda\Lambda \to E \to X^s \to 0$$

such that the connecting homomorphism $\partial : \operatorname{Hom}_\Lambda(X, X^s) \to \operatorname{Ext}^1_\Lambda(X, {}_\Lambda\Lambda)$ is surjective. Let $r = \operatorname{rk}_R \operatorname{Hom}_\Lambda(X,E)$ and let $f_1, \ldots, f_r$ be an R-basis. Consider $f = (f_1, \ldots, f_r)^t : X^r \to E$, where $(f_1, \ldots, f_r)^t$ denotes the transpose of the row vector $(f_1, \ldots, f_r)$. This yields two exact sequences, where $f = \pi\mu$ denotes the canonical factorization:

$$0 \to K = \ker f \longrightarrow X^r \longrightarrow B = \operatorname{im} f \to 0$$

$$0 \to B \longrightarrow E \longrightarrow Q = \operatorname{cok} f \to 0$$

Then Q is a projective generator of $X^{\perp}$.

Before stating some further properties of perpendicular categories we will consider some examples. The proof of the assertions on the perpendicular category in examples (b) and (c) needs some knowledge of the representation theory of hereditary algebras which can be found in [ARS] and [Ri1].

Examples:

(a) Let Λ be an Artin algebra and let P be an indecomposable projective Λ-module, say $P = \Lambda e$ for some primitive idempotent e. Then clearly we have that $P^{\perp} \simeq \operatorname{mod}(\Lambda/\Lambda e\Lambda)$.

(b) Let k be a field and let Λ be the Kronecker algebra. So

$$\Lambda = \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}.$$

Let X be an indecomposable Λ-module whose dimension vector is given as $\mathbf{dim}\, X = (1,1)$. Then $X^{\perp} \simeq \text{mod}^{\text{fd}}\, k[T]$, the polynomial ring in one variable over k and mod^{fd} denotes the finite-dimensional modules. Note that X is not exceptional.

(c) Again let k be a field and let Λ be the following 8-dimensional k-algebra

$$\Lambda = \begin{pmatrix} k & k & k^2 \\ 0 & k & k^2 \\ 0 & 0 & k \end{pmatrix}.$$

Let X be an indecomposable Λ-module whose dimension vector is given as $\mathbf{dim}\, X = (5,5,4)$. It is not hard to see that such an exceptional indecomposable module exists. Then the perpendicular category of X is isomorphic to

$$X^{\perp} \simeq \text{mod} \begin{pmatrix} k & k^4 \\ 0 & k \end{pmatrix}.$$

We now return to the general situation. The following proposition from [H4] contains some relevant homological information.

Proposition. *Let $X \in \text{mod}\,\Lambda$ with $\text{pd}_\Lambda X \leq 1$ and $\text{Ext}^1_\Lambda(X,X) = 0$. Let $X^{\perp} \simeq \text{mod}\,\Lambda_0$, where $\Lambda_0 = \text{End}_\Lambda Q$. If $\text{Ext}^i_\Lambda(Q,Q) = 0$ for $i \geq 1$, then $\text{gl.dim}\Lambda_0 \leq \text{gl.dim}\Lambda$ and $\text{pd}_\Lambda Z - \text{pd}_\Lambda Q \leq \text{pd}_{\Lambda_0} Z$ for all $Z \in X^{\perp}$.*

We will provide some situations where the extra assumption of the last proposition is satisfied.

Lemma. *Let $S \in \text{mod}\,\Lambda$ be a simple module with $\text{pd}_\Lambda S \leq 1$. Let Q be a projective generator of $S^{\perp}$. Then Q satisfies $\text{Ext}^i_\Lambda(Q,Q) = 0$ for $i \geq 1$.*

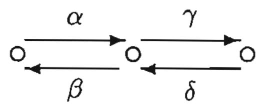
Proof. Observe that S satisfies $\text{Ext}^1_\Lambda(S,S) = 0$ since $\text{pd}_\Lambda S \leq 1$. Recall that the projective generator ${}_\Lambda Q$ was constructed before as the cokernel of a map $f : S^r \to E$, where $r = \dim_R \text{Hom}_\Lambda(S,E)$ and the components of f form an R-basis of $\text{Hom}_\Lambda(S,E)$ and E was constructed as middle term of an exact sequence $0 \to {}_\Lambda\Lambda \to E \to S^t \to 0$. If S is simple, then the map f clearly is injective. In particular we see that $\text{pd}_\Lambda Q \leq 2$. So it is enough to show that $\text{Ext}^2_\Lambda(Q,Q) = 0$. But this follows since $\text{Ext}^1_\Lambda(S,Q) = 0$ and $\text{pd}_\Lambda E \leq 1$.

Recall that it was shown in [Z] that a k-algebra Λ of $\text{gl.dim}\,\Lambda \leq 2$ always has a simple module S with $\text{pd}_\Lambda S \leq 1$.

It is easily seen that for quasi-hereditary algebras there exists an indecomposable projective module P such that the progenerator Q of $P^{\perp}$ satisfies $\text{Ext}^i_\Lambda(Q,Q) = 0$ for $i > 0$. For the definition and further properties of quasi-hereditary algebras we refer to [CPS] and [PS].

Let Λ be a finite-dimensional algebra over an algebraically closed field k. It is also easily seen that a Λ-module $X \in \operatorname{mod}\Lambda$ satisfying the following conditions: $\mathrm{pd}_\Lambda X \leq 1$, $\mathrm{Ext}^1_\Lambda(X,X) = 0$, $\mathrm{End}_\Lambda X \simeq k$ and $\mathrm{Hom}_\Lambda(X, {}_\Lambda\Lambda) = 0$ allows the conclusion of the proposition above. In this case we infer that a projective generator ${}_\Lambda Q$ of $X^\perp$ even satisfies $\mathrm{pd}_\Lambda Q \leq 1$ and coincides with the middle term of the exact sequence $0 \to {}_\Lambda\Lambda \to E \to X^t \to 0$ with $t = \dim_k \mathrm{Ext}^1_\Lambda(X, {}_\Lambda\Lambda)$. Moreover we infer that in this case $\mathrm{Hom}_\Lambda(X, {}_\Lambda Q) = 0$. Note that this is the situation described in [GL].

We will now give an example to show that some assumptions are necessary to obtain a bound as in the proposition above. Consider the algebra Λ given by the following quiver



bound by

$$\alpha\gamma = \delta\beta = \delta\gamma = \beta\alpha - \gamma\delta = 0.$$

So Λ is given as the quotient of the path-algebra over the field k by the two-sided ideal generated by $\alpha\gamma, \delta\beta, \delta\gamma, \beta\alpha - \gamma\delta$.

So the indecomposable projective modules associated with the vertices of the quiver have the following Loewy series:

$$P(1) = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} ; \qquad P(2) = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} ; \qquad P(3) = \begin{array}{c} 3 \\ 2 \end{array} .$$

Then one may easily compute that $\mathrm{gl.dim}\,\Lambda \leq 4$. In fact, we have the following minimal projective resolutions of the simple Λ-modules:

$$0 \to P(3) \to P(2) \to P(1) \to S(1) \to 0$$

$$0 \to P(3) \to P((2) \to P(1) \oplus P(3) \to P(2) \to S(2) \to 0$$

$$0 \to P(3) \to P(2) \to P(1) \oplus P(3) \to P(2) \to P(3) \to S(3) \to 0.$$

Let $X = P(1)$. So X does not satisfy the last two conditions above.

Then the indecomposable summands of the progenerator of $X^\perp$ are easily computed and have the following Loewy series:

$$\begin{array}{c} 2 \\ 3 \end{array} \qquad \begin{array}{c} 3 \\ 2 \end{array}$$

An easy computation also shows that $\mathrm{gl.dim}\,X^\perp = \infty$. For this note that $X^\perp \simeq \operatorname{mod}\Lambda_0$, where Λ_0 is given by the following quiver

$$\circ \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \circ$$

bound by $\gamma\delta = \delta\gamma = 0$.

But if we choose $Y = P(3)$ then one obtains that gl.dim$Y^{\perp} = 2$. In this case the Loewy series of the indecomposable summands of the progenerator of $Y^{\perp}$ are givén as follows:

$$\begin{array}{ccc} 1 & \quad & 2 \\ 2 & & 1 \\ 1 & & \end{array}$$

Thus $Y^{\perp} \simeq \operatorname{mod} \Lambda_0$, where Λ_0 is given by the following quiver

$$\circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$$

bound by $\alpha\beta = 0$.

It seems to be an interesting question to determine conditions for algebras of finite global dimension to admit an indecomposable Λ-module X with $\mathrm{pd}_{\Lambda} X \leq 1$ and $\mathrm{Ext}^1_{\Lambda}(X, X) = 0$ such that gl.dim$X^{\perp} < \infty$.

It is not hard to see that for an algebra Λ of finite global dimension there always exists an indecomposable exceptional module X with $\mathrm{pd}_{\Lambda} X \leq 1$. In fact, let $m =$ gl.dimΛ, then there exists a simple Λ-module S with $\mathrm{id}_{\Lambda} S = m$. Then we may choose $X = \tau_{\Lambda}^{-} S$, where τ_{Λ}^{-} is the inverse of the Auslander-Reiten translation. It is easy to verify that X satisfies our claim.

Note that perpendicular categories are related to the notion of strong idempotents in the sense of [APT].

We also point out that perpendicular categories may be used to show that the bounded derived category of Λ may admit a recollement in certain situations. For details we refer to [BBD] and [H4]. This then may be used to reduce certain questions to problems for algebras with fewer simple modules [H5].

In general it is a difficult problem to compute the perpendicular category to a given module. The situation is somewhat easier in case of hereditary algebras. Moreover most of the recent investigations deal with this particular case. This will be outlined in the next section.

3. Hereditary algebras

For some relevant background in the representation theory of hereditary algebras we refer to [ARS] and [Ri1].

Let $\vec{\Delta}$ be a finite connected quiver without oriented cycle and let k be some field. Let $\Lambda = k\vec{\Delta}$ be the path algebra of $\vec{\Delta}$. Then Λ is a finite-dimensional hereditary k-algebra. Let X be an indecomposable exceptional Λ-module. In this section we will present some results about the structure of $X^{\perp}$. If X is a preprojective or preinjective Λ-module the determination of $X^{\perp}$ is straightforward, compare for example [HHKU]. If X is regular and Λ is tame the computation of $X^{\perp}$ can be read off the results in [Ri1].

Given this we may assume that Λ is representation-infinite or equivalently that Δ is not a Dynkin diagram, where Δ denotes the underlying graph of $\vec{\Delta}$. Recall that an indecomposable Λ-module X is called quasi-simple if X is regular and the middle term M of the Auslander-Reiten sequence ending at X is indecomposable. We are interested in indecomposable regular Λ-modules X with $\mathrm{Ext}^1_\Lambda(X,X) = 0$. It follows that such a module X is a brick, compare section 1. It even holds that it satisfies $\mathrm{End}_\Lambda X \simeq k$, compare [Ri2]. Note that these modules are uniquely determined by their dimension vector and that there exist at most countably many such modules up to isomorphism. If Δ is wild (i.e. properly contains an affine diagram) and has at least three vertices then there are infinitely many indecomposable exceptional Λ-modules up to isomorphism.

Observe that for a quasi-simple module X with $\mathrm{Ext}^1_\Lambda(X,X) = 0$ the middle term M of the Auslander-Reiten sequence ending at X still satisfies $\mathrm{End}_\Lambda M \simeq k$. It is also easily seen that in this case we have an isomorphism $\mathrm{Ext}^1_\Lambda(M,M) \simeq \mathrm{Hom}_\Lambda(X,\tau^2 X)$. Also note that $X^{\perp}$ is again hereditary by the results of the previous section for a module X with $\mathrm{Ext}^1_\Lambda(X,X) = 0$ and that $X^{\perp}$ is equivalent to the module category over a finite-dimensional connected hereditary k-algebra Λ_0 if X is a quasi-simple module with $\mathrm{Ext}^1_\Lambda(X,X) = 0$, see [St]. Moreover if Δ is wild also Δ_0 will be wild in this situation [St]. We point out that in this case Δ_0 will not be a full subgraph of Δ, see [HHKU].

First we will determine necessary and sufficient conditions such that Λ_0 is a tree-algebra (i.e. Δ_0 is a tree). The result shows that this will be rarely the case.

Proposition. *Let X be a quasi-simple module with $\mathrm{Ext}^1_\Lambda(X,X) = 0$ and let M be the middle term of the Auslander-Reiten sequence ending at X. Let $X^{\perp} \simeq \mathrm{mod}k\vec{\Delta}_0$ for some finite and connected quiver $\vec{\Delta}_0$. Then Δ_o is a tree if and only if Δ is a tree and $\mathrm{Ext}^1_\Lambda(M,M) = 0$.*

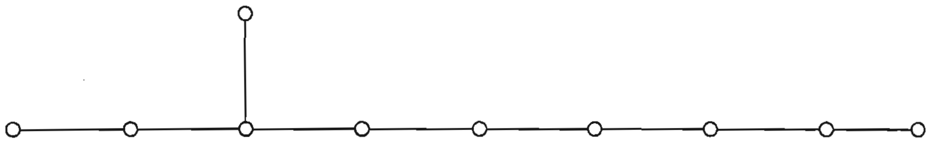
For a proof we refer to [H4]. We point out that this follows from computations of Hochschild cohomology as established in [H2].

Given an arbitrary Artin algebra Λ we may define the Coxeter transformation Φ_Λ. This is an isomorphism of the Grothendieck group uniquely defined by the property that $\Phi_\Lambda(P) = -I$ for each indecomposable projective Λ-module P with simple top S and I is the indecomposable injective Λ-module with simple socle S. Note that we do not distinguish between modules and their isomorphism class. The matrix representation of Φ_Λ with respect to the basis of the simple Λ-modules will also be denoted by Φ_Λ. Let ρ_Λ be the spectral radius of Φ_Λ.

If $\Lambda = k\vec{\Delta}$ for a wild quiver Δ then $\rho_\Lambda > 1$ and actually is a simple eigenvalue [Ri3]. The following is a result in [K1]. Using these and some computations in [X] one may deduce that certain wild quiver algebras will not occur as perpendicular categories. This has been formalized into the concept of domination and is used for establishing natural bijections (Kerner-bijections) between the sets of regular components of a given pair of wild hereditary algebras. For details we refer to [K2].

Theorem. Let $\Lambda = k\vec{\Delta}$ be a wild hereditary algebra and let X be a quasi-simple exceptional Λ-module. Let Λ_0 be wild hereditary with $X^\perp \simeq \operatorname{mod}\Lambda_0$. Then $\rho_{\Lambda_0} > \rho_\Lambda$.

A typical example of a graph Δ_0 such that there is no quiver Δ and no exceptional quasi-simple $k\vec{\Delta}$-module X with $X^\perp \simeq k\vec{\Delta}_0$ is given by the following graph:



We will now discuss some results from [HHKU]. For this let $\vec{\Delta}$ be a wild quiver and $\vec{\Delta}_0$ a quiver such that there exists an exceptional quasi-simple $k\vec{\Delta}$-module X with $X^\perp \simeq \operatorname{mod} k\vec{\Delta}_0$. Note that Δ_0 has to be wild again. We will say that $\vec{\Delta}$ *dominates* $\vec{\Delta}_0$. The main problem dealt with in [HHKU] is the question how many such modules X exist with $X^\perp \simeq \operatorname{mod} k\vec{\Delta}_0$. It is easily seen that with X also $\tau^i X$ for i an integer will have this property. Up to this there will be only finitely many exceptions as the following result from [HHKU] shows.

Theorem. *Given $\vec{\Delta}$ and $\vec{\Delta}_0$ as above. Then there exist only finitely many τ-orbits of exceptional quasi-simple modules Y with $Y^\perp \simeq \operatorname{mod} k\vec{\Delta}_0$.*

The main idea of the proof is to show first that this assertion is independent of the base field. So it will hold for any field if it holds for some field. For this the theory and results for exceptional sequences are used. In a second step the result is proved for a finite field. For this one uses heavily the fact that the automorphism group of the bounded derived category of a wild hereditary algebra is quite 'small'. For details we refer to [HHKU].

The theorem above has the following corollary. It is shown in [HHKU] that a stronger form is actually equivalent to the theorem above.

Corollary. *Let $\vec{\Delta}$ be a wild quiver. Then there exist only finitely many regular components of the Auslander-Reiten quiver of $k\vec{\Delta}$ containing exceptional modules of quasi-length 2.*

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TILTING LINE BUNDLES AND MODULI OF THIN SINCERE REPRESENTATIONS OF QUIVERS

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ABSTRACT. Classification problems of vector bundles on smooth projective varieties over an algebraically closed field k and classification problems of modules over finite dimensional k–algebras are closely related via derived equivalences of the corresponding bounded derived categories. We review known results and provide an introduction to the concept of moduli spaces for a certain class of representations of quivers.

In order to classify vector bundles on algebraic varieties over an algebraically closed field k or modules over finite dimensional k-algebras it is an important problem to classify the so called exceptional objects either in the category $\mathrm{Coh}(\boldsymbol{X})$ of coherent sheaves of $\mathcal{O}_{\boldsymbol{X}}$-modules on a smooth projective algebraic variety $\boldsymbol{X}$ [Ha1] or in the category Λ–mod of finitely generated right modules over a finite dimensional algebra Λ. An object M is exceptional if the endomorphism ring is the field and $\mathrm{Ext}^l(M,M) = 0$ for all $l > 0$. There are several connections between these two classification problems. If $\mathcal{T}$ is a tilting sheaf on $\boldsymbol{X}$ (page 78), then the right derived functor of $\mathrm{Hom}(\mathcal{T}, -)$ is an equivalence of the corresponding bounded derived categories [Ba]

$$\underline{R}\mathrm{Hom}(\mathcal{T}, -) : \mathcal{D}^b(\mathrm{Coh}(\boldsymbol{X})) \longrightarrow \mathcal{D}^b(\Lambda\text{–mod}).$$

For further information on derived categories and derived functors we refer to [Ha2] or [GM]. At present there are only a few examples of tilting sheaves known, and so it is of some interest to construct them.

There is a method to construct partial tilting bundles, which includes direct summands of tilting bundles. For this we consider fine moduli spaces of finite dimensional representations of modules over hereditary algebras [Ki]. A fine moduli space is a vector bundle $\mathcal{U}$ on a variety $\mathcal{M}$ which is an $\Lambda \otimes \mathcal{O}_{\boldsymbol{X}}$-module parametrising the stable modules of a given dimension vector d. So in particular

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each fibre of $\mathcal{U}$ is a finitely generated Λ–module. It is conjectured, that the bundle $\mathcal{U}$ is a partial tilting bundle (page 80). In the case that Λ is hereditary, there is a unique derived equivalence of the above form, in which $\boldsymbol{X}$ is the projective line and Λ is the Kronecker algebra. Thus, for hereditary Λ we are forced to consider partial tilting bundles. If Λ is not hereditary, the construction to be described for partial tilting bundles does not in general work. Already for the dimension vector $d = (1, \dots, 1)$ one can get any projective variety as the moduli space of modules with this dimension vector (page 81). However, most varieties do not admit a tilting bundle. In particular, there is no such bundle if the canonical divisor admits a section.

Assume that $d = (1, \dots, 1)$ and Λ is hereditary. By Morita equivalence we may assume that Λ is the path algebra kQ of some finite quiver Q. In [Hi2] a construction is outlined for any of the moduli spaces $\mathcal{M}^\theta(Q)$ of modules of dimension vector $d = (1, \dots, 1)$ over $\Lambda = kQ$, depending on some weight θ. All these moduli spaces are toric varieties which are well understood. In particular the Picard group, the Grothendieck group and formulas for the cohomology of line bundles are known. This allows us to check in any explicit situation whether a direct sum of line bundles is a partial tilting bundle. Moreover the universal family $\mathcal{U}$ is a direct sum of line bundles. If the cohomology groups of this family vanish, then it is also an exceptional sequence (page 78).

The aim of this paper is to provide an introduction to the concepts mentioned above. The theory of tilting bundles and derived equivalences started with papers of Beilinson [Be] and Bernstein, Gelfand and Gelfand [BGG]. Later Kapranov constructed exceptional sequences on homogeneous varieties and certain intersections of quadrics [K1],[K2]. The notion of an exceptional object was introduced by Drezet and le Portier [DP] in order to classify vector bundles on the projective plane $\boldsymbol{P}^2$. Another approach to this problem was introduced by Rudakov [R1],[R2], who classified exceptional bundles on $\boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ by making use of the solutions of the Markov equation $x^2 + y^2 + z^2 = 3xyz$ for $\boldsymbol{P}^2$, and the Diophantine equation $2x^2 + y^2 + z^2 = 4xyz$ for $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Later Rudakov extended his classification for del Pezzo surfaces. The approach via derived equivalences simplifies parts of his proof substantially. Further Bondal considered exceptional sequences with n members, where $n-1$ is the dimension of $\boldsymbol{X}$, in more detail [Bo] and obtained some remarkable vanishing statement for the extension groups of the simple Λ–modules. Algebras satisfying these vanishing conditions are called Koszul algebras. Bondal's exceptional sequences can be generalized to so called distinguished tilting sequences. These allow us to obtain the analogue of the result of Bernstein, Gelfand and Gelfand [Hi1], an equivalence of the derived category of coherent sheaves with the stable module category over the repetitive algebra. For further results on derived categories of module categories over finite dimensional algebras and on stable module categories we refer the reader to the book of Happel [Hap]. For an introduction to this subject we also mention [SR].

1. Tilting Objects and Exceptional Sequences

Tilting Objects Let $\mathcal{A}$ be an abelian k–category with finite dimensional homomorphism spaces, and of finite global dimension. Let $\mathcal{C}$ be a triangulated k–category (for example the bounded derived category of $\mathcal{A}$) with the property that

$$\bigoplus_i \mathrm{Hom}(M, N[i])$$

is always a finite dimensional vector space where $[i]$ denotes the i-th shift in the triangulated category $\mathcal{C}$. For definitions we refer to [Ha2] or [GM]. The assumptions are fullfiled in case $\mathcal{A}$ is the category of coherent sheaves on a smooth projective variety or the category of finitely generated modules over a directed finite dimensional algebra.

Definition An object $\mathcal{T} \in \mathcal{C}$ is called *tilting object* if it satisfies the following properties:

1.) $\mathrm{Ext}^l(\mathcal{T}, \mathcal{T}) = 0$ for all $l \neq 0$;

2.) the direct summands of $\mathcal{T}$ generate the triangulated category $\mathcal{C}$;

3.) the global dimension of $\mathrm{End}(\mathcal{T})$ is finite.

If $\mathcal{C}$ is the bounded derived category of Λ–modules and $\mathcal{T}$ is a module we call $\mathcal{T}$ *tilting module* (respectively *tilting sheaf* if $\mathcal{T}$ is a sheaf and $\mathcal{C}$ is the bounded derived category of coherent sheaves).

The importance of the notion is due to the fact that by the Lemma of Beilinson [Be] the functor

$$\underline{R}\mathrm{Hom}(\mathcal{T}, -) : \mathcal{C} \longrightarrow \mathcal{D}^b(\Lambda\text{–mod}),$$

where $\Lambda := \mathrm{End}(\mathcal{T})$, is an equivalence of triangulated categories.

Definition Let $\varepsilon = (E_0, E_1, ..., E_D)$ be a sequence of objects in $\mathcal{A}$ or $\mathcal{C}$. The object E_i is called *exceptional* if $\mathrm{Ext}^l(E_i, E_i) = 0$ for all $l > 0$ and $\mathrm{End}(E_i) = k$. The sequence ε together with a surjective monotonic function $s : \{0, \dots, D\} \longrightarrow \{0, \dots, d\}$ is called *leveled exceptional* if E_i is exceptional for $i = 0, \dots, D$ and for $i \neq j$ and $s(i) \geq s(j)$ we have $\mathrm{Ext}^l(E_i, E_j) = 0$ for all l. The sequence ε is called *leveled strong exceptional* if in addition $\mathrm{Ext}^l(E_i, E_j) = 0$ for all $l > 0$ and all $i, j = 0, \dots, D$. A sequence is *full*, if it generates the triangulated category $\mathcal{C}$ or the bounded derived category of $\mathcal{A}$ respectively in the sense, that the smallest full triangulated subcategory which contains all objects E_i for $i = 0, \dots, D$ is the whole category. A full leveled strong exceptional sequence of coherent sheaves on $\boldsymbol{X}$ with $d = \dim \boldsymbol{X}$ is called *distinguished tilting sequence*. In particular the sheaf $E := \bigoplus_{i=0}^{D} E_i$ is a tilting sheaf and the endomorphism algebra is a Koszul algebra [Hil].

Each tilting line bundle $\mathcal{T}$ (the direct summands of $\mathcal{T}$ are line bundles) is a strong exceptional sequence in a natural way, ordered by the partial ordering given by $\mathcal{L} \leq \mathcal{L}'$ if and only if $\operatorname{Hom}(\mathcal{L}, \mathcal{L}') \neq 0$ for any indecomposable direct summands $\mathcal{L}, \mathcal{L}'$ of $\mathcal{T}$.

One of the most important problems we wish to tackle is the construction of tilting bundles or exceptional sequences on a variety $\boldsymbol{X}$. Further one would like to classify all varieties $\boldsymbol{X}$ which admit a tilting bundle or a (full) exceptional sequence. If the variety is one-dimensional, then only the projective line admits a tilting bundle or an exceptional sequence.

The first examples of tilting bundles were found by Beilinson on the projective space $\boldsymbol{P}^n$, $\mathcal{T} = \bigoplus_{i=0}^{n} \Omega^i(i)$, the shifted exterior powers of the cotangent bundle and $\mathcal{T} = \bigoplus_{i=0}^{n} \mathcal{O}(i)$ the tilting line bundle of the shifted structure sheaves [Be]. Later Kapranov [K1], [K2] constructed tilting bundles on homogeneous spaces and intersections of quadrics. It is known that exceptional sequences in the derived category exist on projective space bundles, if the basis admits an exceptional sequence, and under certain conditions also on blowing ups. In general the problems above are unsolved and seem to be very difficult. So the next conjecture is surprising. In contrast the problem to find an exceptional sequence is easy for a finite dimensional directed algebra. For example the sequence of indecomposable projective modules is full strong exceptional. Thus it is a powerful technic to use the derived equivalence to obtain informations on both categories.

2. Moduli Spaces of Representations of Quivers

It would be desirable to exhibit a canonical geometric structure on the set of isomorphism classes of representations of a quivers Q. Such a canonical structure is called a moduli space. If we restrict ourself to stable representations, then such a geometric structure exists. This was worked out in [Ki]. On the other hand it can not exist in general because any moduli space is a quotient of the space of representations and this quotient does in general not exist in the category of varieties. For further definitions and properties of moduli space we refer also to [N]. Let us recall some results. Assume Q is a fixed quiver without oriented cycles, so the path algebra kQ is finite dimensional. We fix a dimension vector d and consider the space $\mathcal{R}(Q,d)$ of all representations

$$\mathcal{R}(Q,d) := \bigoplus_{\alpha \in Q_1} \operatorname{Hom}(k^{d(s(\alpha))}, k^{d(t(\alpha))})$$

(in other words the set of all matrices indexed over the arrows Q_1 of Q of size given by the dimension vector d) and the natural group action of

$$G = G(d) := \prod_{i \in Q_0} \mathrm{GL}_{d(i)},$$

acting by conjugation. The space of all representations is an affine space and we consider the $\boldsymbol{Z}$-graded ring of semi-invariants

$$\bigoplus_l k[\mathcal{R}(Q,d)]^{G,\chi^l} := \bigoplus_l \{f \in k[\mathcal{R}(Q,d)] \mid f(gx) = \chi(g)^l f(x)\}$$

with respect to some character $\chi(g) = \prod \det(g_i)^{-\theta_i} : G \longrightarrow k^*$. This defines a projective algebraic variety

$$\mathcal{M}^\theta(Q,d) := \operatorname{Proj}\left(\bigoplus_l k[\mathcal{R}(Q,d)]^{G,\chi^l}\right).$$

Definition Let $\theta : K_0(\Lambda\text{–mod}) \simeq \boldsymbol{Z}^{Q_0} \longrightarrow \boldsymbol{R}$ be an additive function on the Grothendieck group of the category of finite dimensional modules of Λ. We call θ a *weight* of the category. A module M is called *θ–stable* (*θ–semi-stable*) if $\theta(M) = 0$ and for all proper submodules $N \subset M$ $\theta(N) < 0$ ($\theta(N) \leq 0$ respectively). We remark, that we use a different sign convention than King. Any semi-stable module has a Jordan Hölder filtration with stable quotients. Then two modules are *S–equivalent* if they have the same Jordan Hölder factors.

The points of the space $\mathcal{M}^\theta(Q,d)$ are exactly the θ–semi-stable representations up to S–equivalence [Ki] and $\mathcal{M}^\theta(Q,d)$ is the moduli space of θ–semi-stable modules up to S–equivalence. If d is indivisible and all θ–semi-stable representations are also θ–stable, then there exists a universal family $\mathcal{U}$ on this moduli space, which parametrises the θ–stable representations of Q with dimension vector d. If we forget the structure as a representation of Q, this family is just a vector bundle, the universal bundle $\mathcal{U} = \mathcal{U}(d)$ of this moduli space, and it decomposes into a direct sum $\mathcal{U} = \bigoplus_{i \in Q_0} \mathcal{U}_i$, where $\operatorname{rank}(\mathcal{U}_i) = d_i$. The bundle $\mathcal{U} \longrightarrow \mathcal{M}^\theta(Q,d)$ is the fine moduli space.

Conjecture Let $\mathcal{U} \longrightarrow \mathcal{M}^\theta(Q,d)^s$ be the fine moduli space of θ–stable modules of dimension vector d over the finite dimensional quiver algebra kQ (so we assume Q has no oriented cycle) and assume $\mathcal{M}^\theta(Q,d)^s = \mathcal{M}^\theta(Q,d)$, so it is a smooth projective algebraic variety. Then $\mathcal{U}$ is a partial tilting bundle on $\mathcal{M}^\theta(Q,d)$. This was first conjectured by Schofield. Even he conjectured more, that there exists always a completion of $\mathcal{U}$ to a tilting bundle $\overline{\mathcal{U}}$ by adding certain direct summands to $\mathcal{U}$.

In general it is very hard to compute these moduli spaces $\mathcal{M}^{\theta}(Q,d)$. If $d = (1,\dots,1)$, then there is an inductive way to compute all by starting with some moduli space over a smaller quiver (so it is in particular of smaller dimension) and using projective space bundles and so called simple flip diagrams. Furthermore we obtain an explicit cover of these varieties by open subvarieties U_T, which are isomorphic to the affine space. The index set $\{T\}$ we can completely describe in the combinatoric of the quiver Q, it consists of certain subquivers, which are trees. Furthermore also the change of the moduli space (which means the projective space bundles and the flips) we can understand locally in terms of this covering as well as globally [Hi2]. Of particular interest are further informations about the moduli space like the rank of the Grothendieck group, the structure of the cohomology ring, the Picard group and the cohomology of the line bundles. In the special case $d = (1,\dots,1)$ the moduli space is a toric variety, thus we know the Picard group and the Grothendieck group [O],[F]. Further results in [KW] show, that the Chowring of any moduli space of representations of a quiver is isomorphic to its cohomology ring.

Example We will show, that any projective variety $\boldsymbol{X}$ is a fine moduli space of modules of dimension vector $d = (1,\dots,1)$, even all these modules are uniserial. So assume $\boldsymbol{X}$ is the zero set of homogeneous polynomials $f_1,\dots,f_r \in k[x_0,\dots,x_n]$ of degree $r_1 \leq \dots \leq r_t$. We consider the quiver Q with points $Q_0 = \{0,\dots,r_t\}$ and arrows $\{x_0,\dots,x_n\}$ from $i-1$ to i for all $i = 1,\dots,r_t$. Here we use the same symbol for possibly different arrows, but we may differ them by their starting or ending point. We consider the finite dimensional algebra $\Lambda = kQ/I$, where I is generated by all possible relations defined by the polynomials $f_1,\dots,f_r$ and all commutativity relations $x_l x_m - x_m x_l$. It is an easy exercise to show, that set of θ-stable modules coincides with the set of indecomposable modules for any θ with $\theta_{i-1} > \theta_i$ for all $i = 1,\dots,r_t$ and $\sum_{i=0}^{r_t} \theta_i = 0$. We conclude, that also all moduli spaces are the same for those θ and isomorphic to the quotient $\mathcal{R}(Q)^{\text{ind}}/G$ of the subset of the indecomposable modules. This quotient exists and is isomorphic to $\boldsymbol{X}$, because $\boldsymbol{X}$ is the subvariety in the product of r_t projective spaces defined by the equations $f_1,\dots,f_r$ intersected with the diagonal.

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ON THE EXPONENT OF LATTICES OVER GROUP RINGS

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1. Preliminaries

Let R be a complete, discrete, rank one valuation ring with maximal ideal πR and residue class field $k = R/\pi R$, and let G be a finite group. Let $|G|R = \pi^n R$, and let p be the rational prime belonging to πR. We want to consider RG-lattices, that is R-free finitely generated RG-modules, from the point of view of their exponents. Recall that for RG-lattices L and M and $\varphi \in \mathrm{Hom}_{RG}(L, M)$, the exponent of φ is π^a if a is the least exponent of π such that $\pi^a\varphi$ factors through a projective, while the exponent of an RG-lattice M is defined as the exponent of the identity endomorphism of M. Since $|G|\varphi$ always factors through a projective, the exponent π^a of any RG-lattice satisfies $0 \leq a \leq n$. When $a < b$ we write $\pi^a < \pi^b$.

The lattices with exponent 1 are the projective RG-lattices. The indecomposable lattices with maximum exponent π^n, as shown in ([2] Corollary 2.9), are the splitting trace lattices studied by Auslander and Carlson in [1]. These lattices can be characterized by the property that tensoring the almost split sequence of the trivial module R with an indecomposable splitting trace lattice M gives the almost split sequence of M, modulo projective summands. The almost split sequence of R tensored with any other indecomposable splits (see [1], Theorem 3.6).

The splitting trace lattices have the exponential property considered in [2]. An indecomposable non-projective lattice M has property E when the middle term $B(M)$ of the almost split sequence of M satisfies $\exp B(M) < \exp M$. Among other equivalent statements we mention that M has property E if and only if every homomorphism $\varphi : L \to M$, that is not a split epimorphism satisfies $\exp\varphi < \exp M$. The absolutely indecomposable lattices with property E coincide with the virtually irreducible lattices introduced by Knörr in [6] (see also [8] and [9]).

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By ΩM we denote as usual the Heller translate of the RG-lattice M, then $\exp\Omega M = \exp M$ (see [2]). In order to obtain lattices of a given exponent we also have to consider $(R/\pi^j R)G$-modules V, as RG-modules. Then we denote by $\Omega_j V$ the RG-lattice which is the kernel of an essential epimorphism $P \to V$, with P a projective RG-lattice.

2. Main Result

Theorem 2.1. *Let M be a non-projective RG-lattice with $expM = \pi^a$, and for $j \geq 1$ let $B_j = B_j(M) = \Omega_j\left(M/\pi^j M\right)$.*

(a) *For every j there is an exact sequence*

$$\xi_j(M) : 0 \to \Omega M \to B_j \xrightarrow{\beta} M \to 0,$$

which splits if and only if $j \geq a$.

(b) $B_j/\pi^j B_j \cong M/\pi^j M \oplus \Omega M/\pi^j \Omega M$ *for all j.*

(c) $expB_j = \pi^j$ *for $1 \leq j \leq a$.*

(d) *For every homomorphism $\varphi : L \to M$ with $exp\varphi \leq \pi^j$, there exists a homomorphism $\varphi' : L \to B_j$ such that $\beta\varphi' = \varphi$.*

(e) $\Omega B_j(M) = B_j(\Omega M)$ *for all j.*

(f) *For all j and every RG-lattice L the sequence $\xi_j(L)$ tensored with M coincides with the sequence $\xi_j(L \otimes M)$ modulo projective summands.*

Proof

(a). Let $\theta : P \to M$ be a projective cover and let $\xi_j(M)$ be the pullback of the pair $(\theta, \pi^j \mathrm{Id}_M)$. Then it is well known that $\xi_j(M)$ splits if and only if $\pi^j \mathrm{Id}_M$ factors through a projective, that is if and only if $j \geq a$. It follows from the diagram that the middle term of this sequence is $B_j = \Omega_j\left(M/\pi^j M\right)$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
\xi_j(M) : 0 & \longrightarrow & \Omega M & \longrightarrow & B_j & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \pi^j} & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & P & \xrightarrow{\theta} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M/\pi^j M & = & M/\pi^j M & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(b)

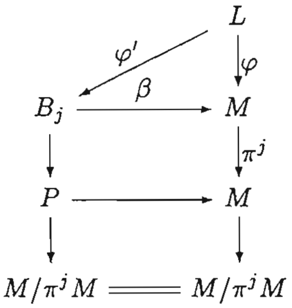
$$\frac{B_j}{\pi^j B_j} \cong \frac{\Omega M + \pi^j P}{\pi^j \Omega M + \pi^{2j} P} \cong$$

$$\frac{\Omega M + \pi^{2j} P}{\pi^j \Omega M + \pi^{2j} P} \oplus \frac{\pi^j P + \pi^j \Omega M}{\pi^j \Omega M + \pi^{2j} P} \cong$$

$$\frac{\Omega M}{\pi^j \Omega M} \oplus \frac{P}{\pi^j P + \Omega M} \cong \frac{\Omega M}{\pi^j \Omega M} \oplus \frac{M}{\pi^j M}$$

(c) $\pi^j \mathrm{Id}_{B_j}$ factors through a projective because $\pi^j B_j \subset \pi^j P \subset B_j$ and $\pi^j P \cong P$, so $\exp B_j \leq \pi^j$. Suppose now that $\pi^i \mathrm{Id}_{B_j}$ factors through a projective. Since $\exp \pi^{a-j} \mathrm{Id}_M = \pi^j$, by (d) there exists $\varphi' : M \to B_j$ such that $\pi^{a-j} \mathrm{Id}_M = \beta\varphi'$. Therefore $\pi^{a-j+i} \mathrm{Id}_M = \beta \pi^i \mathrm{Id}_{B_j} \varphi'$ factors through a projective, hence $a - j + i \geq a$, so $j \leq i$.

(d) If $\exp\varphi \leq \pi^j$ then $\pi^j \varphi$ factors through a projective, so there exists $\psi : L \to P$ such that $\pi^j \mathrm{Id}_M \varphi = \theta\psi$. But from the diagram it is clear that ψ lifts to $\varphi' : L \to B_j$.



(e) Since $\exp B_j = \pi^j$:

$$B_j \oplus \Omega B_j \cong \Omega_j \left(\frac{B_j}{\pi^j B_j} \right) \cong \Omega_j \left(\frac{M}{\pi^j M} \oplus \frac{\Omega M}{\pi^j \Omega M} \right) \cong B_j \oplus B_j(\Omega M)$$

(f) It suffices to observe that, modulo projective summands,

$$\Omega \left(\frac{L}{\pi^j L} \otimes M \right) \cong \Omega_j \left(\frac{L}{\pi^j L} \otimes \frac{M}{\pi^j M} \right)$$

As an example we determine the sequences $\xi_j(R)$ of the trivial lattice R over the cyclic group of order p^n, $C = < g >$, when R is the ring of p-adic integers. We know $\exp R = p^n$, so $1 \leq j \leq n$. From $RC/(g-1)RC \cong R$ we get:

$$\Omega R \cong (g-1)RC \cong \frac{R[X]}{(\phi_1 \ldots \phi_n)} = E_n,$$

where ϕ_i is the cyclotomic polynomial of order p^i and where g acts on E_n by multiplication by X.

Now

$$\frac{RC}{p^jRC+(g-1)RC}\cong\frac{R}{p^jR},$$

hence $B_j = B_j(R) = p^jRC + (g-1)RC = \phi_1(g)\dots\phi_j(g)RC + (g-1)RC$.

This is so because $p^j = (X-1)f + \phi_1\dots\phi_j$ for some $f \in \mathbb{Z}[X]$.

In this example the B_j, for $1 \le j \le n-1$, are indecomposable because $B_j/pB_j \cong R/pR \oplus E_j/pE_j$ as kC-modules, but $E_j/pE_j \cong k[X]/(X-1)^{p^j-1}$, which is an indecomposable kC-module, so if B_j were decomposable it would have at most two indecomposable summands, and one of them would have to be R, but then $p^j = \exp B_j \ge \exp R = p^n$.

In the next section we will see examples of indecomposables kG-modules V such that the RG-lattice $\Omega_1 V$ decomposes.

3. Applications

Corollary 3.1. *If L and M are indecomposables RG-lattices with $expL \le expM = \pi^a$, such that $L/\pi^aL \cong M/\pi^aM$, then either $L\cong M$ or $L \cong \Omega M$ and $\Omega^2M \cong M$.*

Proof. This follows from the Krull-Schmidt theorem because:

$$L\oplus\Omega L \cong \Omega_a\frac{L}{\pi^aL}\cong\Omega_a\frac{M}{\pi^aM}\cong M\oplus\Omega M.$$

We remark that for R the p-adic integers there are indecomposables RC_{p^2}-lattices X and Y with exponent p such that $X/pX \cong Y/pY$, and $X \cong \Omega Y$, but $X \not\cong Y$ (see [5]).

By a theorem proved by Maranda in 1953, for all RG-lattices $L/\pi^{n+1}L \cong M/\pi^{n+1}M$ implies $L \cong M$. (See [3], theorem 30.4 and also [4] for another proof). The same proof shows that if $\exp L \le \exp M = \pi^a$, then $L/\pi^{a+1}L \cong M/\pi^{a+1}M$ implies $L\cong M$. From Corollary 3.1, we conclude that Maranda's theorem is the best possible result of this type.

Corollary 3.2. *If the p-Sylow subgroups of G are cyclic the number of indecomposable RG-lattices of exponent p is finite.*

Proof. We write $M|L$ when M is a direct summand of L. If $\exp M = \pi^a$ then $M|\Omega_a(M/\pi^aM)$, so $M|\Omega_aV$ for some indecomposable $(R/\pi^aR)G$-module V such that $V|M/\pi^aM$. But for $a = 1$, if the p-Sylow subgroups of G are cyclic then the number of indecomposable kG-modules is finite.

As an example we determine the indecomposable RC-lattices of exponent p when R is the ring of p-adic integers. We use the same notation as in section 1 and we write $\phi = \phi_n$.

Let $V_i = k[X]/(X-1)^i$, $i = 1,\dots p^n-1$, then

$$Y_i = \Omega_iV_i = pRC + (g-1)^iRC,$$

and we know from the proof of (c) of theorem 2.1, that $\exp Y_i \leq p$.

Assume first $i \leq p^{n-1}(p-1)$, then since $\phi(g) = (g-1)^{p^{n-1}(p-1)} \pmod{p}$ and $p = (g-1)^{p^{n-1}(p-1)}u \pmod{\phi(g)}$ where u is invertible,

$$Y_i = \phi(g)RC + (g-1)^i RC.$$

Note that as RC-modules $\phi(g)RC \cong RC_{p^{n-1}}$ and

$$\frac{Y_i}{\phi(g)RC} \cong \frac{(X-1)^i R[X]}{\phi} \cong (\varepsilon - 1)^n R_n \cong R_n = R[\varepsilon],$$

where ε is a root of 1 of order p^n. Thus Y_i is an extension:

$$0 \to RC_{p^{n-1}} \to Y_i \to R_n \to 0.$$

If $p^{n-1} \leq i \leq p^{n-1}(p-1)$ this sequence splits and $Y_i \cong RC_{p^{n-1}} \oplus R_n$. This follows from the fact that the idempotent $e = \frac{1}{p}\phi(g)$ verifies $eY_i \subset Y_i$, because $epRC = \phi(g)RC$ and

$$e(g-1)^{p^{n-1}} \in \frac{\phi(g)}{p}(g^{p^{n-1}} - 1) + \phi(g)RC = \phi(g)RC.$$

For $p^{n-1}(p-1) < i < p^n$, if $\bar{\Omega}$ is the Heller operator on kC-modules, then

$$\Omega_i V_i = \Omega_i(\bar{\Omega} V_{p^n - i}) = \Omega(\Omega_i V_{p^n - i}) = \Omega Y_{p^n - i}.$$

Now for $1 \leq i \leq p^{n-1}$ the Y_i are indecomposable (and therefore also the ΩY_i) because otherwise a direct summand of Y_i would have exponent p, but neither $RC_{p^{n-1}}$ nor R_n are direct summands of Y_i, and all other indecomposable lattices of exponent p have rank p^2.

From results of Nazarova (see [7]) it follows that for R the p-adic integers, there exist infinitely many indecomposable RC_{p^3}-lattices of rank prime to p. These are all splitting trace lattices, therefore have exponent p^3. Since these lattices have property E the middle terms of their almost split sequences have exponent p^2, which requires the existence of infinitely many indecomposables of exponent p^2.

In [2] it was shown that an RG-lattice M with exponent π^a has property E if and only if $\xi_{a-1}(M)$ is the almost split sequence of M. This also follows from (d) of theorem 2.1. Thus from (a) and (f) of theorem 2.1 we immediately obtain Corollary 3.3 which extends Theorem 3.6 of [1].

Corollary 3.3. *For every RG-lattice M with exponent π^a the sequence $\xi_j(R)$ tensored with M splits if and only if $j > a-1$. M has the property E if and only if $\xi_{a-1}(R)$ tensored with M is the almost split sequence of M modulo projective summands.*

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SEMI-PERFECT RINGS AND THEIR QUIVERS

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ABSTRACT. The paper is devoted to the study of quivers of semi-perfect rings from the point of view of structural ring theory. It contains an overview of the results of the author and his students.

1. INTRODUCTION

All rings considered in the paper are associative with $1 \neq 0$. Saying a noetherian (etc.) ring we assume that it is a two-sided noetherian (etc.) ring. A ring is called decomposable if it decomposes in to a direct product of two rings, otherwise the ring is indecomposable.

Recall some basic facts on the semi-perfect rings introduced by Bass in 1960 [1]. Let R be the Jacobson radical of ring A. The ring A is called semi-perfect if the factor-ring A/R is artinian and the idempotents can be lifted modulo R. An idempotent e is called local if the ring eAe is local.

Theorem 1.1. [21] *A ring A is semi-perfect if and only if the unity of A can be decomposed into a sum of mutually orthogonal local idempotents.*

Theorem 1.2. *A ring A is semi-perfect if and only if it decomposes into direct sum of right ideals, each of which has exactly one maximal submodule.*

Proof. The proof can be found for example in [13, §7] (see also [11]).

Therefore, a semi-perfect ring A can be represented as a direct sum of right ideals :

$$A = P_1^{n_1} \oplus \cdots \oplus P_s^{n_s},$$

where $P_1, \dots, P_s$ are pairwise nonisomorphic modules and $U_i = P_i/P_iR$, $i = 1, \dots, s$ are simple. The modules $P_1, \dots, P_s$ exhaust up to isomorphism all

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indecomposable projective A-modules, while $U_1, \dots, U_s$ exhaust all nonisomorphic simple A-modules [12, §1]. (X^n denotes a direct sum of n copies of module X and $X^0 = 0$.)

Every projective module over a semi-perfect ring A decomposes into a direct sum of indecomposable projective A-modules. There is a Krull-Schmidt Theorem for projective A-modules [11].

Let $1 = f_1 + \dots + f_s$ be a decomposition of the unity of A into a sum of mutually orthogonal idempotents such that $f_iA = P_i^{n_i}$ $(i = 1, \dots, s)$. Put $A_{ij} = f_iAf_j$, $i, j = 1, \dots, s$. Then A has the following Pierce decomposition:

$$A = \bigoplus_{i,j=1}^{s} A_{ij}, \quad (i, j = 1, \dots, s), \tag{1}$$

and the multiplication is the "matrix-multiplication".

Denote by R_i the Jacobson radical of ring A_{ii}, $i = 1, \dots, s$. Then the radical R of A has the following Pierce decomposition :

$$R = \bigoplus_{i,j=1}^{s} f_iRf_j, \tag{2}$$

where $f_iRf_i = R_i$ and $f_iRf_j = A_{ij}$, $i \neq j$; $i, j = 1, \dots, s$ [12].

A semi-perfect ring A is called reduced if the factor-ring A/R is a direct product of skew-fields. Every semi-perfect ring $A = P_1^{n_1} \oplus \dots \oplus P_s^{n_s}$ is Morita equivalent to the reduced ring

$$B = End_A(P_1 \oplus \dots \oplus P_s).$$

Theorem 1.3. [16] *Every semi-perfect ring A has a unique decomposition into a finite direct product of indecomposable rings, i.e. if*

$$A = B_1 \times \dots \times B_s = C_1 \times \dots \times C_t$$

are two such decompositions then $s = t$ and there exists a permutation σ of $\{1, \dots, s\}$ such that

$$B_i = C_{\sigma(i)} \quad (i = 1, \dots, s).$$

2. Quivers of semi-perfect rings

Following Gabriel a finite oriented graph will be called quiver. A simply laced quiver (no multiple arrows including loops) will be called simple. Denote by $1, \dots, s$ the vertices of the quiver Q and assume that we have t_{ij} arrows between points i and j. Let $[Q]$ denotes the incidence matrix of the quiver Q :

$$[Q] = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1s} \\ \cdots & \cdots & \cdots & \cdots \\ t_{s1} & t_{s2} & \cdots & t_{ss} \end{pmatrix}.$$

The results below can be found in [8, part13] and [20, part 9].

A real matrix $A = (a_{ij})$ is called non - negative if all elements a_{ij} are non-negative.

Denote by $M_n(\mathbb{R})$ the set of all real matrices of order n.

Let τ be a permutation of the numbers $1, 2, \dots, n$ and let

$$P_\tau = \sum_{i=1}^{n} e_{i\tau(i)}$$

be a permutation matrix where e_{ij} are corresponding matrix units. Clearly, $P_\tau^T P_\tau = E$.

Definition 2.1. *A matrix $B \in M_n(\mathbb{R})$ is called permutationally reducible if there exists a permutation matrix P_τ such that*

$$P_\tau^T B P_\tau = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix},$$

where B_1 and B_2 are square matrices of order less that n. Otherwise the matrix is permutationally irreducible.

Definition 2.2. *A quiver is called strongly connected if there is an oriented path between any two of its vertices.*

Proposition 2.3. *A quiver Q is strongly connected if and only if the matrix $[Q]$ is permutationally irreducible* [20, part 9] *(see also* [5]*).*

Note that a renumeration of the vertices of the quiver Q transforms the matrix $[Q]$ into the matrix $P_\tau^T[Q]P$.

Proposition 2.4. *There exists a permutation matrix P such that*

$$P^T[Q]P = \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1t} \\ 0 & B_2 & \cdots & B_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{mt} \end{pmatrix},$$

where the matrices $B_1, B_2, \dots, B_m$ are permutationally irreducible.

Proof. This is obvious.

A maximal (with respect to inclusion) strongly connected subquiver of Q is called a strongly connected component of Q. The Proposition 2.4 immediately implies the following well-known fact about the existence of the partition of a set of vertices of quiver Q into non-intersecting subsets such that the subquivers corresponding to those subsets are strongly connected quivers (strongly connected components of the quiver Q).

Definition 2.5. [6] *Let $Q_1, \dots, Q_m$ be all strongly connected components of the quiver Q. The condensation Q^* of quiver Q is a quiver whose vertices $q_1, \dots, q_m$ corresponds to strongly connected components $Q_1, \dots, Q_m$ and there is an arrow between q_i and q_j if and only if Q has an arrow between Q_i and Q_j.*

Definition 2.6. [24] *A quiver without oriented cycles is called an acyclic quiver.*

Proposition 2.7. [6, §63] *The condensation of any quiver is an acyclic graph.*

Proof. This follows from Proposition 2.3.

Recall that a point of quiver Q is called sink (source) if there is no arrows with an end (beginning) at this point. ([24, §8.6])

Proposition 2.8. [24, §8.6] *Every acyclic quiver has sink (source).*

Proposition 2.9. [24, §8.6] *Suppose that a set of points of an acyclic quiver consists of t elements. Then we can enumerate them by numbers $1, \dots, t$ in such a way that the existence of an arrow from i to j implies $i < j$.*

Let Q be a quiver. Usually the points of Q will be denoted by the numbers $1, 2, \dots, s$. If an arrow σ connects the point i with the point j then i is called the beginning and j the end of the σ. It will be denoted as $\sigma : i \to j$.

A path of quiver Q from a point i to a point j is an ordered set of k arrows $\sigma_1, \cdots, \sigma_k$ such that the beginning of each arrow is the end of the previous one, the point i is the beginning of σ_1, while the point j is the end of σ_k. The number k of arrows is called the length of the path.

An arrow $\sigma : i \to j$ is called extra if there exists a path from i to j of length greater than 1.

Definition 2.10. *Let $S = \{\alpha_1, \dots, \alpha_n\}$ be a finite poset. The diagram of S is a quiver $\Gamma(S)$ with set of points $\{1, \dots, n\}$ and an arrow between i and j $(i \neq j)$ if and only if $\alpha_i < \alpha_j$ and there is no element α_k such that $\alpha_i < \alpha_k < \alpha_j$, $\alpha_k \neq \alpha_i$, $\alpha_k \neq \alpha_j$.*

Clearly, the diagram of a finite poset S is an acyclic simply laced quiver with no extra arrows.

Proposition 2.11. *Let Γ be an acyclic simply laced quiver with no extra arrows. Then Γ is the diagram of a finite poset. Conversely, the diagram of a finite poset is an acyclic simply laced quiver with no extra arrows.*

Proof. By Proposition 2.9 there exists a numbering of the vertices of the quiver Γ by the numbers $\{1, \dots, t\}$ such that $i < j$ whenever there is an arrow from i to j. Since there are no extra arrows, the existence of an arrow $\sigma : i \to j$ implies that there is no k $(k \neq i,\ k \neq j)$ such that there is a path from i to k and an other path from k to j. It follows immediately that Γ is a diagram of a poset consisting of its vertices. The converse statement was discussed above. The proposition is thus proved.

Let J be an ideal of a ring A contained in the Jacobson radical R of A such that the idempotents can be lifted modulo J.

Consider the factor-ring $\overline{A} = A/J = \overline{A_1} \times \cdots \times \overline{A_t}$ where all rings $\overline{A_1}, \dots, \overline{A_t}$ are indecomposable and $\overline{1} = \overline{f_1} + \cdots + \overline{f_t} \in \overline{A}$ is the corresponding decomposition into a sum of mutually orthogonal central idempotents. Put $W = J/J^2$ and substitute the idempotents $\overline{f_1}, \dots, \overline{f_t}$ by the corresponding points $1, \dots, t$. We connect the points i and j by an arrow if and only if $\overline{f_i} W \overline{f_j} \neq 0$. The obtained finite oriented graph $Q(A, J)$ is called the quiver associated with the ideal J. Taking into account Theorem 1.3, one can easily see that the quiver $Q(A, J)$ of the semi-perfect ring A is defined uniquely up to a renumeration of the vertices and does not change for Morita equivalent rings. Moreover,

$$Q(A, J) = Q(A/J^2, W).$$

Since the prime radical I of a ring A is a nil-ideal, it is contained in the Jacobson radical R of A. Using the fact that the idempotents can be lifted modulo any nil-ideal [19, §3.6] one can consider a quiver $Q(A, I)$ associated with the prime radical I.

Definition 2.12. *A quiver $Q(A, I)$ of semi-perfect ring A is called prime. Further we will denote it by $PQ(A)$.*

Let A be a right noetherian semi-perfect ring, R its Jacobson radical, and $P_1, \dots, P_s$ all non-isomorphic projective indecomposable modules. Let

$$P(P_i R) = \bigoplus_{j=1}^{s} P_j^{t_{ij}}, \quad i, j = 1, \dots, s$$

be the projective cover of the module $P_i R$. We establish a correspondence between the modules $P_1, \dots, P_s$ and the points $1, \dots, s$ and connect the vertex i with the vertex

j by t_{ij} arrows. The obtained graph is called the quiver of the right noetherian semi-perfect ring A and will be denoted by $Q(A)$. Analogously, one can define a left quiver $Q'(A)$ of a left noetherian semi-perfect ring A.

Note that the quiver of a semi-perfect ring does not change if we switch to a Morita equivalent ring. Also it is obvious that $Q(A) = Q(A/R^2)$.

Definition 2.13. *Let A be a semi-perfect ring such that A/R^2 is a right artinian ring. The quiver of the ring A/R^2 will be called the quiver of A and will be denoted by $Q(A)$.*

In the case when the factor-ring A/R^2 is left artinian then the left quiver $Q'(A)$ is defined by the formula $Q'(A) = Q'(A/R^2)$.

3. Quivers of some special classes of semi-perfect rings

For convenience, we will consider a quiver consisting of one point only as a strongly connected quiver.

Proposition 3.1. *A strongly connected acyclic quiver is a point.*

It follows from this proposition that strongly connected quivers and acyclic quivers are polar types of quivers.

Proposition 3.2. *A quiver Q is acyclic if and only if there exists a permutation matrix P such that the matrix $P^T[Q]P$ is strictly upper triangular.*

Proof. This follows from Propositions 2.3 and 2.4.

We next consider the quivers of certain types of semi-perfect rings.

(a) **Semi-prime rings.** A ring is called semi-prime if it has no nilpotent ideals.

Theorem 3.3. [16] *The quiver $Q(A)$ of a two-sided noetherian semi-prime indecomposable semi-perfect ring A is strongly connected.*

(b) **Weakly prime rings.**

Definition 3.4. *Let R be the Jacobson radical of the ring A. The ring is called weakly prime if the product of any two ideals that are not in R, is not zero.*

Clearly, any prime ring is weakly prime.

Theorem 3.5. [3] *Let $1 = e_1 + e_2 + \cdots + e_n$ be a decomposition of the unity of semi-perfect ring A into the sum of mutually orthogonal local idempotents and put $A_{ij} = e_iAe_j$ $(i, j = 1, \dots, n)$. The ring A is weakly prime if and only if $A_{ij} \neq 0$ for all i, j.*

Theorem 3.6. [17] *The quiver of a noetherian semi-perfect weakly prime ring is strongly connected.*

(c) **Quasi-Frobenius rings (QF-rings).**

The main result of this section is the following theorem.

Theorem 3.7. [18] *The quiver of an indecomposable QF-ring is strongly connected. Conversely, given a strongly connected quiver Q there exists a finite dimensional QF-algebra A the quiver of which coincides with Q.*

Recall the basic properties of QF-rings needed for the proof of the theorem. A QF-ring is a two-sided artinian ring and all the rings Morita equivalent to a QF-ring are QF-rings.

Let $socM$ be a socle of an A-module M.

Theorem 3.8. [11, 22] *If A is a QF-ring then the socle of any principal A-module is simple. Moreover, if P_1 and P_2 are the non-isomorphic principal A-modules*[1] *then $socP_1 \not\simeq socP_2$. Conversely, if these conditions hold for principal right and left A-modules then A is a QF-ring.*

In the case of finite dimensional algebras the QF-situation is described by theorem 9.3.7 in [4].

Theorem 3.9. *An algebra A is quasi-Frobenius if and only if the socle of each principal A-module is simple and, for any two non-isomorphic principal A-modules P_1 and P_2, $socP_1 \not\simeq socP_2$.*

Let Q be a quiver, i and j are two points of Q and σ_{ij} is an arrow from i to j. A path x_{ij} is called simple if all its points $i_1, i_2, \dots, i_r$ are different. If in addition $i = j$ then x_{ij} is called a simple cycle.

Denote $Q(i) = \{j \in Q \mid \text{there exists a path } x_{ij} \text{ from } i \text{ to } j\}$.

Each arrow σ_{ij} of the quiver $Q(A)$ of the ring A naturally corresponds to a homomorphism $\varphi_{ji} : P_j \to P_i$ and every path $x_{ij} = \sigma_{ii_1}\sigma_{i_1i_2}\dots\sigma_{i_rj}$ naturally corresponds to a homomorphism $\Phi_{ji} : P_j \to P_i$, where $\Phi_{ji} = \varphi_{ji_r}\varphi_{i_ri_{r-1}}\dots\varphi_{i_1i}$.

A path x_{ij} is called maximal if $\Phi_{ji} \neq 0$ but $\varphi_{kj}\Phi_{ji} = 0$ for φ_{kj} corresponding to an arrow σ_{jk}. In this case j is called the end of maximal path x_{ij} with the beginning at i.

Let A be a QF-ring. It follows from Theorem 3.8 that the map $i \to \pi(i)$ which sends every vertex i of $Q(A)$ to the end of the maximal path with the beginning at i is permutation.

Clearly, the permutation $i \to \pi(i)$ satisfies the following conditions :

(α): for any σ_{ij} either $\pi(i) = j$ or $\pi(i) \in Q(j)$;

(β): for any vertex k of Q and any vertex $i \in Q(k)$, we have $\pi(i) \in Q(k)$.

Lemma 3.10. *Suppose that we have a permutation on the set of vertices of a quiver Q satisfying the conditions (α) and (β). Then the quiver Q is strongly connected.*

Proof. Assume that Q is not strongly connected. Then there exist vertices k and l such that there is no path from k to l. Hence $l \notin Q(k)$. Let T be the set of vertices of Q that do not belong to $Q(k)$. Since Q is connected there exists $i \in T$ and an arrow σ_{ij} such that $j \in Q(k)$. Clearly, $Q(j) \subset Q(k)$. Let $Q(k) = \{1, 2, \dots, m\}$. Then it follows from the property (β) that $Q(k) = \{\pi(1), \pi(2), \dots, \pi(m)\}$ and (α) implies that $\pi(i) \in Q(k)$. Therefore, $i \notin Q(k)$ and some vertex from $Q(k)$ is mapped to $\pi(i)$. The obtained contradiction completes the proof.

[1] i. e. indecomposable projective modules

Theorem 3.11. [23, ch.8] *If there is a path from point i to point j of quiver Q and a path from i to j then there exists a sequence of simple cycles $C_1, \dots, C_m$ such that $i \in C_1$, $j \in C_m$ and any pair of neighbour cycles has a common point.*

Corollary 3.12. *Any vertex of a strongly connected quiver belongs to a single cycle.*

Proof of Theorem 3.7. To prove the theorem we construct a quasi-Frobenius algebra $A = K(Q)/I$ where $K(Q)$ is a path algebra of Q with $Q = Q(A)$

By Theorem 3.11 every vertex of Q belongs to a simple cycle.

Consider the ideal I in $K(Q)$ generated by $\{x_{ii}^2 - y_{ii}^2, x_{jk}\}$ for all simple cycles x_{ii} and y_{ii} and paths $x_{j,k}$ not being subpaths of simple cycles. Put $A = K(Q)/I$.

The ideal I generated by the mentioned elements clearly lies between J^2 and J^n for some integer n where J is the ideal of $K(Q)$ generated by the arrows. Thus $Q(A) = Q$.

Clearly, the maximal paths with the beginning at i generate the socle of a principal A-module P_i.

Any subpath of a path x_{ii} is not maximal. At the same time $x_{ii}^2\sigma_{ik} = 0$ since it can not be included in x_{jj}^2 (x_{jj} is a simple cycle) for all j. Thus all maximal paths are x_{ii}^2. Since they coincide in $K(Q)/I$ then $socP_i$ is simple and $\pi(i) = i$. By Theorem 3.9, $A = K(Q)/I$ is QF-algebra.

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SPECIALITY OF JORDAN PAIRS

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ABSTRACT. We discuss some new results concerning the representations of one of the simple Jordan pairs as well as some examples of special and of exceptional varieties of Jordan pairs. These results are the Jordan pair counterparts of the classical theorems for Jordan algebras.

Assume K is an associative and commutative ring with 1, and $V = (V^+, V^-)$ is a pair of K–modules. Let $Q_\sigma\colon V^\sigma \to \mathrm{Hom}\,(V^{-\sigma}, V^\sigma)$ be quadratic maps, and set

$$Q_\sigma(x,z) = Q_\sigma(x+z) - Q_\sigma(x) - Q_\sigma(z), \quad x, z \in V^\sigma,$$

the bilinear maps obtained from Q_σ by linearizations. Define $D_\sigma(x,z)$, $x, z \in V^\sigma$, $y \in V^{-\sigma}$ as

$$Q_\sigma(x,z)y = D_\sigma(x,y)z = \{x,y,z\}, \quad \sigma = \pm.$$

The pair V of K–modules is called a Jordan pair if the following identities (and all their linearizations) hold, for $\sigma = \pm$.

JP1: $D_\sigma(x,y)Q_\sigma(x) = Q_\sigma(x)D_{-\sigma}(y,x)$;
JP2: $D_\sigma(Q_\sigma(x)y,y) = D_\sigma(x, Q_{-\sigma}(y)x)$;
JP3: $Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x)$.

The notion of a Jordan pair was introduced by K. Meyberg in 1969. It was the linear approach to the theory that was developed first. Assume that $V = (V^+, V^-)$ is a pair of K–modules with trilinear compositions $V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ such that $(x,y,z) \mapsto \{xyz\}$. Then V is a Jordan pair if the identities

$$\{xy\{xtx\}\} = \{x\{yxt\}x\},$$
$$\{\{xyx\}yz\} = \{x\{yxy\}z\},$$
$$\{\{xyx\}t\{xyx\}\} = \{x\{y\{xtx\}y\}x\}$$

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and all their linearizations hold for $x,\ z \in V^{\sigma},\ y,\ t \in V^{-\sigma},\ \sigma = \pm$.

If the ring K contains the element $1/2$ then one can set $Q(x)y = \{xyx\}/2$ and thus obtain a Jordan pair according to the first definition. We refer to [4] for all facts and results concerning the basic identities and other properties of Jordan pairs.

We shall give some of the most important examples and constructions of Jordan pairs and related structures. First let us recall the notion of an associative pair. The pair of K–modules $A = (A^+, A^-)$ equipped with the trilinear compositions $A^{\sigma} \times A^{-\sigma} \times A^{\sigma} \to A^{\sigma}$, $(x, y, z) \mapsto \langle xyz \rangle$ is an associative pair if the identities

$$\langle\langle xyz\rangle tu\rangle = \langle x\langle yzt\rangle u\rangle = \langle xy\langle ztu\rangle\rangle$$

hold. (We shall omit the indices $\pm$ unless it may cause misunderstandings.)

1. Let A be an associative K–algebra and let $A = A^+ = A^-$, and $\langle x, y, z\rangle = xyz$. Then (A^+, A^-) is an associative pair.

Let * be an algebra automorphism of order two on A. In this case we can set $\langle x, y, z\rangle = xy^*z$ and thus we obtain another associative pair from A.

2. If J is a Jordan algebra then $J = J^+ = J^-$ and (J^+, J^-) is a Jordan pair with respect to the Jordan triple product $\{x, y, z\} = (x \circ y) \circ z + (y \circ z) \circ x - (z \circ x) \circ y$, where $x \circ y$ stands for the multiplication in the Jordan algebra J. (If J is a quadratic Jordan algebra we obtain a Jordan pair from it in the following way: $Q(x) = U_x$ where U is the quadratic map of J.)

Let us remind the reader that if R is an associative algebra then with respect to the multiplication $a \circ b = (ab + ba)/2$ it becomes a Jordan algebra denoted by $R^{(+)}$. A Jordan algebra J is called special if it is a subalgebra of some $R^{(+)}$ for R an associative algebra; otherwise J is called exceptional. The well-known result of P. Cohn [1] states that every Jordan algebra in two generators is special. Moreover, there exist homomorphic images of the free special Jordan algebra in three generators that are exceptional (see [1]).

3. In a similar fashion we introduce in the associative pair A the new multiplication

$$\{x, y, z\} = (1/2)(\langle x, y, z\rangle + \langle z, y, x\rangle)$$

and thus we obtain a Jordan pair $A^{(+)}$. The pairs $A^{(+)}$ and their subpairs are called special Jordan pairs; otherwise they are exceptional. Some studies of the speciality and exceptionality of Jordan triple systems and Jordan pairs can be found in [5], [6], [2], and [3]

4. Assume that $A = A_{-1} + A_0 + A_1$ is a 3–graded associative algebra. Then (A_{-1}, A_1) is an associative pair with respect to the composition $\langle x, y, z\rangle = xyz$.

5. Let $X = X^- \cup X^+$, $X^- \cap X^+ = \emptyset$ be a set of symbols and $K(X)$ be the free associative algebra over K freely generated by the set X. The algebra $K(X)$ has a standard $\mathbf{Z}$–grading, $K(X) = \sum_{i=-\infty}^{+\infty} K(X)_i$, assuming that the weights of the elements of X^+ and of X^- are $+1$ and -1, respectively.

Denote by I the ideal in $K(X)$ generated by $\sum K(X)_i$, $|i| > 1$, and put $A = K(X)/I$. Therefore the grading on $K(X)$ induces one on A and we have $A = A_{-1} + A_0 + A_1$. Then the subpair of the associative pair $(A_1 + A_0, A_{-1} + A_0)$ generated by the image of the set X under the canonical homomorphism $K(X) \to A$ is the free associative pair freely generated by the set X; it is denoted by $FA(X)$.

6. The free special Jordan pair $FSJ(X)$ freely generated by the set X is the Jordan subpair of $FA(X)^{(+)}$ generated by X (for details, see e.g. [2]). Note that as in the algebra case, the speciality of $FSJ(X)$ means that every special Jordan pair is a homomorphic image of some $FSJ(X)$ for $|X|$ large enough.

7. We can construct another Jordan pair from the free associative pair $FA(X)$ as in the algebra case. Let * be the reversal involution on $K(X)$ i. e. $x^{**} = x$ for $x \in X$ and $(ab)^* = b^*a^*$ for any monomials a and b. Let us denote also by * the involution induced on $FA(X)$. Denote $H(X)$ the set of all symmetric elements in $FA(X)$ under *, i. e.

$$H(X) = \{(u,v) \in FA(X) | u^* = u, v^* = v\}.$$

Then $H(X)$ is a subpair of $FA(X)^{(+)}$. Therefore $H(X)$ is a special Jordan pair.

For a monomial $u \in K(X)$, $u = x_{i_1}x_{i_2}\ldots x_{i_k}$, $x_{i_j} \in X$, we denote

$$[u] = (1/2)\,(x_{i_1}x_{i_2}\ldots x_{i_k} + x_{i_k}\ldots x_{i_2}x_{i_1}).$$

We shall use the same notation for the monomials in the free associative pair $FA(X)$.

A straightforward verification shows that subpairs and direct products of special Jordan pairs are special. A convenient criterion due to P. Cohn [1] is used to recognize whether a homomorphic image of a special Jordan algebra (pair, triple system) is special or not.

Cohn's Criterion. Let I be an ideal in the free special Jordan pair $J = FSJ(X)$. Then the factor pair J/I is special if and only if $\overline{I} \cap J = I$ where $\overline{I}$ is the ideal in the free associative pair generated by the set I.

O. Loos and K. McCrimmon [5] gave an example of a special Jordan triple system having some exceptional homomorphic images.

8. Now let K be a field of a characteristic not two, and W an infinite dimensional vector space over K. Denote by $B = K + W$ the Jordan algebra of the symmetric

bilinear form $(\, , \,) : W \times W \to K$. The multiplication in B is defined in the following way:

$$(\alpha + u) \circ (\beta + v) = (\alpha\beta + (u,v)) + (\alpha v + \beta u)$$

for α, $\beta \in K$, u, $v \in W$. If the form $(\, , \,)$ is nondegenerate then B is a simple Jordan algebra, and, in addition, the Jordan pair $PB = (B, B)$ is also simple.

The Jordan algebra B is special and its associative enveloping algebra is the Clifford algebra C of the form $(\, , \,)$ on the space W. Analogously the Jordan pair PB is special and $PB \subset (C, C)^+$.

9. Assume again that K is a field, $\operatorname{char} K \neq 2$, and let M_n be the $n \times n$ matrix algebra over K, and H_n its subspace of all symmetric matrices.

Then the Jordan pairs (M_n, M_n) and (H_n, H_n) (with respect to the multiplication $\{xyz\} = (1/2)(xyz + zyx)$ are simple ones, see [4].

Now we need the concept of a variety. Let A be an associative algebra. The variety $\operatorname{var} A$ consists of all associative algebras that satisfy all polynomial identities of A. (Note that the polynomial $f \in K(X)$ is called a polynomial identity of A if f vanishes when one substitutes the variables in f with arbitrary elements of A.) The set of all polynomial identities of A forms an ideal in the free associative algebra $K(X)$ that is invariant under all endomorphisms of $K(X)$. This ideal is called the T–ideal of the algebra A (and of the variety $\operatorname{var} A$).

Similar notation will be used for Jordan algebras, pairs, etc.

S. Sverchkov [7] showed that the variety of Jordan algebras $\operatorname{var} B$ is special (i.e. every algebra in $\operatorname{var} B$ is special) in the case when the field K is of characteristic 0.

1. Representations

Let A be a unitary associative algebra with an idempotent e_+, and set $e_- = 1 - e_+$. Then $A = \oplus A^{\sigma\tau}$, σ, $\tau = \pm$, is the Peirce decomposition of A with respect to the orthogonal idempotents e_+ and e_-; here $A^{\sigma\tau} = e_\sigma A e_\tau$. Assume that V is a Jordan pair. Let $d_\sigma : V^\sigma \times V^{-\sigma} \to A^{\sigma\sigma}$ be bilinear maps, and let $q_\sigma : V^\sigma \to A^{\sigma,-\sigma}$ be quadratic maps. We say that (d, q) is a representation of V in A if the following identities (and their linearizations) hold, for $q_\sigma(x, z) = q_\sigma(x + z) - q_\sigma(x) - q_\sigma(z)$.

$$\begin{gathered}
d(x,y)q(x) = q(x)d(y,x) = q(x, Q(x)y), \\
q(x)d(y,z) + d(z,y)q(x) = q(x, \{xyz\}), \\
d(x,y)d(x,z) = d(Q(x)y, z) + q(x)q(y,z), \\
d(z,x)d(y,x) = d(z, Q(x)y) + q(y,z)q(x), \\
q(Q(x)y) = q(x)q(y)q(x).
\end{gathered}$$

(Here we omitted the corresponding indices $\pm$.)

Let $M = (M^+, M^-)$ be a pair of K–modules, and let A be the algebra of the endomorphisms of the module $M^+ \oplus M^-$. Then we can consider the elements of this direct sum as column vectors, and e_+, $e_- \in A$ as the identity maps of M^+ and of M^-, respectively. In other words e_σ is the identity map on M^σ, and it kills $M^{-\sigma}$, $\sigma = \pm$. Thus the elements of A can be viewed as 2×2 matrices. Then we say that M is a V–module (and that a representation of V on M is given).

Therefore we can consider the regular representation of V (onto itself). It is defined via the equalities

$$q_+(x) = \begin{pmatrix} 0 & Q_+(x) \\ 0 & 0 \end{pmatrix}, \qquad q_-(y) = \begin{pmatrix} 0 & 0 \\ Q_-(y) & 0 \end{pmatrix},$$

$$d_+(x,y) = \begin{pmatrix} D_+(x,y) & 0 \\ 0 & 0 \end{pmatrix}, \qquad d_-(y,x) = \begin{pmatrix} 0 & 0 \\ 0 & D_-(y,x) \end{pmatrix}.$$

Then the subalgebra of the endomorphisms of $V^+ \oplus V^-$ generated by $e_\pm$ and by all d's and all q's is the multiplication algebra $M(V)$ of V.

Denote $U(V)$ the universal representation algebra of V (see, e.g. [4]), then the following decomposition holds: $U(V) = Ke_+ \oplus Ke_- \oplus U_0(V)$ where $U_0(V)$ is the subalgebra of $U(V)$ generated by all d and q. Set $M_0(V)$ the image of $U_0(V)$ under the canonical homomorphism $U(V) \to M(V)$. It can be checked that $U_0(V)$ and $M_0(V)$ are ideals.

Proposition 1.1. *Let the set X generate the Jordan pair V. Then the universal representation algebra $U(V)$ of V is generated by the idempotents $e_\pm$ and by all d's and q's in the elements of the set X. The same is true for the multiplication algebra $M(V)$ as well.*

Proof. We shall prove that all $q_\sigma(u)$ and $d_\sigma(v,w)$ can be expressed by the given elements where u, $v \in V^\sigma$, and $w \in V^{-\sigma}$. We can consider the elements u, v, w as monomials in the elements of X. Next we can make use of the identities for the representations in [4], Section 2.3, and continue with an induction on the degrees of these monomials.

Proposition 1.2. *$M(FP) \cong U(FP)$ where FP is the free Jordan pair freely generated by an infinite set X.*

Proof. It is obvious that the assertion follows from the isomorphism between $M_0(FP)$ and $U_0(FP)$. We consider the restriction of the canonical homomorphism $\varphi : U(FP) \to M(FP)$ on $U_0(FP)$. Therefore $\varphi(U_0(FP)) = M_0(FP)$. If $f \in \ker\varphi$ is a polynomial in $U_0(FP)$ then f can be expressed by d's and q's in terms of the elements of X. Since φ "capitalizes" the letters d and q we obtain an identity $\varphi(f)$ for the regular representation of FP. According to [4], Proposition 2.8, it is an identity for all representations of Jordan pairs. Hence it holds for the universal representation of FP, and the Permanence Principle for Jordan pairs shows that $f = 0$ as an element of $U(FP)$. Therefore φ is an isomorphism.

This proposition gives some "standard" characteristic of the free Jordan pairs. Note that some "unusual" properties of them were discovered by E. Zelmanov [8]. Namely, it was proved that FP contains nontrivial zero divisors, and, moreover, its quasiregular radical is not zero.

Now we shall describe the multiplication algebra of the Jordan pair $H = (H_n, H_n)$. It is sufficient to describe the subalgebra $M_0^{++}(H)$ only. Let $p = n(n+1)/2$.

Theorem 1.3. $M_0^{++}(H) \cong M_p(K)$.

Sketch of the proof. The crucial role in the proof plays the fact that the frame of H (i.e. the maximal set of orthogonal idempotents) defines one dimensional Peirce spaces in H. One can construct a collection of homomorphisms $\phi_{ij}^{kl} : H \to H$ such that $\phi_{ij}^{kl}(H_{ij}) = H_{kl}$. Here H_{ij} are the Peirce spaces of H, and ϕ_{ij}^{kl} kills all H_{rs} with $\{r,s\} \neq \{i,j\}$. Thus all elementary matrices E_{ij} belong to $M_0^{++}(H)$, and this yields the isomorphism $M_0^{++}(H) \cong M_p(K)$.

2. Speciality and exceptionality

It was proved in [5] that every Jordan triple system in one generator is special. (The algebra case was dealt with in [1], and the result states that every Jordan algebra in two generators is special.) This comes to show that the Jordan structures are not so close to the associative ones. In fact, there does not exist an analog of the theorem of Poincare – Birkhoff – Witt in the case of Jordan algebras, pairs, and triple systems.

Proposition 2.1. *Every Jordan pair in two generators is special.*

Proof. It is sufficient to prove that the free Jordan pair $J = FP(X)$ is special where $X = X^+ \cup X^-$, $X^+ = \{x\}$, and $X^- = \{y\}$. Let J_n^σ be the submodule of J^σ generated by all monomials of degrees $2n-1$, $\sigma = \pm$. Then $J_1^+ = Kx$, $J_1^- = Ky$. We use the notation given in [4], Section 1.9, for the homotopes and for the Jordan algebras connected to a Jordan pair. Then J_y^+ and J_x^- are Jordan algebras. One can easily check that $J_2^+ = K\{x,y,x\} = Kx^{(2,y)}$, and analogously $J_2^- = Ky^{(2,x)}$.

An induction argument shows that $J_t^+ = Kx^{(t,y)}$, and $J_t^- = Ky^{(t,x)}$. Therefore the Jordan pair J is embedded into the matrices of order two over the polynomial ring $K[z]$ in the following way.

$$(x^{(t,y)}, y^{(s,x)}) \to \begin{pmatrix} 0 & z^{2t-1} \\ z^{2s-1} & 0 \end{pmatrix}.$$

Thus the free Jordan pair in two generators, and hence every Jordan pair in two generators is special.

Theorem 2.2. *Let $J = (J^+, J^-)$ be the free special Jordan pair freely generated by the sets $X^+ = \{x, y\}$ and $X^- = \{z, t\}$. Denote I its ideal generated by the element (u, v) where $u = \{xzx\} - \{yty\}$, and $v = \{zxz\} - \{tyt\}$. Then the Jordan pair J/I is exceptional.*

Sketch of the proof. According to Cohn's Criterion it suffices to prove that the element

$$w = \{\{\{xzx\}zx\}ty\} - \{y\{t\{yzx\}t\}y\}$$

belongs to the ideal $\overline{I}$ but $w \notin I$.

First compute the multidegrees of the homogeneous components of the elemet w. Then find which linear combinations of multiplication operators must be applied to u and to v in order to obtain w. This yields a system of (linear) equations for the coefficients of these multiplication operators. The last system has no solution.

Corollary 2.3. *Let K be a field of zero characteristic. Then the varieties of Jordan pairs generated by the matrix pairs (M_n, M_n) and by (H_n, H_n), respectively, are not special when $n > 2$.*

Proof. This corollary follows from the above theorem since one can prove that the Jordan pair (H_n, H_n) does not satisfy multilinear polynomial identities of degrees less than or equal to 7 when $n > 2$. The argument is similar to that of the well-known Staircase Lemma for the associative algebras $M_n(K)$, and therefore omitted. (The Staircase Lemma is used in order to show that the matrix algebra $M_n(K)$ does not satisfy any polynomial identities of degrees less than $2n$.)

Remark 2.4. *It should be noted that the relatively free pairs in these varieties are special.*

Corollary 2.5. *The variety of all nilpotent of a degree > 7 Jordan pairs is not special.*

Proof. Let N_p be a relatively free pair in the variety of the Jordan pairs that are nilpotent of degree p. Then N_p is the quotient pair of the free Jordan pair by the ideal of all elements of degrees $> p$. Thus the Corollary follows immediately from the above theorem since $\deg(w) = 7$.

It would be interesting to find some special varieties of Jordan pairs. It is an open problem whether the above matrix varieties are special when $n = 2$. In order to find an example of a special variety we need the notion of a weak polynomial identity. Assume that $A = (A^+, A^-)$ is an associative pair, $W = (W^+, W^-)$ a pair of subspaces of A such that the set W generates A as an associative pair. The polynomial $f(x_1, y_1, \dots, y_{k-1}, x_k) \in FA(X)^\sigma$ is called a weak polynomial identity for the pair $\langle A, W\rangle$ if $f(a_1, b_1, \dots, b_{k-1}, a_k) = 0$ for each $a_1, \dots, a_k \in W^\sigma$ and $b_1, \dots, b_{k-1} \in W^{-\sigma}$. It is obvious that the weak polynomial identities for $\langle A, W\rangle$ form an ideal in $FA(X)$. Denote $PC = (C, C)$ the associative pair consisting of two copies of the Clifford algebra on the space W equipped with a symmetric bilinear form. Let T be the ideal of all weak identities of $\langle PC, PW\rangle$ where $PW = (W, W)$.

In the algebra case P. Cohn [1] described the connection between the free special Jordan algebra, and the set of the reversible elements in the free associative algebra when the element 2 is invertible in K (i.e. $1/2 \in K$).

The connection between the free special Jordan pair $FSJ(X)$ and the Jordan pair of the symmetric elements in the free associative pair $H(X)$ played a crucial role in several applications, see e.g. [3]. A system of generators of $H(X)$ was found in [8], and subsequently in [3] it was shown that the above system of generators cannot be reduced significantly. All these results hold for the case $1/2 \in K$.

It is obvious that $FSJ(X) \subset H(X)$.

Theorem 2.6. *$FSJ(X) = H(X)$ if and only if $|X| < 5$.*

Sketch of the proof. First we consider the case $|X| = 4$. If $|X^+| = 3$ then a homotopy argument reduces the case to that of a Jordan algebra. Therefore we can apply the description of the symmetric elements in the free associative pair given in [1]. The case $|X^+| = 2$ consists of a straightforward computation similar to that of the algebra case, see [1]. The main point in the proof of the rest of the theorem is to establish that the pentad $[xyztu]$ does not belong to $FSJ(X)$ when x, y, z, t, and u are different free generators. One can use methods similar to those of the theorem for the exceptional images of a special Jordan pair (Theorem 2) in order to prove the last statement.

Theorem 2.7. *The variety of Jordan pairs defined by all weak polynomial identities from the ideal T (where T is the ideal of all weak identities of $\langle PC, PW\rangle$ where $PW = (W, W)$) is special.*

The technique used in the proof of this theorem relies on an explicit construction of a closing module for this variety, see [7] for details. In addition one needs the description of the set of the symmetric elements in the free associative pair given in the preceding parts.

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GRADED EQUIVALENCES AND BROUÉ'S CONJECTURE

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This paper is an extended version of a talk given at the Constanţa conference. Our aim is to present some results obtained in [M1] and [M2] concerning the reduction of Broué's conjecture to the case of simple groups. This reduction involves an investigation of Morita equivalences, derived equivalences and stable equivalences of Morita type between two G-graded algebras R and S, which are induced by G-graded R, S-bimodules or complexes of G-graded bimodules.

1. Introduction and preliminaries

Let G be a finite group, $\mathcal{O}$ a complete discrete valuation ring with residue field $k =$ of characteristic $p > 0$. for all the finite groups considered here. An $\mathcal{O}G$-module will be a unitary, finitely generated, $\mathcal{O}$-free, and (unless otherwise stated) left $\mathcal{O}G$-module. We denote by $\mathcal{O}G$-mod (respectively mod-$\mathcal{O}G$) the category of left (respectively right) $\mathcal{O}G$-modules.

In [Br1] and [Br2], the following conjectures were stated:

1.1. *If D is an abelian Sylow p-subgroup of G and $H = N_G(D)$, then the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ are isotypic.*

1.2. *If D is an abelian Sylow p-subgroup of G and $H = N_G(D)$, then the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ are Rickard equivalent.*

A Rickard equivalence is a derived equivalence that takes into account the presence of groups. Similarly, a Morita equivalence "with groups" is called a Puig equivalence. We shall give later the precise definitions.

In [FH], Conjecture 1.1 was reduced, by using the classification of finite simple groups, and by developing a Clifford theory for isometries and isotypies, to the case of simple components of G, and the conjecture was proved in the case $p = 2$.

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By [Br2] and [R3, Section 6], conjecture (1.2) implies conjecture (1.1), and we have shown in [M1] that it also can be reduced to the case of the simple components of G. The reduction steps are similar to those of [FH], and may be considered as a Clifford theory for tilting complexes.

In Section 2 we present, following [FH], the structure of groups with abelian Sylow p-subgroups. This structure gives the motivation for the reduction steps.

The main step is the lifting of an equivalence (Morita or derived) between the blocks e and f of normal subgroups of two groups with isomorphic quotient groups, to equivalences between the blocks lying over e and f.

Let us introduce some notation. Let $\tilde{X}$ and $\tilde{Y}$ be finite groups with normal subgroups X and Y respectively, such there is an isomorphism $\alpha\colon \tilde{X}/X \to \tilde{Y}/Y$. We identify via α these two quotient groups and denote them by G. Assume further that e and f are G-invariant blocks of $\mathcal{O}X$ and $\mathcal{O}Y$, and consider the fully G-graded $\mathcal{O}$-algebras $R = e\mathcal{O}\tilde{X}$ and $S = f\mathcal{O}\tilde{Y}$. The blocks of R, respectively S, are precisely the blocks of $\mathcal{O}\tilde{X}$, respectively $\mathcal{O}\tilde{Y}$, covering e, respectively f.

We shall regard here the opposite algebra S^{op} as a G-graded algebra with components $S^{op}_g = S_{g^{-1}}$, for $g \in G$. Then $A = R \otimes_{\mathcal{O}} S^{op}$ is a fully $G \times G$-graded algebra, and let $\Delta = \bigoplus_{g\in G}(R_g \otimes_{\mathcal{O}} S^{op}_g) = A_{\delta(G)}$ be the diagonal subalgebra of A, where $\delta(G) = \{(g,g) \mid g \in G\}$, so Δ may be regarded as a fully G-graded algebra. Many of our results will be stated in this more general setting. Assume further that M_1 (respectively C_1) is an R_1, S_1-bimodule (respectively a bounded complex of R_1, S_1-bimodules) inducing a Morita (respectively derived) equivalence between R_1 and S_1. Then by [M1, Theorem 3.4], this equivalence can be lifted to an equivalence between R and S provided that M_1 (respectively C_1) extends to Δ. This condition is similar to the condition imposed in [FH] on perfect isometries.

The above equivalences between R and S are induced by G-graded bimodules (respectively complexes of G-graded bimodules). One simple, but important observation is that a G-graded R, S-bimodule is the same thing as a left A-module graded by the $G \times G$ set $G \times G/\delta(G)$. We develop in Section 3 the relationship between graded bimodules and modules graded by G-sets.

Section 3 the relationship between graded bimodules and modules graded by G-sets.

In Section 4 we study Morita equivalences induced by graded bimodules. In Section 5 we obtain similar results for derived equivalences induced by complexes of G-graded bimodules, and in Section 6, for stable equivalences of Morita type which are induced by graded bimodules. We shall also find some "graded invariants" and other properties of graded equivalences. For example, we show that if R and S are graded derived equivalent, then the centralizers $C_R(R_1)$ and $C_S(S_1)$ are isomorphic as G-algebras and as G-graded algebras.

We look to some peculiarities of symmetric algebras in Section 7, and in Section 8 we come back to the case of blocks and we discuss Puig and Rickard equivalences. In Section 9 we deal with wreath products. We show that a Rickard or a Puig equivalence between the principal blocks of the base groups of two wreath products can always be lifted. The problem is making the symmetric group to act on a tensor power of a complex. Finally, the conclusions which follow from these reduction steps are drawn in Section 10. We also give here some examples in which the lifting conditions are satisfied.

In this paper, rings will always be associative with unity, and modules are unitary and left, unless otherwise stated. We denote by R-Mod, Mod-S and R-Mod-S the category of all R-modules, right S-modules and R,S-bimodules respectively. If k is a commutative ring, then by a G-graded k-algebra we mean that the elements of k have degree 1. If R and S are k-algebras, then an R,S-bimodule is a module over $R\otimes_k S^{op}$. Beginning with Section 4, R and S will be G-graded k-algebras which are projective as k-modules. We refer to [D1], [D2] and [NRV] for general facts on graded rings and modules, to [Gr] for derived categories, and to [Br1, Section 5] and [Li] for stable equivalences of Morita type.

2. Motivation: groups with abelian Sylow p-subgroups

Consider a finite group G and let D be an abelian Sylow p-subgroup of G. Assume now that $O_{p'}(G) = 1$. We shall see that this is no loss in the context of Broué's conjectures. By the results of [FH, Section 5] (obtained by using at certain stages the classification of finite simple groups), there is an embedding ι of G such that:

(2.1) $$\prod_{i=0}^{s} X_i^{\Omega_i} \leq \iota(G) \leq \prod_{i=0}^{s}(\tilde{X}_i \wr \Sigma_i)$$

where:

- $X_0 = O_p(G)$, the maximal normal p-subgroup of G,
- $\tilde{X}_0$ is a split extension of X_0 by a p'-subgroup of $\mathrm{Aut}(X_0)$,
- $|\Omega_0| = |\Sigma_0| = 1$,
- $X_1, \dots, X_s$ is a complete system of representatives for the isomorphism classes of simple components of G,
- X_i is a normal subgroup of $\tilde{X}_i$ and $\tilde{X}_i/X_i$ is a Hall p'-subgroup of $\mathrm{Aut}(X_i)/X_i$, for $1 \leq i \leq s$,
- $\Omega_1, \dots, \Omega_s$ are disjoint finite sets,
- Σ_i is a p'-subgroup of the symmetric group $S(\Omega_i)$, for $1 \leq i \leq s$.

We change now the notations setting $G' = \iota(G)$, $G = \prod_{i=0}^{s} X_i^{\Omega_i}$, $\tilde{G} = \prod_{i=0}^{s}(\tilde{X}_i \wr \Sigma_i)$, $H = N_G(D)$, $H' = N_{G'}(D)$ and $\tilde{H} = N_{\tilde{G}}(D)$. If $D_i = X_i \cap D$, then D_i is a Sylow p-subgroup of X_i and $D = \prod_{i=0}^{s} D_i$. Denote $Y_i = N_{X_i}(D_i)$

and $\tilde{Y}_i = N_{\tilde{X}_i}(D_i)$, so $H = \prod_{i=0}^{s} Y_i^{\Omega_i}$ and $\tilde{H} = \prod_{i=0}^{s}(\tilde{Y}_i \wr \Sigma_i)$. (Remark that for $i = 0$ we have $D_0 = O_p(G) = X_0 = Y_0$ and $\tilde{X}_0 = \tilde{Y}_0$.) By the Frattini argument we have $\tilde{X}_i = \tilde{Y}_i X_i$, hence the map $\alpha_i \colon \tilde{X}_i/X_i \to \tilde{Y}_i/Y_i$, $\alpha_i(y_i X_i) = y_i Y_i$ is an isomorphism. Moreover, α_i, $1 \le i \le s$, induce the isomorphism

$$\alpha = \prod_{i=0}^{s} \alpha_i^{\wr\Sigma_i} \colon \prod_{i=0}^{s}(\tilde{X}_i/X_i) \wr \Sigma_i \to \prod_{i=1}^{s}(\tilde{Y}_i/Y_i) \wr \Sigma_i, \quad ((y_i X_i^{\Omega_i}, \sigma_i)) \mapsto ((y_i Y_i^{\Omega_i}, \sigma_i))$$

such that $\alpha(G'/G) = H'/H$.

In the next sections we shall give conditions under which Broué's Conjecture 1.2 holds for G' provided that it holds for the simple components X_i of G'. The reduction steps are motivated by the structure theorem above. For each i, an equivalence between the principal blocks of $\mathcal{O}X_i$ and $\mathcal{O}Y_i$ has to be lifted to an equivalence between the principal blocks of $\mathcal{O}\tilde{X}_i$ and $\mathcal{O}\tilde{Y}_i$, then to an equivalence between the principal blocks of $\mathcal{O}[\tilde{X}_i \wr \Sigma_i]$ and $\mathcal{O}[\tilde{Y}_i \wr \Sigma_i]$, then to an equivalence for the direct products $\prod_{i=0}^{s}(\tilde{X}_i \wr \Sigma_i)$ and $\prod_{i=0}^{s}(\tilde{Y}_i \wr \Sigma_i)$, and finally to an equivalence for subgroups of these direct products (corresponding under the isomorphism α). We shall see that restrictive conditions have to be imposed only at the first step.

3. Graded bimodules and modules graded by G-sets

Let G be a group, $R = \bigoplus_{g\in G} R_g$ a G-graded ring, and X a left G-set. An R-module M is called X-graded if $M = \bigoplus_{x\in X} M_x$ (as additive subgroups), and $R_g M_x \subseteq M_{gx}$ for all $x \in X$, $g \in G$. If Y is a subset of X, we denote $M_Y = \bigoplus_{x\in Y} M_x$. The category of X-graded R-modules and grade-preserving R-homomorphisms is denoted (G, X, R)-Gr. We are interested in the special case when $X \simeq G/H$, where H is a subgroup of G, and then we denote $(G/H, R)$-Gr $= (G, G/H, R)$-Gr. This is a Grothendieck category with a projective generator. Notice that for $H = G$ we obtain the category R-Mod and for $H = \{1\}$, the category R-Gr of G-graded R-modules.

We recall from [NRV] some basic results results.

3.1. A G/H-graded R module is a projective object of $(G/H, R)$-Gr if and only if it is projective when regarded as an R-module. If G is finite, then the same holds for injectives.

3.2. If R is fully graded (that is, $R_g R_h = R_{gh}$ for all $g, h \in R$), then the functors $R \otimes_{R_H} - \colon R_H$-Mod $\to (G/H, R)$-Gr and $(-)_H \colon (G/H, R)$-Gr $\to R_H$-Mod are inverse equivalences of categories. Notice also that in this case, for all $g \in G$, R_g is a progenerator of R_1-Mod.

Modules graded in this way arise naturally when we consider G-graded bimodules.

Assume that k is a commutative ring, and that R and S are fully G-graded k-algebras. By an R, S-bimodule we always mean a module over the k-algebra $A = R \otimes_k S^{op}$. We regard S^{op} as a G-graded k-algebra with components $S^{op}_g = S_{g^{-1}}$, and A as a $G \times G$-graded k-algebra with $A_{(g,h)} = R_g \otimes_k S^{op}_g$. Let $\delta(G) = \{(g,g) \mid g \in G\}$ be the diagonal subgroup of $G \times G$, and let $\Delta = \bigoplus_{g \in G} R_g \otimes_k S^{op}_g = A_{\delta(G)}$, which is a fully $\delta(G)$-graded (or G-graded) k-subalgebra of A.

Multiplication in A gives the following bimodule isomorphisms.

(3.3.a) $$A \simeq R \otimes_{R_1} \Delta \simeq \Delta \otimes_{R_1} R$$

(3.3.b) $$A \simeq S^{op} \otimes_{S_1^{op}} \Delta \simeq \Delta \otimes_{S_1^{op}} S^{op},$$

where Δ is an R_1-bimodule and an S_1 bimodule in an obvious way. For instance, the first isomorphism comes from the decomposition $G \times G = \bigcup_{g \in G} (g, 1)\delta(G)$, and it is graded in the sense that it respects the bijection $G \times \delta(G) \to G \times G$, $(g, (x, x)) \mapsto (gx, x)$.

Let now M be a G-graded R, S-bimodule, that is, $M = \bigoplus_{x \in G} M_x$ and $R_g M_x S_h = M_{gxh}$ for all $g, h, x \in G$. We denote by R-Gr-S the category of G-graded R, S-bimodules and graded preserving R, S-linear maps. In other words, regarding G as a left $G \times G$- set M is an object of $(G \times G, G, A)$-Gr. Using the isomorphism $G \simeq G \times G/\delta(G)$ of $G \times G$-sets, we obtain the equivalence of categories $\mathcal{F}\colon R\text{-Gr-}S \to (G \times G/\delta(G), A)\text{-Gr}$, where $\mathcal{F}(M) = M$ regarded as above.

If N is in turn a Δ-module, then (3.3) provides the isomorphisms

(3.4.a) $$A \otimes_\Delta N \simeq (R \otimes_{R_1} \Delta) \otimes_\Delta N \simeq R \otimes_{R_1} N \qquad (\text{of } R, S_1\text{-bimodules})$$

(3.4.b) $$A \otimes_\Delta N \simeq (S^{op} \otimes_{S_1^{op}} \Delta) \otimes_\Delta N \simeq N \otimes_{S_1} S \qquad (\text{of } R_1, S\text{-bimodules}),$$

and these isomorphisms preserve the gradings via the isomorphism $G \times G/\delta(G) \simeq G$ of $G \times G$-sets.

By making suitable transport of structure, we conclude:

Lemma 3.5. *The functors $\mathcal{F}^{-1} \circ (A \otimes_\Delta -), R \otimes_{R_1} -, - \otimes_{S_1} S\colon \Delta\text{-Mod} \to R\text{-Gr-}S$ are naturally isomorphic equivalences of categories, and their inverse is $(-)_{\delta(G)} \circ \mathcal{F} = (-)_1$.*

We analyse further tensor products and homomorphisms of graded bimodules, so let R, S and T be fully G-graded k-algebras. We need to introduce the notations $A(R, S) = R \otimes_k S^{op}$ and $\Delta(R, S) = A(R, S)_{\delta(G)}$.

Let first M be a G-graded R, S-bimodule and N a G-graded S, T-bimodule. Then $M \otimes_S N$ is a G-graded R, T-bimodule with $(M \otimes_S N)_x = \sum_{yz=x} M_y \otimes_S N_z$, and we have the isomorphisms

(3.6.a) $$M \otimes_S N \simeq M \otimes_S (S \otimes_{S_1} N) \simeq R \otimes_{R_1} (M_1 \otimes_{S_1} N)$$

Let now M be a G-graded S,T-bimodule, N a G-graded S,R-bimodule, and assume that G is finite or that M is a finitely generated S-module. Then, using [D1, Corollary 3.10], one can easily verify that $\mathrm{Hom}_S(M,N)$ is a G-graded R,T-bimodule, where for $g \in G$, $\mathrm{Hom}_S(M,N)_g = \{f \in \mathrm{Hom}_S(M,N) \mid f(M_x) \subseteq N_{xg}$ for all $x \in G\}$, and $(rft)(m) = f(mt)r$ for all $r \in R$, $t \in T$, $m \in M$ and $f \in \mathrm{Hom}_S(M,N)$. Again, we have the isomorphisms

(3.6.b) $$\mathrm{Hom}_S(M,N) \simeq \mathrm{Hom}_S(S \otimes_{S_1} M_1, N) \simeq R \otimes_{R_1} \mathrm{Hom}_{S_1}(M_1,N_1).$$

Observe that the isomorphisms (3.6) suggest that the 1-component of $M \otimes_S N$ can be identified with $M_1 \otimes_{S_1} N_1$, and the 1-component of $\mathrm{Hom}_S(M,N)$ can be identified with $\mathrm{Hom}_{S_1}(M_1,N_1)$. Let us state this precisely.

Recall that since S is fully graded, for all $g \in G$, $S_{g^{-1}}S_g = S_1$, hence there are elements $s'_i \in S_{g^{-1}}$, $s_i \in S_g$, $1 \leq i \leq l$ such that

(3.7.) $$\textstyle\sum_{i=1}^{l} s'_i s_i = 1$$

Lemma 3.8. *Let $g \in G$, $r_g \in R_g$, $t_{g^{-1}} \in T_{g^{-1}} = T_g^{op}$, and let s'_i, s_i chosen as in (3.7).*

a) *Assume that M is a graded R,S-bimodule and N is a G-graded S,T-bimodule. Then $M_1 \otimes_{S_1} N_1$ is a $\Delta(R,T)$-module with multiplication*

$$r_g(m \otimes_{S_1} n)t_{g^{-1}} = \sum_{i=1}^{l} r_g m s'_i \otimes_{S_1} s_i n t_{g^{-1}},$$

for $m \in M_1$, $n \in N_1$. This definition does not depend on the choice made in (3.7), and the map $M_1 \otimes_{S_1} N_1 \to (M \otimes_S N)_1$, $m \otimes_{S_1} n \mapsto m \otimes_S n$ is an isomorphism of $\Delta(R,T)$-modules. Moreover we have that

$$M \otimes_S N \simeq A(R,T) \otimes_{\Delta(R,T)} (M_1 \otimes_{S_1} N_1).$$

b) *Assume that M is a G-graded S,T-bimodule, N is a G-graded S,R-bimodule, and that G is finite or M is a finitely generated S-module. Then $\mathrm{Hom}_{S_1}(M_1,N_1)$ is a $\Delta(T,R)$-module with multiplication*

$$(t_{g^{-1}} f r_g)(m) = \sum_{i=1}^{l} s'_i f(s_i m t_{g^{-1}}) r_g$$

for $m \in M$ and $f \in \mathrm{Hom}_{S_1}(M_1,N_1)$. Again this definition is independent on the choice made in (3.7), and the map $\mathrm{Hom}_S(M,N) \to \mathrm{Hom}_{S_1}(M_1,N_1)$, $f \mapsto f_1$ is a $\Delta(R,T)$-isomorphism. Moreover, if G is finite, then $\mathrm{Hom}_S(M,N) \simeq A(T,R) \otimes_{\Delta(T,R)} \mathrm{Hom}_{S_1}(M_1,N_1)$.

The explicit R,S-bimodule structure of $R\otimes_{R_1} M$ and $M\otimes_{S_1} S$ in Lemma 3.5, where M is a $\Delta(R,S)$-module, is given by

$$(r_g\otimes_{R_1} m)s_h=\sum_{j=1}^{q} r_gr_j\otimes_{R_1}(r'_j\otimes_k s_h)m,$$

where for $1\le j\le q$, $r_j\in R_h$ and $r'_j\in R_{h^{-1}}$ are chosen such that $\sum_{j=1}^{q} r_jr'_j=1$, and

$$r_g(m\otimes_{S_1} s_h)=\sum_{i=1}^{l}(r_g\otimes_k s'_i)m\otimes_{S_1} s_is_h,$$

where s_i, s'_i are as in (3.7).

4. Morita equivalences

Let G be a finite group, k a commutative ring, and let R and S be two G-graded k algebras, projective over k. These assumptions will also be in force in the next sections (although we do not need them everywhere). We shall use the notations $A=A(R,S)$ and $\Delta=\Delta(R,S)$ introduced in the previous section, and denote $A(R)=A(R,R)$ and $\Delta(R)=\Delta(R,R)=A(R)_{\delta(G)}$.

We shall say that R and S are *graded Morita equivalent*, if there is a G-graded R,S-bimodule M and a G-graded S,R-bimodule N inducing a Morita equivalence between R and S such that the bimodule isomorphisms $\alpha\colon M\otimes_S N\to R$ and $\beta\colon N\otimes_R M\to S$ are grade preserving (that is, $\alpha(M_x\otimes_S N_y)\subseteq R_{xy}$ and $\beta(N_x\otimes_R M_y)\subseteq S_{xy}$ for all $x,y\in G$).

There are examples which show that it is possible to have a Morita equivalence between R_1 and S_1 (and hence an equivalence between R-Gr and S-Gr) without R and S being Morita equivalent.

The constructions presented in Section 3 are the main ingredients in the proof of the following theorem.

Theorem 4.1. *Let M_1 be an R_1,S_1-bimodule, N_1 an S_1,R_1-bimodule, and denote $M=R\otimes_{R_1}M_1$ and $N=N_1\otimes_{S_1}S$ The following statements are equivalent:*

(i) *There is a structure of a G-graded R,S-bimodule on M and a structure of a G-graded S,R-bimodule on N (extending the given structures), such that M and N induce a graded Morita equivalence between R and S.*

(ii) *M_1 and N_1 induce a Morita equivalence between R_1 and S_1, and M_1 extends to a Δ-module.*

Certainly, graded Morita equivalences have some specific properties. In order to state them as consequences of the above theorem, we need to recall some well-known facts.

4.2.a. Assume that M is a G-graded R,S-bimodule, H is a subgroup of G and that V is a G/H-graded S-module. Then $M\otimes_S V$ is a G/H-graded R-module with components $(M\otimes_S V)_{gH}=\sum_{xyH=gH}(M_x\otimes_S V_{yH})$ for all $x,y,g\in G$.

4.2.b. If U is a G/H-graded R-module, then $\mathrm{Hom}_R(M,U)$ is a G/H-graded S-module with $\mathrm{Hom}_S(M,U)_{gH}=\{f\in\mathrm{Hom}_R(M,U)\mid f(M_x)\subseteq U_{xgH}$ for all $x\in G\}$. In particular, $E=\mathrm{End}_R(M)^{op}$ is a G-graded k-algebra and M is a G-graded R,E-bimodule.

4.3.a. Let now U an V be R-modules and $H\le G$. Then by [D2, Theorem 2.1], $\mathrm{Hom}_{R_1}(U,V)$ is a kG-module, and $\mathrm{Hom}_{R_H}(U,V)=\mathrm{Hom}_{R_1}(U,V)^H$, where for $f\in\mathrm{Hom}_{R_1}(U,V)$ and $g\in G$, if $r_j\in R_g$, $r'_j\in R_{g^{-1}}$, $1\le j\le q$ are chosen such that $\sum_{j=1}^q r_jr'_j=1$, then ${}^gf(u)=\sum_{j=1}^q r_jf(r'_ju)$, for all $u\in U$, and this action is compatible with the composition. If we start with G-graded bimodules, then we obtain modules with suitable G-gradings. For example, $C_R(R_1)=\mathrm{End}_{R_1,R}(R)$ is a G-algebra and a G-graded algebra, and $C_R(R_1)^G=Z(R)$ is a graded subalgebra of R.

4.3.b. An R-module W is called relatively R_H-projective if it is a direct summand of a G/H-graded R-module, or equivalently, by Higman's criterion [D2, Proposition 3.3], if $id_W\in\mathrm{Tr}_H^G(\mathrm{End}_{R_1}(W))$. An R-linear map $f\colon U\to V$ is called relatively R_H-projective if it factorises through a relatively R_H-projective R-module, or equivalently, if $f\in\mathrm{Tr}_H^G\,\mathrm{Hom}_{R_1}(U,V)$. Denote by $\mathrm{Hom}_{R_1}^{pr}(U,V)$ the set of R_1-linear maps which factorise through a projective R_1-module. Then it is not difficult to see that $\mathrm{Hom}_{R_1}^{pr}(U,V)$ is a kG-submodule of $\mathrm{Hom}_{R_1}(U,V)$ (invariant under composition too), and $\mathrm{Tr}_1^G(\mathrm{Hom}_{R_1}^{pr}(U,V))=\mathrm{Hom}_{R_1}^{pr}(U,V)$. The G-invariant graded ideal $C_R^{pr}(R_1)$ of $C_R(R_1)$ is defined analogously, and we have $C_R^{pr}(R_1)^G=Z^{pr}(R)$.

Corollary 4.4. *Assume that the equivalent conditions of Theorem 3.4 hold. Then we have:*

a) *For each subgroup H of G, R_H and S_H are graded Morita equivalent, and the categories $(G/H,R)$-Gr and $(G/H,S)$-Gr are equivalent.*

b) *If U and V are R-modules, then there is an isomorphism between the kG-modules $\mathrm{Hom}_{R_1}(U,V)$ and $\mathrm{Hom}_{S_1}(N_1\otimes_{R_1}U,N_1\otimes_{R_1}V)$, which restricts to an isomorphism $\mathrm{Hom}_{R_1}^{pr}(U,V)\simeq\mathrm{Hom}_{S_1}^{pr}(N_1\otimes_{R_1}U,N_1\otimes_{R_1}V)$.*

c) *For any subgroups H,K of G, the categories R_H-Mod-R_K and S_H-Mod-S_K are equivalent.*

d) *There is an isomorphism $C_R(R_1)\simeq C_S(S_1)$ of G-algebras and G-graded algebras, which restricts to an isomorphism $C_R^{pr}(R_1)\simeq C_S^{pr}(S_1)$.*

5. Derived equivalences

If $\mathcal{A}$ is an abelian category, we denote by $D^b(\mathcal{A})$ the derived category of bounded complexes of objects of $\mathcal{A}$, and notice that the constructions given in Sections 3 and 4, (especially (3.2), (3.5), (3.8), (4.2) and (4.3) extend to the derived categories.

The algebras R and S are said to be *derived equivalent* if the categories $D^b(R\text{-Mod})$ and $D^b(S\text{-Mod})$ are equivalent as triangulated categories. By the results of [Ri2, Sections 3,4], R and S are derived equivalent if and only if there is an object $C \in D^b(R\text{-Mod-}S)$ and an object $D \in D^b(S\text{-Mod-}R)$ such that $C \overset{\mathbf{L}}{\otimes}_S D \simeq R$ in $D^b(R\text{-Mod-}R)$ and $D \overset{\mathbf{L}}{\otimes}_R C \simeq S$ in $D^b(S\text{-Mod-}S)$. In this case, the equivalence is given by the functors $D \overset{\mathbf{L}}{\otimes}_R - \simeq \mathbf{R}\mathrm{Hom}_R(C,-)$ and $C \overset{\mathbf{L}}{\otimes}_S -$, and we also have that $D \simeq \mathbf{R}\mathrm{Hom}_R(C,R)$ in $D^b(S\text{-Mod-}R)$. Then C is called a *two-sided tilting complex* for R and S, and D is an *inverse* of C.

The purpose of this section is to show that results similar to those of Section 4 hold for derived equivalences induced by complexes of G-graded bimodules.

We say that R and S are *graded derived equivalent* if there are objects $C \in D^b(R\text{-Gr-}S)$ and $D \in D^b(S\text{-Gr-}R)$ inducing a derived equivalence between R and S such that $\alpha\colon C \overset{\mathbf{L}}{\otimes}_S D \to R$ and $\beta\colon D \overset{\mathbf{L}}{\otimes}_R C \to S$ are isomorphisms in $D^b(R\text{-Gr-}R)$, respectively in $D^b(S\text{-Gr-}S)$. We have first a rather straightforward consequence of the fact that the categories $D^b(R\text{-Gr-}S)$ and $D^b(\Delta\text{-Mod})$ are equivalent.

Theorem 5.1. *The following statements are equivalent:*

i) *R and S are graded derived equivalent, and $D \in D^b(S\text{-Gr-}R)$.*

ii) *There are objects $C_1 \in D^b(\Delta(R,S)\text{-Mod})$, $D_1 \in D^b(\Delta(S,R)\text{-Mod})$, and isomorphisms $\alpha_1 : C_1 \overset{\mathbf{L}}{\otimes}_{S_1} D_1 \to R_1$ in $D^b(\Delta(R)\text{-Mod})$ and $\beta_1 : D_1 \overset{\mathbf{L}}{\otimes}_{R_1} C_1 \to S_1$ in $D^b(\Delta(S)\text{-Mod})$.*

If we only know that a two-sided tilting complex for R_1 and S_1 extends to $\Delta(R,S)$, then additional hypotheses are needed.

Theorem 5.2. *Assume that the order of G is invertible in k. Then the following statements are equivalent:*

(i) *R and S are graded derived equivalent.*

(ii) *There is an object $C_1 \in D^b(\Delta\text{-Mod})$ which, regarded as an object of $D^b(R_1\text{-Mod-}S_1)$, is a two-sided tilting complex for R_1 and S_1.*

Corollary 5.3. *If the equivalent conditions of Theorem 5.1 hold, then:*

a) *For each subgroup H of G, R_H and S_H are graded derived equivalent, and the categories $D^b((G/H,R)\text{-Gr})$ and $D^b((G/H,S)\text{-Gr})$ are equivalent.*

b) *If* $X, Y \in D^b(R\text{-Mod})$, *then*

$$\mathbf{R}\mathrm{Hom}_{R_1}(X,Y) \simeq \mathbf{R}\mathrm{Hom}_{S_1}(C_1 \overset{\mathbf{L}}{\otimes}_{R_1} X, C_1 \overset{\mathbf{L}}{\otimes}_{R_1} Y) \text{ in } D^b(kG\text{-Mod}).$$

c) *For any* $H, K \leq G$, *the categories* $D^b(R_H\text{-Mod-}R_K)$ *and* $D^b(S_H\text{-Mod-}S_K)$ *are equivalent.*

d) *There is an isomorphism* $C_R(R_1) \simeq C_S(S_1)$ *of* G*-algebras and* G*-graded algebras which restricts to an isomorphism* $C_R^{pr}(R_1) \simeq C_S^{pr}(S_1)$.

Assume that k is a field of characteristic $p > 0$ or a complete discrete valuation ring with residue field of characteristic p. Then vertices of R-modules and defect groups of blocks of R can be defined (see [D2] and [B, Section 3]). It follows by our results that a graded derived equivalence between R and S preserve vertices of modules and induces a bijection between the blocks of R and S such the defect groups are preserved.

6. Stable equivalences of Morita type

In order to simplify the statements of our results, we shall assume in this section that k is a field, and that R, S are finite dimensional fully G-graded k-algebras. We denote by R-mod the category of finitely generated R-modules (in general, by a module we shall understand a finitely generated module), and by $R\text{-}\overline{\mathrm{mod}}$ its stable category. Analogously, $(G/H, R)\text{-}\overline{\mathrm{gr}}$ is the stable category of $(G/H, R)$-gr.

Since in general there is no obvious relation between M and N, the following theorem is an easy consequence of the results of Section 3.

Theorem 6.1. *Let* M_1 *be an* R_1, S_1*-bimodule,* N_1 *an* S_1, R_1*-bimodule, both projective as left and right modules, and denote* $M = R \otimes_{R_1} M_1$, $N = N_1 \otimes_{S_1} S$. *The following statements are equivalent:*

(i) M *extends to a* G*-graded* R, S*-bimodule and* N *extends to a* G*-graded* S, R*-bimodule such that* M *and* N *induce a graded stable equivalence of Morita type between* R *and* S.

(ii) M_1 *extends to a* $\Delta(R, S)$*-module,* N_1 *extends to a* $\Delta(S, R)$*-module, and there are isomorphisms* $M_1 \otimes_{S_1} N_1 \simeq R_1 \oplus U_1$ *in* $\Delta(R)$-mod *and* $N_1 \otimes_{R_1} M_1 \simeq S_1 \oplus V_1$ *in* $\Delta(S)$-mod, *where* U_1 *is a projective* $\Delta(R)$*-module and* V_1 *is a projective* $\Delta(S)$*-module.*

Corollary 6.2. *If the equivalent conditions of Theorem 5.4 hold, then*

a) *For each subgroup* H *of* G, R_H *and* S_H *are graded stably equivalent, and the categories* $(G/H, R)\text{-}\overline{\mathrm{gr}}$ *and* $(G/H, S)\text{-}\overline{\mathrm{gr}}$ *are equivalent.*

b) *If* U *and* V *are* R*-modules, then the map* $f \mapsto \mathrm{Hom}_{R_1}(N_1, f)$ *induces an isomorphism between* $\overline{\mathrm{Hom}}_{R_1}(U, V)$ *and* $\overline{\mathrm{Hom}}_{S_1}(N_1 \otimes_{R_1} U, N_1 \otimes_{R_1} V)$ *as* kG*-modules.*

c) *For all subgroups H, K of G, the categories R_H-$\overline{\text{mod}}^{pr}$-$R_K$ and S_H-$\overline{\text{mod}}^{pr}$-$S_K$ are equivalent, where the objects of R_H-$\overline{\text{mod}}^{pr}$-$R_K$ are bimodules which are projective as left and as right modules.*

d) *$C_R(R_1)/C_R^{pr}(R_1)$ and $C_S(S_1)/C_S^{pr}(S_1)$ are isomorphic as G-algebras and as G-graded algebras.*

We discuss further a particular case of stable equivalences of Morita type - equivalences which are induced by adjoint functors at the level of module categories. This situation frequently occurs (see [R2, Corollary 5.5] and [Br1, Section 5 and Example 6.4]).

Let M be a G-graded R, S-bimodule, $\text{Hom}_R(M, -)\colon R\text{-mod} \to S\text{-mod}$ and $M \otimes_S -\colon S\text{-mod} \to R\text{-mod}$. The adjunction induces the natural maps

$$\alpha\colon\ M \otimes_S Hom_R(M, R) \to R \text{ of } G\text{-graded } R\text{-bimodules, and}$$

$$\beta\colon S \to \text{Hom}_R(M, M \otimes_S S) \text{ of } G\text{-graded } S\text{-bimodules.}$$

Denote $N = \text{Hom}_R(M, R)$, which is an object of S-gr-R. Then we have a natural homomorphism of G-graded S-bimodules $\gamma\colon N \otimes_R M \to \text{Hom}_R(M, M)$.

Taking now the 1-components of these maps, by the results of Section 3, we deduce that

$$\alpha_1\colon M_1 \otimes_{S_1} N_1 \to R_1 \text{ is } \Delta(R)\text{-linear,}$$

$$\beta_1\colon S_1 \to \text{Hom}_{R_1}(M_1, M_1 \otimes_{S_1} S_1) \text{ is } \Delta(S)\text{-linear, and}$$

$$\gamma_1\colon N_1 \otimes_{R_1} M_1 \to \text{Hom}_{R_1}(M_1, M_1) \text{ is } \Delta(S)\text{-linear.}$$

Notice also that if M is projective in R-mod and in mod-S, then γ is an isomorphism and the functors $\text{Hom}_R(M, -)$ and $N \otimes_R -$ are isomorphic.

The next proposition can now be deduced without difficulty from the above remarks and from the following well-known consequences of (3.1), (3.2) and (4.3).

6.3. The k-algebra R is selfinjective if and only if R_1 is selfinjective.

6.4. If the order of G is invertible in k, then an R-module U is projective if and only if U is projective as an R_1-module.

Assume that M is a G-graded R, S-bimodule and N is a G-graded S, R-bimodule, both projective as left and right modules. We say that M and N induce a *graded stable equivalence of Morita type* between R and S, if there are isomorphisms $M \otimes_S N \simeq R \oplus U$ in R-Gr-R, where U is a projective R-bimodule, and $N \otimes_R M \simeq S \oplus V$ in S-Gr-S, where V is a projective S-bimodule.

Proposition 6.5. *Assume that R and S are selfinjective algebras and that the order of G is invertible in k. Let M_1 be an R_1, S_1-bimodule, projective in R_1-mod and in mod-S_1, and denote $M = R \otimes_{R_1} M_1$. The following conditions are equivalent:*

(i) *M extends to a G-graded R, S-bimodule, α is an epimorphism with* $\operatorname{Ker}\alpha$ *projective R-bimodule, and β is a monomorphism with* $\operatorname{Coker}\beta$ *projective S-bimodule.*

(ii) *M_1 extends to a Δ-module, α_1 is an epimorphism with* $\operatorname{Ker}\alpha_1$ *projective R_1-bimodule, and β is a monomorphism with* $\operatorname{Coker}\beta_1$ *projective S_1-bimodule.*

7. A remark on symmetric algebras

We return to our original situation: $R = e\mathcal{O}\tilde{X}$ and $S = f\mathcal{O}\tilde{Y}$, where $e \in \mathcal{O}X$ and $f \in \mathcal{O}Y$ are G-invariant block idempotents. Then R is a symmetric crossed product algebra with symmetrizing form $\lambda\colon R \to \mathcal{O}$ (where $\lambda(\sum_{\alpha\in\tilde{X}} \alpha_x x) = \alpha_1$), and the restriction $\lambda_1\colon R_1 \to \mathcal{O}$ of λ to R_1 is a symmetrizing form for R_1. Moreover, for all $r \in R_1$ and $g \in G$, we have that $\lambda_1(\bar{g}r\bar{g}^{-1}) = \lambda_1(r)$. Similar statements hold for S too, where we denote by $\mu\colon S \to \mathcal{O}$ the symmetrizing form. The following statements are true under these more general assumptions.

Let M be an (R_1, S_1)-bimodule. Then its $\mathcal{O}$-dual M^* is an (S_1, R_1)-bimodule, and $\operatorname{Hom}_{\mathcal{O}}(R\otimes_{R_1} M, \mathcal{O}) \simeq \operatorname{Hom}_R(R\otimes_{R_1} M, R) \simeq M^*\otimes_{R_1} R$ as G-graded (S, R)-bimodules.

Let (C, d) be a bounded complex of (R_1, S_1)-bimodules. Then C^* is a complex of (S_1, R_1)-bimodules, naturally isomorphic to $\operatorname{Hom}_{R_1}(C, R_1)$, and $(R \otimes_{R_1} C)^* \simeq \operatorname{Hom}_R(R \otimes_{R_1} C, R) \simeq C^* \otimes_{R_1} R$ as complexes of G-graded (S, R)-bimodules.

8. Local structure

The *Frobenius category* $\mathfrak{Fr}_p(G)$ of a finite group G has as objects the p-subgroups of G, and the morphisms between two p-subgroups are those group-homomorphisms which are induced by inner automorphisms of G. Then two finite groups G and H are said to *have the same p-local structure* if they have a common Sylow p-subgroup D, and the embedding of D in G and H induce an equivalence of the categories $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(H)$. This means that:

(8.1) For any subgroups P, Q of D and any isomorphism $\phi\colon P \to Q$, there is an element $g \in G$ such that $\phi(x) = gxg^{-1}$, for $x \in P$, if and only if there is an element $h \in H$ such $\phi(x) = hxh^{-1}$ for $x \in P$.

It is well-known that in the situation of Broué's conjecture (when D is abelian and $H = N_G(D)$), the groups G and H do have the same p-local structure.

Let $e \in Z(\mathcal{O}G)$ and $f \in Z(\mathcal{O}H)$ be the principal block idempotents. By a *Rickard equivalence* between $\mathcal{O}Ge$ and $\mathcal{O}Hf$ we mean that it is given a bounded complex (C, d) of $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodules such that:

(8.2.a) For each integer i, C^i is projective in $\mathcal{O}Ge$-mod and in mod-$\mathcal{O}Hf$.

(8.2.b) If we denote $C^* = \operatorname{Hom}_{\mathcal{O}}(C, \mathcal{O})$, then there are homotopy equivalences of complexes of bimodules $C \otimes_{\mathcal{O}Hf} C^* \sim \mathcal{O}Ge$ and $C^* \otimes_{\mathcal{O}Ge} C \sim \mathcal{O}Hf$.

(8.2.c) For each integer i, C^i is a p-permutation $\mathcal{O}[G \times H^{op}]$-module with vertex contained in $\delta(D) = \{(x, x^{-1}) \mid x \in P\} \subseteq G \times H^{op}$.

If these conditions hold, then C will be called a *Rickard tilting complex* for $\mathcal{O}Ge$-mod and $\mathcal{O}Hf$-mod. These complexes were called "splendid" in [R3], and their main feature is that they induce derived equivalences between the principal blocks of the local subgroups $C_G(P)$ and $C_H(P)$ for all subgroups P of D. The local tilting complexes are obtained in the following way (see [R3, Sections 4 and 5]):

(8.3.a) Apply to C the Brauer functor

$$\mathrm{Br}_{\delta(P)}^{G\times H^{op}}(-) = (-)(\delta(P)) \colon \mathcal{O}[G \times H^{op}]\text{-mod} \to k[N_{G\times H^{op}}(\delta(P))]\text{-mod}.$$

(Recall that for an $\mathcal{O}G$-module M,

$$\mathrm{Br}_P^G(M) = M(P) = M^P/(\sum_{Q<P} \mathrm{Tr}_Q^P(M^Q) + \mathfrak{p}M^P) = k\otimes_{\mathcal{O}}(M^P/\sum_{Q<P} \mathrm{Tr}_Q^P(M^Q)),$$

where M^P is the set of points fixed by P, and $\mathrm{Tr}_Q^P\colon M^Q \to M^P$ is the trace map.)

(8.3.b) Regard $C(\delta(P))$, by restriction, as a complex of p-permutation $k[C_G(P) \times C_H(P)^{op}]$-modules.

(8.3.c) Finally, lift $C(\delta(P))$ to obtain a complex C_P of p-permutation $\mathcal{O}[C_G(P) \times C_H(P)^{op}]$-modules, which will be a Rickard tilting complex for the principal blocks of $\mathcal{O}[C_G(P)]$ and $\mathcal{O}[C_H(P)]$.

Similarly, the blocks $\mathcal{O}Ge$ and $\mathcal{O}Hf$ are said to be *Puig equivalent* if there is a Morita equivalence between them defined by an $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodule M (and by its $\mathcal{O}$-dual M^*) such that M is a relatively $\delta(D)$-projective p-permutation $\mathcal{O}[G \times H^{op}]$-module.

In particular, if $\mathcal{O}Ge$ and $\mathcal{O}Hf$ are Puig equivalent, then they are Rickard equivalent and moreover, one obtains Morita equivalences between the principal blocks of the local subgroups $C_G(P)$ and $C_H(P)$, where $P \leq D$, by applying to the bimodule M the same algorithm as in (8.3).

We point out that we may assume that $O_{p'}(G) = 1$, where $O_{p'}(G)$ is the maximal normal p'-subgroup of G. Denote $\bar{G} = G/O_{p'}(G)$, and let $\pi\colon G \to \bar{G}$, $\pi(g) = \bar{g}$ be the canonical map. Then D may be viewed as a common Sylow p-subgroup of G and $\bar{G}$. By [FH, Proposition 7.C], π induces an equivalence between $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(\bar{G})$, and it is not difficult to see that the principal blocks of $\mathcal{O}G$ and $\mathcal{O}\bar{G}$ are Puig equivalent. Denote by $\mathcal{O}Ge$ the principal block of $\mathcal{O}G$ and $\mathcal{O}\bar{G}\bar{e}$ the principal block of $\mathcal{O}\bar{G}$.

Therefore, if H is another group having D as a Sylow p-subgroup, $\mathcal{O}Hf$ the principal block of $\mathcal{O}H$, and $\bar{H} = H/O_{p'}(H)$, then G and H have the same p-local structure if and only if $\bar{G}$ and $\bar{H}$ have the same p-local structure; further *if*

$\mathcal{O}\tilde{G}\tilde{e}$ and $\mathcal{O}\tilde{H}\tilde{f}$ are Puig (respectively Rickard equivalent), then $\mathcal{O}Ge$ and $\mathcal{O}Hf$ are Puig (respectively Rickard) equivalent.

We shall refine the conclusions of the preceding sections by showing the relationship between the different local equivalences which are obtained applying the Brauer functor. In view of [R3, Section 5], it is no loss to consider only modules over k. We need to know the behaviour of the Brauer functor with respect to induction.

Lemma 8.4. *Assume that $H \le G$ and that $P \le D$ are p-subgroups of H. Let M be a relatively D-projective p-permutation kH-module. If $C_G(P)T_H(P,D) = T_G(P,D)$, then there is a natural isomorphism of $k[C_G(P)]$-modules*

$$\mathrm{Ind}_{C_H(P)}^{C_G(P)}(\mathrm{Res}_{C_H(P)}^{N_H(P)}(\mathrm{Br}_P^H(M))) \simeq \mathrm{Res}_{C_G(P)}^{N_G(P)}(\mathrm{Br}_P^G(\mathrm{Ind}_H^G(M))).$$

We return now to the original situation, and assume that $e \in Z(kX)$ and $f \in Z(kY)$ are the principal block idempotents. Let $R = ek\tilde{X}$, $R_1 = ekX$, $S = fk\tilde{Y}$, $S_1 = fkY$, $A = R \otimes_k S^{op}$ and $\Delta = A_{\delta(G)}$. We also denote for short $\delta(G) = \{(x, y^{-1}) \in \tilde{X} \times \tilde{Y}^{op} \mid \alpha(xX) = yY\}$.

We shall make the following additional assumptions:

(8.5.a) G is a p'-group.

(8.5.b) D is a common Sylow p-subgroup of X and Y, and $\tilde{X}$, $\tilde{Y}$ have the same p-local structure in a way compatible with the isomorphism $\alpha\colon \tilde{X}/X \to \tilde{Y}/Y$, that is, (see(8.1)): for any subgroup P, Q of D and any $\sigma \in \mathrm{Hom}(P,Q)$, there is $x \in \tilde{X}$ such that $\sigma(u) = xux^{-1}$ for $u \in P$ if and only if there is $y \in \tilde{Y}$ such that $\sigma(u) = yuy^{-1}$ for $u \in P$, and x, y are related by $\alpha(xX) = yY$.

(8.5.c) For any subgroup P of D, $\alpha(C_{\tilde{X}}(P)X/X) = C_{\tilde{Y}}(P)Y/Y$.

If P is a subgroup of D, we denote $X_P = C_X(P), Y_P = C_Y(P), \tilde{X}_P = C_{\tilde{X}}(P)$, $\tilde{Y}_P = C_{\tilde{Y}}(P)$. Then $e_P = \mathrm{Br}_P^X(e) \in Z(kX_P)$ and $f_P = \mathrm{Br}_P^Y(f) \in Z(kY_P)$ are the principal block idempotents. The condition (8.5.c) imply that α induces the isomorphism $\alpha_P\colon \tilde{X}_P/X_P \to \tilde{Y}_P/Y_P$, $\alpha_P(xX) \cap \tilde{Y}_P$ for $x \in \tilde{X}_P$. Denote by G_P these two isomorphic groups, and consider the fully G_P-graded k-algebras $R^P = e_Pk\tilde{X}_P$ and $S^P = f_Pk\tilde{Y}_P$ with $R_1^P = e_PkX_P$ and $S_1^P = f_PkY_P$. Then $A^P = R^P \otimes_k S^{P\,op}$ is a fully $G_P \times G_P^{op}$-graded k-algebra, and let $\Delta^P = A^P_{\delta(G_P)}$, where we also identify $G_P = \{(x, y^{-1}) \in \tilde{X}_P \times \tilde{Y}_P \mid \alpha_P(xX_P) = yY_P\}$. Denote finally $\delta\colon D \to X \times Y^{op}$, $\delta(u) = (u, u^{-1})$, and notice that $C_{X\times Y^{op}}(\delta(P)) = X_P \times Y_P^{op}$, $C_{\delta(G)}(\delta(P)) = \delta(G_P)$ and $C_{\tilde{X}\times\tilde{Y}^{op}}(\delta(P)) = \tilde{X}_P \times \tilde{Y}_P^{op}$.

Corollary 8.6. *With the above notations, assume that conditions (8.5) hold.*

a) Let M be a relatively $\delta(D)$-projective p-permutation (R_1, S_1)-bimodule such that (M, M^*) defines a Puig equivalence between R_1 and S_1. Assume that M

extends to a Δ-module ${}_{\Delta}M = M$. Then $\tilde{M} = A \otimes_{\Delta} M$ is a relatively $\delta(D)$-projective p-permutation A-module and $(\tilde{M}, \tilde{M}^*)$ defines a Puig equivalence between R and S. Moreover, the modules inducing the local equivalences are related in the following way (where $P \leq D$):

$$\mathrm{Br}_{\delta(P)}^{\tilde{X}\times\tilde{Y}^{op}}(\tilde{M}) \simeq A^P \otimes_{\Delta^P} \mathrm{Br}_{\delta(P)}^{X\times Y^{op}}(M) \quad (\text{as } A^P\text{-modules}).$$

b) Let C be a Rickard tilting complex for R_1 and S_1. Assume that C extends to a complex ${}_{\Delta}C = C$ of Δ-modules. Then $\tilde{C} = A \otimes_{\Delta} C$ is a Rickard tilting complex for R and S. Moreover, we have:

$$\mathrm{Br}_{\delta(P)}^{\tilde{X}\times\tilde{Y}^{op}}(\tilde{C}) \simeq A^P \otimes_{\Delta^P} \mathrm{Br}_{\delta(P)}^{X\times Y^{op}}(C) \quad (\text{as complexes of } A^P\text{-modules}).$$

Observe also that $\mathrm{Br}_{\delta(P)}^{\tilde{X}\times\tilde{Y}^{op}}(\tilde{M})$ is actually a G_P-graded (A^P, B^P)-bimodule, and $\mathrm{Br}_{\delta(P)}^{\tilde{X}\times\tilde{Y}^{op}}(\tilde{C})$ is a complex of G_P-graded (A^P, B^P)-bimodules.

9. Equivalences for wreath products

Let $\Omega = \{1, \dots, n\}$ and Σ a subgroup of the symmetric group $S(\Omega)$. We will show in this section that equivalences between two blocks of $\mathcal{O}X$ and $\mathcal{O}Y$ lift to an equivalence between blocks of $\mathcal{O}[X \wr \Sigma]$ and $\mathcal{O}[Y \wr \Sigma]$.

If M is an ($\mathcal{O}$-free) $\mathcal{O}$-module, denote $M^{\otimes\Omega} = M \otimes \cdots \otimes M$ (n times), and if (C, d) is a (bounded) complex of $\mathcal{O}$-modules, let again $C^{\otimes\Omega} = C \otimes \dots \otimes C$ (n times) with the differential defined by:

$$d(c_{i_1} \otimes \dots \otimes c_{i_n}) = \sum_{l=1}^{n} (-1)^{i_1+\cdots+i_{l-1}} c_{i_1} \otimes \dots \otimes dc_{i_l} \otimes \dots \otimes c_{i_n},$$

where c_{i_l} belongs to the l-th factor C of $C^{\otimes\Omega}$ and has degree $i_l \in \mathbb{Z}$. We shall denote for short $c_{i_1} \otimes \dots \otimes c_{i_n} = c_{i_1} \dots c_{i_n} \in (C^{\otimes\Omega})^{i_1+\cdots+i_n}$.

The next lemma shows that $S(\Omega)$ acts naturally on $M^{\otimes\Omega}$ and on $C^{\otimes\Omega}$.

Lemma 9.1. *a) Let M be an $\mathcal{O}$-module. Then $M^{\otimes\Omega}$ becomes a left $\mathcal{O}[S(\Omega)]$-module by defining, for $\sigma \in S(\Omega)$ and $m = m_1 \otimes \dots \otimes m_n \in M^{\otimes\Omega}$,*

$$\sigma \cdot m = m_{\sigma^{-1}(1)} \otimes \dots \otimes m_{\sigma^{-1}(n)}.$$

b) Let (C, d) be a complex of $\mathcal{O}$-modules. Then there is a function $\epsilon\colon S(\Omega) \times \mathbb{Z}_2^{\Omega} \to \mathbb{Z}_2$ such that by defining:

$$\sigma \cdot (c_{i_1} \dots c_{i_n}) = (-1)^{\epsilon_\sigma(i_1,\dots,i_n)} c_{i_{\sigma^{-1}(1)}} \dots c_{i_{\sigma^{-1}(n)}},$$

$C^{\otimes\Omega}$ becomes a complex of $\mathcal{O}[S(\Omega)]$-modules.

For $m = m_1 \otimes \dots \otimes m_n \in M^{\otimes\Omega}$, $c = c_{i_1} \otimes \dots \otimes c_{i_n} \in C^{\otimes\Omega}$ and $\sigma \in \Sigma$, we shall also denote ${}^{\sigma}m = \sigma \cdot m$, ${}^{\sigma}c = \sigma \cdot c$, $m^{\sigma} = \sigma^{-1} \cdot m$ and $c^{\sigma} = \sigma^{-1} \cdot c$. If X is a group and A is an $\mathcal{O}$-algebra such that are given group-homomorphisms

$\Sigma \to \operatorname{Aut} X$ and $\Sigma \to \operatorname{Aut} A$, then we denote by $X \rtimes \Sigma = \{x\sigma \mid x \in X,\ \sigma \in \Sigma\}$, $A * \Sigma = \{a\sigma \mid a \in A,\ \sigma \in \Sigma\}$ with multiplications $(x\sigma)(y\tau) = x \cdot {}^{\sigma}y \cdot \sigma\tau$ in $X \rtimes \Sigma$, and $(a\sigma)(b\tau) = a \cdot {}^{\sigma}b \cdot \sigma\tau$ in $A * \Sigma$.

We return now to our basic situation. Let X, Y be two groups and $e \in Z(\mathcal{O}X)$, $f \in Z(\mathcal{O}Y)$ be block idempotents. Then $\tilde{e} = e^{\otimes\Omega} = e \otimes \cdots \otimes e \in (\mathcal{O}X)^{\otimes\Omega} \simeq \mathcal{O}[X^{\Omega}]$ and $\tilde{f} = f^{\otimes\Omega} \in (\mathcal{O}Y)^{\otimes\Omega} \simeq \mathcal{O}[Y^{\Omega}]$ are Σ-invariant block idempotents. Denote $R_1 = \tilde{e}\mathcal{O}[X^{\Omega}] \simeq (e\mathcal{O}X)^{\otimes\Omega}$, $S_1 = \tilde{f}\mathcal{O}[Y^{\Omega}] \simeq (f\mathcal{O}Y)^{\otimes\Omega}$, and let $R = R_1 * \Sigma \simeq \tilde{e}\mathcal{O}[X \wr \Sigma]$ and $S = S_1 * \Sigma \simeq \tilde{f}\mathcal{O}[Y \wr \Sigma]$. Then R is a fully Σ-graded $\mathcal{O}$-algebra, and its elements are of the form $r\sigma = (r_1 \otimes \ldots \otimes r_n)\sigma$, where $r_i \in e\mathcal{O}X$ and the multiplication is given by

$$(r\sigma)(r'\sigma') = r \cdot {}^{\sigma}r' \cdot \sigma\sigma' = (r_1 r'_{\sigma^{-1}(1)} \otimes \ldots r_n r'_{\sigma^{-1}(n)})\sigma\sigma',$$

and similar statements hold for S.

We may apply the results of the preceding section with $\tilde{e}$, $\tilde{f}$ in the place of e, f, with X^{Ω}, Y^{Ω} in place of X, Y, with $X \wr \Sigma$, $Y \wr \Sigma$ in place of $\tilde{X}$, $\tilde{Y}$ and with Σ in place of G. Since Σ is naturally isomorphic to $(X \wr \Sigma)/X^{\Omega}$ and to $(Y \wr \Sigma)/Y^{\Omega}$, the isomorphism α may be regarded to be just the identity. Denote again $A = R \otimes_{\mathcal{O}} S^{op}$ and $\Delta = A_{\delta(\Sigma)}$, where $\delta(\Sigma) = \{(\sigma, \sigma^{-1}) \mid \sigma \in \Sigma\} \subseteq \Sigma \times \Sigma^{op}$. There are isomorphisms $A \simeq (R_1 \otimes_{\mathcal{O}} S_1^{op}) * (\Sigma \times \Sigma^{op})$ and $\Delta \simeq (R_1 \otimes_{\mathcal{O}} S_1^{op}) * \delta(\Sigma) \simeq \Delta_1 * \Sigma$, so we may write $\Delta = \{(r \otimes s)\sigma \mid r \in R_1,\ s \in S_1^{op},\ \sigma \in \Sigma\}$, with multiplication defined by

$$(r \otimes s)\sigma \cdot (r' \otimes s')\sigma' = (r \cdot {}^{\sigma}r' \otimes {}^{\sigma}s' \cdot s)\sigma\sigma'.$$

Let now M be an $(e\mathcal{O}X, f\mathcal{O}Y)$-bimodule. Then $M^{\otimes\Omega}$ is an $R_1 \otimes_{\mathcal{O}} S_1^{op}$-module. The point is that due to Lemma 9.1, $M^{\otimes\Omega}$ extends to a Δ-module by defining

$$(r \otimes s)\sigma \cdot m = (r \otimes s) \cdot {}^{\sigma}m = r_1 m_{\sigma^{-1}(1)} s_1 \otimes \ldots \otimes r_n m_{\sigma^{-1}(n)} s_n,$$

hence we may talk about the (R, S)-bimodule $M \wr \Sigma = A \otimes_{\Delta} M$. As we have seen in Section 3, $M \wr \Sigma$ is naturally isomorphic to $R \otimes_{R_1} M$ and to $M \otimes_{S_1} S$.

If C is a complex of $(e\mathcal{O}X, f\mathcal{O}Y)$-bimodules, we define similarly, using Lemma 9.1, the complex of (R, S)-bimodules $C \wr \Sigma = A \otimes_{\Delta} C \simeq \mathcal{O}\Sigma \otimes_{\mathcal{O}} C \simeq C \otimes_{\mathcal{O}} \mathcal{O}\Sigma$.

The following theorem is just a particular case of the previous results.

Theorem 9.2. *a) Assume that (M, M^*) defines a Morita equivalence between $e\mathcal{O}X$ and $f\mathcal{O}Y$. Then $(M \wr \Sigma, (M \wr \Sigma)^*)$ defines a Morita equivalence between R and S, inducing Morita equivalences between the corresponding blocks of $\mathcal{O}[X \wr \Sigma]$ and $\mathcal{O}[Y \wr \Sigma]$ covering $\tilde{e}$ and $\tilde{f}$.*

b) Assume that Σ is a p'-group and that (C, C^*) defines a derived equivalence between $e\mathcal{O}X$ and $f\mathcal{O}Y$. Then $(C \wr \Sigma, (C \wr \Sigma)^*)$ defines a derived equivalence between R and S, inducing derived equivalences between the corresponding blocks of $\mathcal{O}[X \wr \Sigma]$ and $\mathcal{O}[Y \wr \Sigma]$ covering $\tilde{e}$ and $\tilde{f}$.

Assume in addition that: Σ is a p'-group, $e\mathcal{O}X$ and $f\mathcal{O}Y$ are the principal blocks, D is a common Sylow p-subgroup of X and Y and X, Y have the same p-local structure, M is a relatively $\delta(D)$-projective p-permutation $\mathcal{O}[X \times Y^{op}]$-module and C is a bounded complex of relatively $\delta(D)$-projective p-permutation $\mathcal{O}[X \times Y^{op}]$-modules. Then, in Theorem 9.2, the expression "*Morita equivalence*" can be replaced by "*Puig equivalence*" and "*derived equivalence*" by "*Rickard equivalence*". Indeed, it is easy to verify that the following statements are true: $e^{\otimes\Omega} \in (\mathcal{O}X)^{\otimes\Omega}$ and $f^{\otimes\Omega} \in (\mathcal{O}Y)^{\otimes\Omega}$ are the principal block idempotents; D^Ω is a common Sylow p-subgroup of X^Ω and Y^Ω and X^Ω, Y^Ω, respectively $X \wr \Sigma$, $Y \wr \Sigma$ have the same p-local structure as required in (8.5.b); for any subgroup P of D^Ω, the obvious isomorphism $\alpha\colon X \wr \Sigma/X^\Omega \to Y \wr \Sigma/T^\Omega$ has the property $\alpha(C_{X\wr\Sigma}(P)X^\Omega/X^\Omega) = C_{Y\wr\Sigma}(P)Y^\Omega/Y^\Omega$ (as required in (8.5.c)); $M^{\otimes\Omega}$ and $C^{\otimes\Omega}$ are relatively $\delta(D^\Omega)$-projective and p-permutation over $k[X^\Omega \times Y^{\Omega\, op}]$, and apply Corollary 8.6.

10. Conclusions and examples

In this section we put the things together in order to obtain equivalences between blocks in the situation presented in Section 2.

For $0 \le i \le s$ let X_i be a normal subgroup of $\tilde{X}_i$ and Y_i a normal subgroup of $\tilde{Y}_i$, such that there is an isomorphism $\alpha_i\colon \tilde{X}_i/X_i \to \tilde{Y}_i/Y_i$. We identify these two factor groups and we denote them by G_i. Let further $\Omega_0, \ldots, \Omega_s$ be disjoint finite sets, and let $\Sigma_i \le S(\Omega_i)$. Then α_i induces the isomorphism $\alpha_i^{\Omega_i}\colon \tilde{X}_i^{\Omega_i}/X_i^{\Omega_i} \to \tilde{Y}_i^{\Omega_i}/Y_i^{\Omega_i}$, and these factor groups may be identified with $G_i^{\Omega_i}$; α_i also induces the isomorphism $\alpha_i^{\wr\Sigma_i}\colon \tilde{X}_i \wr \Sigma_i/X_i^{\Omega_i} \to \tilde{Y} \wr \Sigma_i/Y_i^{\Omega_i}$, and we identify these factor groups with $G_i \wr \Sigma_i$. Consider the isomorphism

$$\alpha = \prod_{i=0}^{s} \alpha_i\colon \prod_{i=0}^{s}(\tilde{X}_i \wr \Sigma_i)/\prod_{i=0}^{s} X_i^{\Omega_i} \to \prod_{i=0}^{s}(\tilde{Y}_i \wr \Sigma_i)/\prod_{i=0}^{s} X_i^{\Omega_i},$$

and let $G = \prod_{i=0}^{s} G_i \wr \Sigma_i$. Denote $X = \prod_{i=0}^{s} X_i^{\Omega_i}$, $Y = \prod_{i=0}^{s} Y_i^{\Omega_i}$, $\tilde{X} = \prod_{i=0}^{s} \tilde{X}_i \wr \Sigma_i$, $\tilde{Y} = \prod_{i=0}^{s} \tilde{Y}_i \wr \Sigma_i$, and consider the subgroups $X \le X' \le \tilde{X}$, $Y \le Y' \le \tilde{Y}$ such that $\alpha(X'/X) = Y'/Y$. Finally, identify $X'/X = Y'/Y = G'$ via the restriction of α.

Let $e_i \in \mathcal{O}X_i$ and $f_i \in \mathcal{O}Y_i$ be block idempotents, and denote $R^i = e_i^{\otimes\Omega_i}\mathcal{O}[\tilde{X}_i \wr \Sigma_i]$, $S^i = f_i^{\otimes\Omega_i}\mathcal{O}[\tilde{Y}_i \wr \Sigma_i]$, $R_1 = e\mathcal{O}X$, $S_1 = f\mathcal{O}Y$ (where $e = \otimes_{i=0}^{s} e_i^{\otimes\Omega_i}$ and $f = \otimes_{i=0}^{s} f_i^{\otimes\Omega_i}$), $R = e\mathcal{O}\tilde{X} \simeq \prod_{i=0}^{s} R^i$ and $S = f\mathcal{O}\tilde{Y} \simeq \prod_{i=0}^{s} S^i$ (where the isomorphisms are of G-graded rings).

Furthermore, let $A = R \otimes_{\mathcal{O}} S^{op} \simeq \prod_{i=0}^{s} A^i$ (where $A^i = R^i \otimes_{\mathcal{O}} S^{i\ op}$), and let $\Delta = A_{\delta(G)} \simeq \prod_{i=0}^{s}(\Delta^i)^{\otimes\Omega}$ (where $\Delta^i = (e_i\mathcal{O}\tilde{X}_i \otimes_{\mathcal{O}} f_i\mathcal{O}\tilde{Y}_i^{op})_{\delta(G_i)}$). We shall also consider the G'-components $R' = R_{G'} = \bigoplus_{g\in G'} R_g$, $S' = S_{G'}$, $A' = A_{G'\times G'^{op}} \simeq R' \otimes_{\mathcal{O}} S'^{op}$ and $\Delta' = A'_{\delta(G')}$.

For $0 \le i \le s$, let M_i be an $(e_i\mathcal{O}X_i, f_i\mathcal{O}Y_i)$-bimodule and $C_i^\cdot$ a (bounded) complex of $(e_i\mathcal{O}X_i, f_i\mathcal{O}Y_i)$-bimodules, and denote $M = \otimes_{i=0}^s M_i^{\otimes\Omega_i}$ and $C^\cdot = \otimes_{i=0}^s C_i^{\cdot\otimes\Omega_i}$.

Theorem 10.1. *Assume that for $0 \le i \le s$, M_i extends to a Δ^i-module and $C_i^\cdot$ extends to a complex of Δ^i-modules.*

a) If for $0 \le i \le s$, (M_i, M_i^*) (respectively $(C_i^\cdot, C_i^{\cdot *})$) defines a Morita (respectively derived) equivalence between $e_i\mathcal{O}X_i$ and $f_i\mathcal{O}Y_i$, then M (respectively C) extends to Δ, and $(A' \otimes_{\Delta'} M, (A' \otimes_{\Delta'} M)^*)$ (respectively $(A' \otimes_{\Delta'} C^\cdot$, $(A \otimes_{\Delta'} C^\cdot)^*)$, provided that G_i and Σ_i are p'-groups) defines a Morita (respectively derived) equivalence between R' and S'.

b) Assume in addition that for each i, the conditions (3.8.a,b,c) are satisfied by $X_i, Y_i, \tilde{X}_i, \tilde{Y}_i, G_i$ and α_i, and that Σ_i is a p'-group. If for $0 \le i \le s$, M_i (respectively $C_i^\cdot$) defines a Puig (respectively Rickard) equivalence between $e_i\mathcal{O}X_i$ and $f_i\mathcal{O}Y_i$, then $A' \otimes_{\Delta'} M$ (respectively $A' \otimes_{\Delta'} C^\cdot$) defines a Puig (respectively Rickard) equivalence between R' and S'.

The hypothesis if Theorem 5.1 has to be verified for each $0 \le i \le s$. We discuss here two situations when this checking is easy.

10.2. With the notations of Section 2, the case $i = 0$ is trivial. Suppose that $X = Y$, $\tilde{X} = \tilde{Y}$, D is a Sylow p-subgroup of X and that $G = \tilde{X}/X$ is a p'-group. Let $e\mathcal{O}X$ be the principal block, and denote $R_1 = S_1 = e\mathcal{O}X$, $R = S = e\mathcal{O}\tilde{X}$, and let $M = e\mathcal{O}X$ regarded as an R_1, S_1-bimodule. Then M is in fact a Δ-module simply because X is normal in $\tilde{X}$, and it is clearly a relatively $\delta(D)$-projective p-permutation Δ -module. It follows that M defines a Puig autoequivalence of $e\mathcal{O}X$ and $\tilde{M} = A \otimes_\Delta M \simeq R \otimes_{R_1} M \simeq R$ defines a Puig autoequivalence of $e\mathcal{O}\tilde{X}$.

10.3. Another nice situation which may occur for $1 \le i \le s$ is that of *isomorphic blocks*. Let X be a normal subgroup of $\tilde{X}$ such that $\tilde{X} = C_{\tilde{X}}(D)X$ where D is a Sylow p-subgroup of X and $G = \tilde{X}/X$ is a p'-group. Let e and Then the principal blocks $e\mathcal{O}X$ and $\tilde{e}\mathcal{O}\tilde{X}$ are Puig equivalent.

Let now Y be a normal subgroup of another group $\tilde{Y}$ such that D is also a Sylow p-subgroup of Y, $\tilde{Y} = C_{\tilde{Y}}(D)Y$ and $\tilde{X}/X \simeq \tilde{Y}/Y$, and let f, $\tilde{f}$ be the principal block idempotents of $\mathcal{O}Y$ and $\mathcal{O}\tilde{Y}$, respectively. It follows that if $e\mathcal{O}X$ and $f\mathcal{O}Y$ are Puig (respectively Rickard) equivalent, then $\tilde{e}\mathcal{O}\tilde{X}$ and $\tilde{f}\mathcal{O}\tilde{Y}$ are Puig (respectively Rickard) equivalent.

Remark that if X, $\tilde{X}$ and D are as above, and $\tilde{Y} = N_{\tilde{X}}(D)$, $Y = N_X(D)$, then $\tilde{Y} = C_{\tilde{Y}}(D)Y$, so these observations apply.

10.4. Examples a) *The principal* 3-*block of* A_5. Although this example fits into the situation of [P] (see also [Br2, p.14]), we shall deal with permutations instead of regarding $A_5 \simeq \mathrm{SL}_2(4)$. So let $X = A_5$, $D = \langle(1\ 2\ 3)\rangle$ a Sylow

3-subgroup of X, $Y = N_X(D) = D \rtimes \langle y \rangle$, where $y = (1\ 2)\sigma$, $\sigma = (4\ 5)$, and let $U = \{e, (1\ 2)(3\ 4), (1\ 4)(2\ 3), (1\ 3)(4\ 2)\}$ be a Sylow 2-subgroup of X. If $M = k[X/U]$ is the k-space with basis X/U, then M is a relatively $\delta(D)$-projective 3-permutation $k[X \times Y^{op}]$-module, and defines a Puig equivalence between the principal 3-blocks of kX and kY.

Let now $\tilde{X} = S_5 = X \rtimes G$, where $G = \langle g \rangle$, $g = (1\ 2)$, and $\tilde{Y} = N_{\tilde{X}}(D)$. We see that M extends to a Δ-module by defining an action of G on M: $g \cdot xU = x^g U$, which is correct, since g normalizes U. Consequently, Corollary 8.6 applies.

b) *Symmetric groups.* This example was considered in [Rou1]. Let X be the symmetric group S_p, and let $w < p$ so the wreath-product $X \wr S_w$ is a subgroup of S_{pw}. Let D be a (cyclic) Sylow p-subgroup of X and let e (respectively f) be the principal block idempotents of $\mathcal{O}X$ (respectively $\mathcal{O}Y$, where $Y = N_X(D)$). Then by [Rou2, Theorem 10] and the results of Section 9 above, we deduce that $e^{\otimes w}\mathcal{O}[X \wr S_w]$ and $f^{\otimes w}\mathcal{O}[Y \wr S_w]$ are Rickard equivalent.

c) In [Rou2], several examples of derived equivalences between the principal blocks e and f of the groups X and $Y = N_X(D)$ respectively (where D is an abelian Sylow p-subgroup of X) coming from stable equivalences between $e\mathcal{O}X$ and $f\mathcal{O}Y$ were constructed. All of them satisfy the liftability conditions. Let $R = \mathcal{O}\tilde{X}$ and $R = f\mathcal{O}\tilde{Y}$, where $\tilde{Y} = N_{\tilde{X}}(D)$ and assume that $G = \tilde{X}/X \simeq \tilde{Y}/Y$ is a p'-group. Suppose that $e\mathcal{O}Xf$ induces a stable equivalence of Morita type between R_1 and S_1. By [Li, Theorem 2.1], $e\mathcal{O}Xf$ has, up to isomorphism, a unique indecomposable nonprojective R_1, S_1-direct summand. It is obvious that $e\mathcal{O}Xf$ extends to a Δ-module, and M also extends to Δ. Let $\pi\colon P \to M$ be a projective cover in Δ-mod of M. Since G is a p'-group, we have that $J(\Delta) = J(\Delta_1)\Delta = \Delta J(\Delta_1)$, hence P is a projective cover of M in Δ_1-mod too. Under suitable hypotheses, a tilting complex

$$C = (0 \to P' \xrightarrow{\pi'} M \to 0)$$

for R_1 and S_1 was constructed in [Rou2], where P' is a Δ_1-direct summand of P, with the property that P' and P/P' have no common direct summands, and π' is the restriction of π to P'. Then P' and P/P' extend to Δ such that $P \simeq P' \oplus P/P'$ in Δ-mod. Consequently, C extends to a complex of Δ-modules and $A \otimes_\Delta C$ is a tilting complex for R and S.

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CROSSED COPRODUCTS

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ABSTRACT. This is an expository paper. We provide a motivation for the study of crossed coproducts by discussing the connection between the algebra and coalgebra structures of a Hopf algebra with an eye to its representation theory.

It happens frequently that meaningful mathematical objects are "marriages" between two notions or concepts, each of them having a substantial contribution to the success of the couple.

Let us look, as an example from commutative algebra, at Cohen-Macaulay rings. They are defined, roughly speaking, by an equality between *height* and *depth*. *Depth*'s bad behaviour under localization is "repaired" by its being equal to *height*. On the other hand, it can happen (in general) that the *heights* of two prime ideals with none between them differ by more than 1. This is not possible in the case of Cohen-Macaulay rings, because the equality of *height* and *depth* doesn't allow it to happen. In conclusion, *height*'s contribution is the good behaviour under localization at a prime ideal, while *depth* brings catenarity.

The purpose of this note is to discuss another example of such a coexistence, namely the one of the algebra and coalgebra structures in a Hopf algebra. For obvious reasons, the algebra structure will play the main part, so we will try to focus on the contribution of the coalgebra structure to a better understanding of the algebra, and in particular of its representation theory.

We try to keep the exposition at a highly non-technical level: there will be few rigorous definitions, no diagrams, and (perhaps the most surprising of all) no "sigma notation" summations.

The interested reader is referred to the expository papers [4] and [25]. Many of the results we are quoting from the original papers may be also found in S. Montgomery's monograph [24].

We begin by recalling a few definitions that may be found in [31] or [24, Ch. 5].

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A vector space C over the field k is called a *coalgebra* provided there exist k-linear maps $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$, making two diagrams commute: the commutativity of the first diagram says that Δ is *coassociative* (which means that after applying it once, it doesn't matter on which of the components one applies it again), while the commutativity of the second one ensures that ε is a *counit* (meaning that after applying it on any component of a $\Delta(c)$ $(c \in C)$, one gets c again (after the standard identifications $C \simeq C \otimes k \simeq k \otimes C$)). A coalgebra is said to be *cocommutative* if $\Delta = T \circ \Delta$, where T is the twist map $T : C \otimes C \longrightarrow C \otimes C$, $T(c \otimes d) = d \otimes c$. A *subcoalgebra* of a coalgebra C is a k-subspace C' of C such that $\Delta(C') \subseteq C' \otimes C'$. C is *simple* if it has no proper subcoalgebras, and *irreducible* if it has a unique simple subcoalgebra. If C is a coalgebra, then the set of *group-like* elements of C is defined by

$$G(C) = \{g \in C \mid g \neq 0 \text{ and } \Delta(g) = g \otimes g\}.$$

If $g \in G(C)$, then $\varepsilon(g) = 1$, and a simple subcoalgebra of C is one-dimensional $\Leftrightarrow$ it is equal to kg for some $g \in G(C)$. A coalgebra is said to be *pointed* if all its simple subcoalgebras are one-dimensional.
If $g, h \in G(C)$, then the set of g, h – *primitive* elements of C is defined by

$$P_{g,h} = \{c \in C \mid \Delta(c) = c \otimes g + h \otimes c\}.$$

If $c \in P_{g,h}$, then $\varepsilon(c) = 0$. The 1,1-primitive elements are called simply *primitive*, the other ones are called *skew-primitive*.
For each pair $g, h \in G(C)$, let $P'_{g,h}$ be a subspace of $P_{g,h}$ such that

$$P_{g,h} = k(g - h) \oplus P'_{g,h}.$$

The *coradical* C_0 of the coalgebra C is the sum of all simple subcoalgebras of C. If C is pointed, then $C_0 = kG(C)$. The coradical is the bottom piece of a filtration, called the *coradical filtration* of the coalgebra C, which is defined inductively as follows: for each $n \geq 1$ put

$$C_n = C_{n-1} \wedge C_0 := \Delta^{-1}(C_{n-1} \otimes C + C \otimes C_0).$$

The following important result tells us more about the coradical filtration of a pointed coalgebra. Roughly, it says that in this case the quotients of the coradical filtration consist of skew-primitives.

Theorem 1.2. *(Taft-Wilson,* [33] *or* [24, 5.4.1]*) Let C be a pointed coalgebra, with $G = G(C)$. Then*
1) $C_1 = kG \oplus (\oplus_{g,h \in G} P'_{g,h})$
2) for any $n \geq 1$ and $c \in C_n$,

$$c = \sum_{g,h \in G} c_{g,h}, \quad \textit{where } \Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w,$$

for some $w \in C_{n-1} \otimes C_{n-1}$.

Let $\mathcal{S}$ denote the set of simple subcoalgebras of a coalgebra C.

Definition 1.3. *The quiver Γ_C is the oriented graph whose vertices are the elements of $\mathcal{S}$; there exists an edge $S_1 \to S_2$ $(S_i \in \mathcal{S}) \Leftrightarrow S_2 \wedge S_1 \neq S_1 + S_2$. C is called* link-indecomposable *(L.I.) if Γ_C is connected. When C is pointed we write $x \to y$ $(x, y \in G(C))$ if $kx \to ky$. By Theorem 1.2 we have that $x \to y \Leftrightarrow$ there exists a non-trivial x,y-primitive element.*

Remark 1.4. *It was proved in* [26] *that Γ_C is isomorphic (as a directed graph) to the Ext quiver of simple (right) C-comodules.*

The following dual to a classical theorem of Brauer was also proved in [26].

Theorem 1.5. *Any coalgebra C decomposes as a direct sum of its link-indecomposable components (which are the maximal subcoalgebras with respect to their quiver being connected).*

Example 1.6. *If $C = U(L)$, then $\mathcal{S} = \{k1\}$, $P_{1,1} = L \neq 0$, so Γ_C is a loop (one vertex, 1, and an arrow from it to itself).*

Example 1.7. *Let $H = U_q(sl(2))$ for q not a root of 1, as described by Drinfeld and Jimbo (see* [15]*). As an algebra, it is*

$$k\langle E, F, K, K^{-1} \mid KE = q^2 EK, KF = q^{-2} FK, EF - FE = \frac{K^2 - K^{-2}}{q^2 - q^{-2}}\rangle.$$

Its coalgebra structure is given by $K \in G(H)$ and $E, F \in P_{K^{-1},K}$. It is known that H is pointed, with $G(H) = \langle K \rangle$, that the skew-primitives are in the $kG(H)$-module spanned by $1, E$, and F and that for all n, EK^n and $FK^n \in P_{K^{n-1},K^{n+1}}$. Consequently, Γ_H consists of two connected components ($U_q(sl(2))$ is not link-indecomposable):

$$\begin{array}{ccccccccccc} \cdots & \to & K^{-3} & \to & K^{-1} & \to & K & \to & K^3 & \to & \cdots \\ \cdots & \to & K^{-2} & \to & 1 & \to & K^2 & \to & K^4 & \to & \cdots \end{array}$$

A k-algebra A is defined by reversing the arrows in the diagrams defining a coalgebra[1]: we have thus k-linear maps $M : A \otimes A \longrightarrow A$ (multiplication), and $u : k \longrightarrow A$ (unit). A k-vector space H which is in the same time an algebra with multiplication M and unit u, and a coalgebra with comultiplication Δ and counit ε is a *bialgebra* if M and u are coalgebra maps, or, equivalently, Δ and ε are algebra maps (the definitions of coalgebra maps and tensor products of coalgebras being obvious; also note that k itself is a coalgebra and an algebra with all maps equal to the identity). A bialgebra H with an *antipode* S (which is a two-sided inverse of the identity of H under an operation on $Hom_k(H, H)$

[1]This is an old coalgebraists joke which I quote from a W. Nichols and M. Sweedler paper, which is "a revision by the second author to notes taken by the first author at lectures given by the second author".

(named convolution) determined by the algebra and coalgebra structures of H) is called a *Hopf algebra.*

Let us look now at the definitions of representations of groups and Lie algebras. Let G be a finite group. A representation of G is a group morphism

$$G \longrightarrow GL(V),$$

where V is a k-vector space, or, equivalently, a module over the group ring kG. A representation of the Lie algebra L is a Lie map

$$L \longrightarrow gl(V),$$

or, equivalently, a module over the universal enveloping algebra $U(L)$.
Both kG and $U(L)$ have additional structures. Our main goal is now to see how these structures help to study the representations. Let us see what these additional structures are. First, both of them are coalgebras, and even Hopf algebras. The coalgebra structures are given as follows:
- In the case of kG we let the elements of G be group-like, i.e. we put $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ for all $g \in G$. The antipode S is defined by $S(g) = g^{-1}$. Now extend all the above functions by linearity.
- In the case of $U(L)$ we make all elements x of L primitive, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$. The antipode is given by $S(x) = -x$. Now extend the above functions first on a PBW basis of $U(L)$ by asking Δ and ε to be multiplicative and S to be anti-multiplicative, and then, by linearity, to the whole of $U(L)$.

Remark 1.8. *Both Hopf algebras described above are cocommutative, but since we will be interested also in dealing with some quantum groups, this property is not relevant for us. What will be really significant in the sequel is that they are both pointed. Note that any cocommutative coalgebra becomes pointed if we replace the base field by an algebraic closure of it.*

The first example (we shall look at more important ones later) of how the coalgebra structures described above are being helpful is the definition of the tensor product of representations:

Example 1.9. - *In the case of kG we let V and W be kG-modules and we define a kG-module structure on $V \otimes W$ by*

$$g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$$

for all $g \in G$, $v \in V$, $w \in W$.
- *In the case of $U(L)$ we let V and W be $U(L)$-modules and we define a $U(L)$-module structure on $V \otimes W$ by*

$$x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$$

for all $x \in L$, $v \in V$, $w \in W$.
Hence in both cases the correct module structure on $V \otimes W$ *is obtained by applying* Δ *of an element componentwise on a* $v \otimes w \in V \otimes W$.

Let us see now what other structures do kG and $U(L)$ have. We begin by $U(L)$.

Remark 1.10. *$U(L)$ is a filtered algebra, the filtration being the obvious filtration by degree. When k has characteristic 0, this filtration is exactly the coradical filtration of $U(L)$.*

We look next at kG, which is a G-graded k-algebra, so we can speak of graded representations, or G-graded kG-modules. It is known from [6] that the category of G-graded kG-modules is isomorphic to the category of modules over the smash product $kG\#kG^*$.

Remark 1.11. *Connecting graded and ungraded properties of a graded representation is the same as studying the transfer of module properties between kG and $kG\#kG^*$. This is similar to the classical problem of studying the transfer of module properties between kH and kG, where H is a normal subgroup of G (the case when H is not normal has been systematically studied in* [8]*, then more generally in* [27] *and* [28] *and also admits a general Hopf algebraic point of view according to* [13]*).*

In order to explain why these problems are similar we need some more definitions.

Let A be an algebra and M a left (or right) A-module. We can write this by defining a "multiplication by scalars" map from $A \otimes M$ (or $M \otimes A$) to M which makes two diagrams commute. We can then reverse the arrows in these diagrams to obtain the definition of a comodule over a coalgebra. The category of comodules over a coalgebra is also a Grothendieck category [29], the comodule maps being also defined by dualizing the definition of module maps. An algebra A which is also a left module over the Hopf algebra H is said to be a *left H-module algebra* if the multiplication of A is H-linear (the H-module structure on $A \otimes A$ is defined like in Example 1.9, i.e. via the comultiplication of H). If A is a left H-module algebra, then we can form the *smash product* (or *semidirect product*) $A\#H$, and also define the subalgebra *of invariants* of A, denoted by A^H [31]. Right module algebras are defined in a similar manner.

Example 1.12. *If G is a finite group and A is an algebra, then G acts on A as automorphisms $\Leftrightarrow$ A is a left kG-module algebra. The subalgebra of invariants in this case is the usual invariant subalgebra of A, while the smash product is exactly the skew group-ring of G over A.*

Example 1.13. *If L is a Lie algebra and A is an algebra on which L acts as derivations, then A is a left $U(L)$-module algebra.*

If an algebra A is also a right comodule over the Hopf algebra H, then we call it a *right H-comodule algebra* if the multiplication of A is a comodule map, or, equivalently, if the comodule structure map of A is an algebra map. For a right H-comodule algebra A we can define its subalgebra *of coinvariants* of A, denoted by A^{coH}. If the Hopf algebra H is finite dimensional, then A is a right H-comodule algebra $\Leftrightarrow$ A is a left H^*-module algebra, and in this case we have $A^{coH} = A^{H^*}$.

Example 1.14. *If G is a group, then the algebra A is graded by G $\Leftrightarrow$ A is a right kG-comodule algebra. The subalgebra of coinvariants in this case is just the homogeneous part of A of degree the unit of G. If G is finite, we can form in this case the smash product $A\#kG^*$ as in* [6].

If A is a right H-comodule algebra and with comodule structure map

$$\rho_A : A \longrightarrow A \otimes H,$$

and we denote $B := A^{coH} = \{a \in A \mid \rho_A(a) = a \otimes 1\}$, then we say that $B \subset A$ is a *faithfully flat H-Galois extension* if the map

$$A \otimes_B A \longrightarrow A \otimes H, \quad a \otimes b \mapsto a\rho_A(b)$$

is bijective (i.e. the extension is Galois, see [17]) and A is faithfully flat as a right B-module.

Example 1.15. *A G-graded ring is a kG-Galois extension of its homogeneous part of degree unit of G $\Leftrightarrow$ A is strongly graded (see* [35] *or* [24]*). In this case it is a faithfully flat kG-Galois extension.*

Example 1.16. *G is a group, H a normal subgroup, G/H the quotient group and $B := kH \subset A := kG$, $H := k(G/H)$.*

Example 1.17. *L is a (p-)Lie algebra, $L' \subset L$ is a (p-)Lie ideal, L/L' the quotient Lie algebra and $B := U(L') \subset A := U(L)$, $H := U(L/L')$ are the enveloping (p-enveloping) algebras.*

Example 1.18. *B is any algebra, H any Hopf algebra with bijective antipode acting (weakly) on B, $\sigma : H \otimes H \longrightarrow B$ is an invertible 2-cocycle, and $A := B\#_\sigma H$ is the* crossed product *(cf.* [2], [14]*). This example generalizes smash products (hence skew group rings) and twisted group rings (hence modules over a crossed product generalize projective representations).*

Remark 1.19. *Examples 1.16 and 1.18 make now clear that "similar" in Remark 1.11 means that both extensions considered there are faithfully flat Galois extensions.*

A study of the representation theory of faithfully flat Hopf Galois extensions is performed by H.-J. Schneider in [30]. Several classical results from representation theory are obtained in [30] as corollaries: Green's indecomposability theorem for groups, a version of Blattner's simplicity theorem for induction from ideals of Lie algebras, and work of Dade on strongly graded algebras.

Example 1.18 will be of particular importance in the sequel. It arises as a natural generalization of group crossed products, which are in fact group extensions as described by Schreier in 1926. The Hopf algebraic definition was given in [32] for the particular case of cocommutative Hopf algebras acting on commutative algebras, and then independently in [2] and [14] for the general case.

We are now in a position to give our examples of how a condition on the coalgebra structure can influence the algebra structure. The first such example was proved in [7] using Theorem 1.2.

Theorem 1.20. *Let H be a finite dimensional pointed Hopf algebra, A a left H-module algebra and M a left $A\#H$-module. Then:*
i) ${}_{A\#H}M$ is Noetherian $\Leftrightarrow$ ${}_AM$ is Noetherian.
ii) ${}_{A\#H}M$ is Artinian $\Leftrightarrow$ ${}_AM$ is Artinian.
iii) ${}_{A\#H}M$ has a Krull dimension $\Leftrightarrow$ ${}_AM$ has a Krull dimension, and the dimensions are equal if either of them exists.

Corollary 1.21. *Let H be a finite dimensional pointed Hopf algebra, A a right H-comodule algebra which is a faithfully flat Galois extension of $B := A^{coH}$, and M a left A-module. Then:*
i) ${}_AM$ is Noetherian $\Leftrightarrow$ ${}_BM$ is Noetherian.
ii) ${}_AM$ is Artinian $\Leftrightarrow$ ${}_BM$ is Artinian.
iii) ${}_AM$ has a Krull dimension $\Leftrightarrow$ ${}_BM$ has a Krull dimension, and the dimensions are equal if either of them exists.

Our second (and main) example will be the following beautiful result due to S. Montgomery [26]. It extends similar results obtained in the cocommutative case by Cartier and Gabriel and by Kostant in the early 1960's.

Theorem 1.22. *Let H be a pointed Hopf algebra, and set $G = G(H)$. Denote by $H_{(1)}$ the indecomposable component containing 1. Then:*
i) $H_{(1)}$ is a Hopf algebra.
ii) $N := G(H_{(1)})$ is a normal subgroup of G.
iii) H is isomorphic to a crossed product $H_{(1)}\#_\sigma k(G/N)$, with cocycle $\sigma : G/N \times G/N \longrightarrow N$.

Example 1.23. *For $H = U_q(sl(2))$, the indecomposable component of 1 is*

$$H_{(1)} = k\langle EK, FK, K^2, K^{-2}\rangle,$$

with the same relations as in Example 1.7. In this case we have $N = \langle K^2\rangle$, $G/N = \langle \overline{K}\rangle \simeq \mathbf{Z}_2$ (here $\overline{K}$ is the coset KN), and the cocycle

$$\sigma : G/N \times G/N \longrightarrow N$$

is given by $\sigma(\overline{K},\overline{K}) = K^2$, *and* $\sigma(\overline{K},\overline{1}) = \sigma(\overline{1},\overline{K}) = \sigma(\overline{1},\overline{1}) = 1$. *Then*

$$U_q(sl(2)) \simeq H_{(1)}\#_\sigma k(G/N) \simeq H_{(1)}\#_\sigma k\mathbf{Z}_2.$$

If we set $E_1 = EK$, $F_1 = K^{-1}F$, *and* $K_1 = K^2$, *then*

$$H_{(1)} = k\langle E_1, F_1, K_1, K_1^{-1}\rangle,$$

which is the "new version" of $U_q(sl(2))$, *used in* [18], [11], [34].

Remark 1.24. *By Example 1.18 we see that the "old version" is a faithfully flat Galois extension of the "new" one, so their representations are strongly related* [30]. *Moreover, the extension "new"-*$U_q(sl(2)) \subset$ *"old"-*$U_q(sl(2))$ *satisfies the hypotheses of Corollary 1.21 (since* $k\mathbf{Z}_2$ *is pointed), thus all assertions there apply in this case.*

Remark 1.25. *A similar decomposition holds for* $U_q(sl(n))$, *(M. Takeuchi, private communication to S. Montgomery, following* [34]*).*

We have seen how the coalgebra structure can affect the algebra structure. In order to study the converse, what we need first is an object with good properties that can be used for proving structure theorems for coalgebras. This object is the dual of the crossed product, namely the crossed coproduct. The nice part is that both notions arise together in a natural way when studying extensions of Hopf algebras. We explain this in the sequel.

Let us consider first the notions which are dual to the ones for algebras introduced above, to obtain the notions of *module coalgebras* and *comodule coalgebras*. If C is a left H-comodule coalgebra we can form the smash coproduct $C \rtimes H$ as in [23]. If the Hopf algebra H coacts weakly on C and $\alpha : C \longrightarrow H \otimes H$, then we can form the smash coproduct $C \rtimes_\alpha H$. This notion was introduced in [12] by dualizing the crossed product from [32]. Then it appeared in the general form in several papers: [10, 16, 22, 19, 20]. The following example extends [12, Example (5.2.2)] and appears in [9]:

Example 1.26. *Let* G_2 *be an affine algebraic group over an algebraically closed field* k, *and let* L *be its coordinate ring. Let* G_1 *be a closed normal subgroup of* G_2 *and let* K *be its coordinate ring. If* G_3 *denotes the quotient algebraic group of* G_2 *by* G_1, H *is its coordinate ring, and if the canonical map of algebraic groups from* G_2 *to* G_3 *splits as a morphism of varieties, then* L *is a crossed coproduct of* H *and* K.

Let us look now at extensions of Hopf algebras.

Example 1.27. *(*[16, 22, 9]*) Assume that* $K \xrightarrow{f} L \xrightarrow{g} H$ *is a sequence of Hopf algebras such that there exists* $\lambda : C \longrightarrow K \otimes H$, *an isomorphism of left* K*-modules and right* H*-comodules. Denote by* $i_K = Id_K \otimes u_H$, $i_H = u_K \otimes Id_H$, $p_K = Id_K \otimes \varepsilon_H$ *and* $p_H = \varepsilon_K \otimes Id_H$, *and assume moreover that* $f = \lambda^{-1} i_K$,

$g = p_H\lambda$, and that $k := p_K\lambda$ and $h := \lambda^{-1}i_H$ are invertible under convolution (note that this condition is very natural, since any function between two groups extends to an invertible function between their group rings).
Then L is a right H-comodule algebra and a left K-module coalgebra, and L is isomorphic, as an algebra, to a crossed product, and, as a coalgebra, to a crossed coproduct (of K and H). This is what is called a bicrossproduct *in* [20].

Crossed coproducts have properties which are dual to the ones of crossed products proved in [3] or [14]: they are Galois coextensions (the dual notion of a Galois extension, introduced in [30]), they have a dual property to the normal basis property, etc. On the other hand, there is no result dual to Theorem 1.22 available yet.

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AN INTRODUCTION TO QUASITILTED ALGEBRAS

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Let Λ be an artin algebra, for example for simplicity a finite dimensional algebra over an algebraically closed field k. Let $\operatorname{mod}\Lambda$ be the category of finitely generated (left) Λ-modules and let $D = \operatorname{Hom}_k(\ ,k)$ be the ordinary duality between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\text{op}}$. We shall in this writeup give an introduction, together with basic properties and connections, to the theory of quasitilted algebras as introduced and developed in our joint work with Smalø, with later contributions by others. We also include some of our own recent work.

We assume the reader is familiar with the definition and basic properties of quivers, almost split sequences and AR-quivers. On the other hand we recall the background material we need on hereditary algebras and tilted algebras, in order to motivate the quasitilted algebras and the properties of those which we discuss. For the background material we refer to the books [R1],[ARS] and the survey [A]. Our exposition does not necessarily follow the order in which the material was developed.

The hereditary finite dimensional algebras over an algebraically closed field k are Morita equivalent to path algebras $k\Gamma$ of finite quivers. It is one of the best understood classes of finite dimensional algebras, but nevertheless there are still interesting open questions. For a recent survey we refer to [K2].

1. Hereditary algebras

Let k be an algebraically closed field and Γ a finite quiver without oriented cycles. Let $k\Gamma$ be the corresponding path algebra. Recall that an artin algebra Λ is said to be of finite representation type if there is only a finite number of indecomposable objects in $\operatorname{mod}\Lambda$ up to isomorphism. The following classification theorem of Gabriel from the early seventies is an important result in modern representation theory.

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Theorem 1.1. *With the above notation $k\Gamma$ is of finite representation type if and only if the underlying graph $|\Gamma|$ of Γ is a finite disjoint union of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8.*

The k-algebras of infinite representation type fall into two main classes: tame and wild. For the purpose of these notes we do not need the general definitions. We just remark that for a connected quiver Γ we have that $k\Gamma$ is tame if and only if $|\Gamma|$ is one of the extended Dynkin diagrams $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. And $k\Gamma$ is wild if it is neither tame nor of finite representation type.

From a homological point of view the path algebras $k\Gamma$ are simplest possible, amongst the non semisimple algebras. For we have $\mathrm{gl.dim}\, k\Gamma \leq 1$, so that $k\Gamma$ is hereditary, and in fact each hereditary finite dimensional k-algebra is isomorphic to some $k\Gamma$.

For an artin algebra Λ the category $\mathrm{mod}\,\Lambda$ has almost split sequences, and consequently an associated AR-quiver. A lot of work in representation theory has been devoted to describing possible shapes of AR-quivers for various classes of algebras. For an indecomposable algebra of finite representation type there is only one connected component. In the hereditary case this component is of a particularly nice shape. It is what is called a preprojective component.

A *preprojective component* $\mathcal{C}$ of an AR-quiver has the property that there are no oriented cycles, and for each C in $\mathcal{C}$ there is some projective P in $\mathcal{C}$ and $i \geq 0$ with $C = \tau^{-i}P$, where τ is the translation on the AR-quiver induced from the operation $D\,\mathrm{Tr}$.

We also have the dual notion of a *preinjective* component. When $k\Gamma$ is indecomposable of finite representation type, the unique component is both preprojective and preinjective.

Assume now that $k\Gamma$ is indecomposable and of infinite representation type. Then there is a unique preprojective component, which in this case contains all indecomposable projective modules, and a unique preinjective component, which contains all indecomposable injective modules.

If $\Lambda = k\Gamma$ is tame, all other components, an infinite number, are tubes, that is of the form $ZA_\infty/\langle\tau^n\rangle$ for some $n \geq 1$, where n is 1 for all but a finite number of the tubes.

If $\Lambda = k\Gamma$ is wild, all other components are of the form ZA_∞.

For infinite representation type we have the following picture:

where $\mathcal{P}$ denotes the preprojective component, $\mathcal{Q}$ the preinjective component and $\mathcal{R}$ the union of the other components, which are called the regular components. Then the triple $(\mathcal{P}, \mathcal{R}, \mathcal{Q})$ defines $\mathcal{R}$ to be a *separating* subcategory, in the sense of Ringel, that is, we have the following properties:

(i) Each indecomposable module in $\operatorname{mod}\Lambda$ is in $\mathcal{P}$, $\mathcal{R}$ or $\mathcal{Q}$.

(ii) $\operatorname{Hom}(\mathcal{R},\mathcal{P})=0$, $\operatorname{Hom}(\mathcal{Q},\mathcal{P})=0$, $\operatorname{Hom}(\mathcal{Q},\mathcal{R})=0$.

(iii) Every map $f: P \to Q$ with P in $\mathcal{P}$ and Q in $\mathcal{Q}$ factors through an object in $\mathcal{R}$.

In addition we have the following.

(iv) Each indecomposable projective module is in $\mathcal{P}$ and each indecomposable injective module is in $\mathcal{Q}$.

Since hereditary algebras are so well understood, it is useful to try to exploit this fact by investigating algebras which in some way are built up from hereditary algebras, by taking endomorphism rings or one-point extensions. Our aim is twofold, and "contradictory". We want algebras which are close enough to the hereditary ones in order to be able to transfer information from the hereditary case, but far enough away so that we get something as different as possible.

On the other hand, from a homological point of view, global dimension at most two gives the next level.

The algebras we shall investigate fall into both these frames. We go on to discuss the by now classical theory of tilting, from a point of view suitable for the subsequent treatment of quasitilted algebras.

2. Tilted algebras

Let Λ be an artin algebra, for example a finite dimensional k-algebra. A module T in $\operatorname{mod}\Lambda$ is said to be a *tilting module* if

(i) $\operatorname{pd} T \leq 1$

(ii) $\operatorname{Ext}^1(T,T)=0$

(iii) the number of nonisomorphic indecomposable summands of T is equal to the number of nonisomorphic simple Λ-modules.

A module U in $\operatorname{mod}\Lambda$ is said to be a *cotilting module* if $D(U)$ is a tilting module in $\operatorname{mod}\Lambda^{\mathrm{op}}$.

There is an interesting connection between tilting/cotilting modules and torsion theories. Recall that a subcategory $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is a *torsion class* if $\mathcal{T}$ is closed under extensions and factors. A subcategory $\mathcal{F}$ of $\operatorname{mod}\Lambda$ is a *torsionfree class* if $\mathcal{F}$ is closed under extensions and submodules.

A torsion class $\mathcal{T}$ determines a torsionfree class $\mathcal{F}$ by $\mathcal{F}=\{C;\operatorname{Hom}(\mathcal{T},C)=0\}$, and a torsionfree class $\mathcal{F}$ determines a torsion class $\mathcal{T}$ by $\mathcal{T}=\{C;\operatorname{Hom}(C,\mathcal{F})=0\}$. It is easy to see that if we start with a torsion class $\mathcal{T}$, and construct the associated torsionfree class $\mathcal{F}$, then the associated torsion class of $\mathcal{F}$ is again the $\mathcal{T}$ we started with. We say that $(\mathcal{T},\mathcal{F})$ is a *torsion pair* if $\mathcal{T}$ is a torsion class and $\mathcal{F}$ the associated torsionfree class. The torsion pair is said to be split if each indecomposable Λ-module is in $\mathcal{T}$ or $\mathcal{F}$.

If T is a tilting module, it can be shown that the category $\operatorname{Fac} T$, whose objects are the factors of finite direct sums of copies of T, is closed under extensions. Hence $\mathcal{T} = \operatorname{Fac} T$ is a torsion class in this case, and we can associate a torsion pair $(\mathcal{T}, \mathcal{F})$ with T.

If U is a cotilting module, then dually the category $\operatorname{Sub} U$, whose objects are submodules of finite direct sums of copies of U, is closed under extensions. Hence $\mathcal{Y} = \operatorname{Sub} U$ is a torsionfree class, and we have an associated torsion pair $(\mathcal{X}, \mathcal{Y})$.

We state the following basic result.

Theorem 2.1. *Let T be a tilting module over an artin algebra Λ, and let $\Gamma = \operatorname{End}_\Lambda(T)^{\text{op}}$.*

(a) ${}_\Gamma U = {}_\Gamma D(T)$ *is a cotilting module in* $\operatorname{mod}\Gamma$, *and* $\operatorname{End}_\Gamma(D(T))^{\text{op}} \simeq \Lambda$.

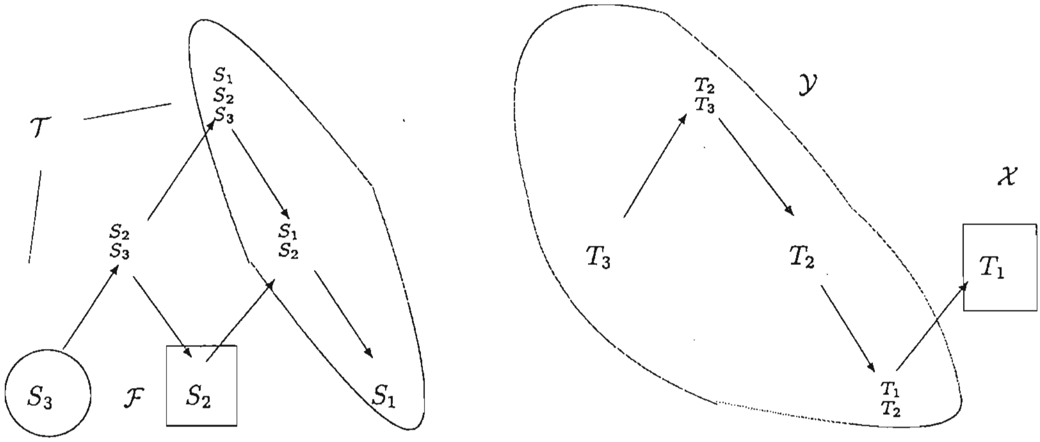
(b) $\operatorname{Hom}_\Lambda(T, \ ) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ *induces an equivalence of categories* $\operatorname{Hom}_\Lambda(T, \ ) : \mathcal{T} = \operatorname{Fac} T \to \mathcal{Y} = \operatorname{Sub} U$.

(c) $\operatorname{Ext}^1_\Lambda(T, \ ) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ *induces an equivalence of categories* $\operatorname{Ext}^1_\Lambda(T, \ ) : \mathcal{F} \to \mathcal{X}$, *where $(\mathcal{T}, \mathcal{F})$ is a torsion pair in* $\operatorname{mod}\Lambda$ *and $(\mathcal{X}, \mathcal{Y})$ a torsion pair in* $\operatorname{mod}\Gamma$.

(d) *If Λ is hereditary, then the torsion pair $(\mathcal{X}, \mathcal{Y})$ is split.*

We give an example to illustrate.

Example Let Λ be the 3×3 lower triangular matrix ring $\begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}$ over the field k, or equivalently, the path algebra $k\Gamma$ where Γ is the quiver $\cdot_1 \longrightarrow \cdot_2 \longrightarrow \cdot_3$. We have $\Lambda = P_1 \amalg P_2 \amalg P_3$, where P_i is indecomposable projective given by the i^{th} column, and $S_i = P_i/\underline{r}P_i$ is the corresponding simple module. Here $\underline{r}$ denotes the radical of Λ. Then $T = P_1 \amalg S_1 \amalg S_3$ is a tilting module, and $\Gamma = \operatorname{End}_\Lambda(T)^{\text{op}}$ is isomorphic to $\begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ 0 & k & k \end{pmatrix}$, that is Λ/I where $I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}$. Denote the indecomposable projective Γ-modules corresponding to the i^{th} column by Q_i and the corresponding simple module by T_i. We then have the AR-quivers



where we have marked the corresponding torsion and torsionfree classes.

A torsion theory $(\mathcal{T},\mathcal{F})$ in $\operatorname{mod}\Lambda$ is said to be hereditary if the torsion class $\mathcal{T}$ is closed under submodules. It is easy to see that $(\mathcal{T},\mathcal{F})$ is hereditary if and only if $\mathcal{F}=\operatorname{Sub}I$ for an injective Λ-module I. For let $S_1,\dots,S_t$ be the nonisomorphic simple modules which are not composition factors of any modules in $\mathcal{T}$, and let I be the injective envelope of $S_1\amalg\cdots\amalg S_t$.

There is the following information on when the torsion theories coming from tilting or cotilting modules are hereditary.

Proposition 2.2. *Let Λ be an artin algebra.*

(a) *If T is a tilting module, then the torsion pair $(\mathcal{T},\mathcal{F})$ with $\mathcal{T}=\operatorname{Fac}T$ is hereditary if and only if $\mathcal{T}=\operatorname{mod}\Lambda$.*

(b) *The torsion pair $(\mathcal{X},\mathcal{Y})$ is a hereditary torsion pair determined by a cotilting module if and only if $\mathcal{Y}=\operatorname{Sub}I$ for a faithful injective module I, that is I has zero annihilator.*

Proof: (a) Since $\mathcal{T}$ is known to contain all injective modules, it would have to contain $\operatorname{mod}\Lambda$ if it is closed under submodules.

(b) If $\mathcal{Y}=\operatorname{Sub}U$ for a cotilting module U and $(\mathcal{X},\mathcal{Y})$ is the corresponding torsion theory, then we have seen that $\mathcal{Y}=\operatorname{Sub}I$ for an injective module I, and it is known that $\mathcal{Y}$ contains Λ, so that I must be faithful.

If conversely $\mathcal{Y}=\operatorname{Sub}I$ for a faithful injective module I, it is easy to see that $\operatorname{Sub}I$ is closed under extensions, so that it is a torsionfree class. Then there is some X in $\operatorname{mod}\Lambda$ such that $U=I\amalg X$ is a cotilting module with the property that $\operatorname{Sub}I=\operatorname{Sub}U$ [AS]. □

An artin algebra Γ is a *tilted algebra* if $\Gamma\simeq\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ where Λ is hereditary and T is a tilting Λ-module. It follows from Theorem 2.1 that a tilted algebra $\Gamma=\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ is of finite representation type if Λ is. That a tilted algebra is homologically close to hereditary algebras is seen from the following.

Proposition 2.3. *Let Γ be a tilted algebra. Then there is a split torsion pair $(\mathcal{X},\mathcal{Y})$ satisfying the following.*

(a) $\operatorname{pd}\mathcal{Y}\le 1$ *(that is,* $\operatorname{pd}Y\le 1$ *for Y in $\mathcal{Y}$).*

(b) $\operatorname{id}\mathcal{X}\le 1$.

(c) *Γ is in $\mathcal{Y}$.*

(d) *$\mathcal{Y}=\operatorname{Sub}V$ for some V in* $\operatorname{mod}\Gamma$.

(e) $\operatorname{gl.dim}\Gamma\le 2$.

Actually, we can show the following, using [HRS1].

Proposition 2.4. *An artin algebra Γ is a tilted algebra if and only if Γ satisfies the properties (a), (c), (d) in Proposition 2.3.*

Also some of the properties of the AR-quiver for hereditary algebras can be transferred to tilted algebras [St, R2].

Theorem 2.5. *Let Γ be a tilted artin algebra.*

(a) *There is always a preprojective and a preinjective component in the AR-quiver of Γ.*

(b) *If C is a regular component, that is containing neither projective nor injective modules, then C is of the form ZA_∞ or $ZA_\infty/\langle\tau^n\rangle$ for some $n \geq 1$ and at most one component is of the form $Z\Delta$ where Δ is the underlying graph of the original hereditary algebra.*

There is then only a finite number of components left, and these have also been described [K1, Li].

Also for tilted algebras we have natural separating subcategories.

Proposition 2.6. *Let T be a tilting module over the hereditary algebra Λ, and $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ the associated tilted algebra. Let $D({}_\Lambda T) = \operatorname{Hom}_\Lambda(T, D\Lambda)$ and $\mathcal{J} = \operatorname{add} D(T)$. Then $\operatorname{ind}\mathcal{J}$ is a separating subcategory of $\operatorname{ind}\Lambda$, the indecomposable modules in $\operatorname{mod}\Lambda$.*

3. Quasitilted algebras

We now introduce the class of quasitilted algebras, as a generalization of tilted algebras. To avoid some technicalities we give a definition related to our original definition. The relationship can be seen from the proof of II Theorem 2.3 in [HRS1]. We also give an indication of the definition in [HRS1].

Dropping the "finiteness" condition on $\mathcal{Y}$ in the characterization of tilted algebras from the previous section, we say that an artin algebra Γ is quasitilted if there is a split torsion pair $(\mathcal{X}, \mathcal{Y})$ with $\operatorname{pd}\mathcal{Y} \leq 1$ and Γ in $\mathcal{Y}$.

It turns out that there is additional homological information on the torsion class $\mathcal{X}$, which is seen by using results from Chapter II in [HRS1].

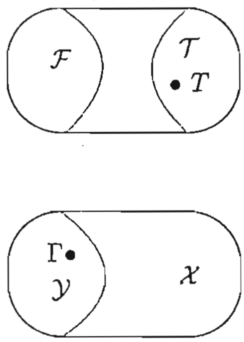
Proposition 3.1. *If for Γ there is a split torsion pair $(\mathcal{X}, \mathcal{Y})$ with $\operatorname{pd}\mathcal{Y} \leq 1$ and Γ in $\mathcal{Y}$, then $\operatorname{id}\mathcal{X} \leq 1$.*

The following is a direct consequence of the definition and Proposition 2.4.

Proposition 3.2. *If Γ is a quasitilted algebra of finite representation type, then Γ is a tilted algebra.*

Our definition of a quasitilted algebra Γ in [HRS1] was that $\Gamma = \operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}}$, where $\mathcal{H}$ is a hereditary abelian k-category where the morphisms are finite dimensional k-spaces and T is a tilting object in $\mathcal{H}$, with properties similar to the properties of a tilting module.

To try to improve the understanding of what is going on, we make the following remark. Consider first the following picture where $(\mathcal{T}, \mathcal{F})$ is a torsion pair for the hereditary algebra Λ, where T is a tilting Λ-module and $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ the corresponding tilted algebra and $(\mathcal{X}, \mathcal{Y})$ the induced torsion pair.



Here $\mathcal{Y} = \mathrm{Sub}\, U$ where U is the image of the injective cogenerator $D\Lambda$ under the functor $\mathrm{Hom}_\Lambda(T, \ ) : \mathrm{mod}\, \Lambda \to \mathrm{mod}\, \Gamma$ and Γ in $\mathcal{Y}$ is the image of the tilting module T. When we drop the finiteness condition on $\mathcal{Y}$, that is, do not require that $\mathcal{Y}$ is of the form $\mathrm{Sub}\, \mathcal{U}$, it is then reasonable that $\mathrm{mod}\, \Lambda$ has to be replaced by a category which is not a module category. But since $\Gamma \in \mathcal{Y}$, it is reasonable that we still have the tilting object T.

It is actually possible to construct an appropriate hereditary category $\mathcal{H}$, starting with a quasitilted algebra Γ. The construction is carried out inside the bounded derived category $D^b(\mathrm{mod}\, \Gamma)$. It is inspired from the "switch" between torsion and torsionfree classes which occurs in tilting theory and is illustrated by the above picture. It is the smallest abelian subcategory of $D^b(\mathrm{mod}\, \Gamma)$ containing $\mathcal{X}$, and $\mathcal{Y}$ shifted one step to the right.

In our definition of quasitilted algebras we have required that the torsion pair $(\mathcal{X}, \mathcal{Y})$ is split. Formulated as a condition on $\mathcal{X}$ and $\mathcal{Y}$ this is easily seen to amount to the following. Denote by $\mathrm{ind}\, \Gamma$ the indecomposable modules in $\mathrm{mod}\, \Gamma$. We say that $\mathcal{Y}$ is closed under predecessors if whenever there is a chain of nonzero maps $Z \to Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y$ between modules in $\mathrm{ind}\, \Gamma$ with Y in $\mathcal{Y}$, then also Z is in $\mathcal{Y}$. The property of being closed under successors is defined dually.

Proposition 3.3. *The following are equivalent for a torsion pair $(\mathcal{X}, \mathcal{Y})$ for an artin algebra Γ.*
(a) *The torsion pair $(\mathcal{X}, \mathcal{Y})$ splits.*
(b) *$\mathcal{Y}$ is closed under predecessors.*
(c) *$\mathcal{X}$ is closed under successors.*

Proof: (a) $\Rightarrow$ (b) Assume that $(\mathcal{X}, \mathcal{Y})$ splits, and let Y and U be in $\mathrm{ind}\, \Gamma$ with Y in $\mathcal{Y}$ and $\mathrm{Hom}(U, Y) \neq 0$. Then U is not in $\mathcal{X}$, and is consequently in $\mathcal{Y}$ since the torsion pair splits. This shows that $\mathcal{Y}$ is closed under predecessors.

(b) $\Rightarrow$ (a) Assume that $\mathcal{Y}$ is closed under predecessors, and let U be in $\operatorname{ind}\Gamma$ and not in $\mathcal{Y}$. Then $\operatorname{Hom}(U,\mathcal{Y}) = 0$, so that U is in $\mathcal{X}$, and consequently the pair $(\mathcal{X},\mathcal{Y})$ splits.

The rest of the proof is similar. □

For quasitilted algebras there are some natural choices of split torsion pairs $(\mathcal{X},\mathcal{Y})$ having our desired properties. For an arbitrary artin algebra Λ denote by $\mathcal{L}$ the objects in $\operatorname{ind}\Lambda$ such that all predecessors have projective dimension at most one. Denote by $\mathcal{R}$ the objects in $\operatorname{ind}\Lambda$ such that all successors have injective dimension at most one. Let $\widetilde{\mathcal{L}} = \operatorname{add}\mathcal{L}$ and $\widetilde{\mathcal{R}} = \operatorname{add}\mathcal{R}$ be the corresponding additive categories. Then $\widetilde{\mathcal{L}}$ is clearly in general a torsionfree class closed under predecessors, and hence gives rise to a split torsion pair in $\operatorname{mod}\Lambda$. Similarly $\widetilde{\mathcal{R}}$ is a torsion class giving rise to a split torsion pair. Actually the only missing condition for $\widetilde{\mathcal{L}}$ to give rise to a quasitilted algebra is that $\widetilde{\mathcal{L}}$ contains the projectives. Since clearly any torsionfree class which can be used in the definition must be contained in $\widetilde{\mathcal{L}}$, we get the following.

Proposition 3.4. *An artin algebra Λ is quasitilted if and only if $\widetilde{\mathcal{L}}$ contains the projective modules, which is the case if and only if $\widetilde{\mathcal{R}}$ contains the injective modules.*

For an artin algebra Λ the predecessors $\mathcal{U}$ of the indecomposable projective modules give rise to a torsionfree class, but here the missing property is for the modules to have projective dimension at most one. This is the case for quasitilted algebras. For quasitilted algebras which are not tilted we have the following. We here first recall that for a quasitilted algebra any indecomposable module is either in $\mathcal{L}$ or in $\mathcal{R}$ [HRS1].

Proposition 3.5. *Let Λ be a quasitilted algebra which is not tilted. Then $\widetilde{\mathcal{L}}$ is the largest torsionfree class satisfying the conditions of the definition and $\widetilde{\mathcal{L}\setminus\mathcal{R}}$ is the smallest one.*

Proof: The first claim is obvious from the previous remark. For the second claim, we know from [HRS1] that $\mathcal{L}\cap\mathcal{R}$ contains no projectives, that is all projectives are in $\widetilde{\mathcal{L}\setminus\mathcal{R}}$. If X is in $\mathcal{L}\setminus\mathcal{R}$, then X has a successor Y with $\operatorname{id}Y = 2$. Then we have that $\operatorname{Hom}(\operatorname{Tr}DY,\Lambda) \neq 0$, so that X is a predecessor of a projective module, and we are done. □

For a tilted algebra Λ we always have $\mathcal{L}\cap\mathcal{R} \neq \emptyset$, but for an arbitrary quasitilted algebra this is an open question. From the previous proposition this question is equivalent to the question whether there is more than one torsionfree class with the desired properties, for a quasitilted algebra which is not tilted.

Before going on we recall some basic facts on quasitilted algebras [HRS1].

Theorem 3.6. *An artin algebra Λ is quasitilted if and only if* $\mathrm{gl.dim}\,\Lambda \leq 2$ *and if X is in* $\mathrm{ind}\,\Lambda$, *then* $\mathrm{pd}\,X \leq 1$ *or* $\mathrm{id}X \leq 1$.

Proposition 3.7. *Let Λ be an artin algebra.*

(a) *There is no oriented cycle in the quiver of a quasitilted algebra Λ.*

(b) *If Λ is quasitilted and P is a projective Λ-module, then* $\mathrm{End}_\Lambda(P)^{\mathrm{op}}$ *is quasitilted.*

(c) *If G is a finite group whose order is invertible in Λ, then Λ is quasitilted if and only if ΛG is quasitilted, where ΛG is the skew group ring.*

We now discuss existence of separating subcategories for quasitilted algebras. Note that we have a natural possibility of a trisection of $\mathrm{ind}\,\Lambda$, as $(\mathcal{L} \setminus \mathcal{R}, \mathcal{L} \cap \mathcal{R}, \mathcal{R} \setminus \mathcal{L})$. We then have properties (i) and (ii) in the definition for $\mathcal{L} \cap \mathcal{R}$ to be a separating subcategory from section 1, and we have seen that we have (iv) if Λ is quasitilted, and not tilted. But (iii) does not necessarily hold. There may be objects in $\mathcal{L} \setminus \mathcal{R}$ with no nonzero map to $\mathcal{L} \cap \mathcal{R}$.

It is an interesting question whether there is some other natural trisection for quasitilted algebras such that the separation conditions (i), (ii) and (iii) are satisfied. Note that in general a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with (i) and (ii) gives rise to split torsion pairs $(\mathcal{C}, \mathcal{A} \cup \mathcal{B})$ and $(\mathcal{B} \cup \mathcal{C}, \mathcal{A})$. And if we in addition have (iv) we have the following observation.

Proposition 3.8. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a trisection for Λ satisfying (i), (ii), (iv). Then Λ is quasitilted.*

Proof: If $\mathrm{id}X = 2$ for X in $\mathrm{ind}\,\Lambda$, then X is a predecessor of a projective module, which is in $\mathcal{A}$ by (iv), and hence X is in $\mathcal{A}$. Then $\mathrm{id}X \leq 1$ for X in $\mathcal{B} \cup \mathcal{C}$. Similarly $\mathrm{pd}\,X \leq 1$ for X in $\mathcal{A} \cup \mathcal{B}$. Since now every submodule of a projective module then has projective dimension at most one, it follows from Theorem 3.6 that Λ is quasitilted. □

As for the components of the AR-quiver it is interesting that there is a connection with the trisection $(\mathcal{L} \setminus \mathcal{R}, \mathcal{L} \cap \mathcal{R}, \mathcal{R} \setminus \mathcal{L})$ for a quasitilted algebra which is not tilted.

Proposition 3.9. *Let Λ be a quasitilted algebra which is not tilted.*

(a) *$\mathcal{R}$ has no Ext-projective objects and $\mathcal{L}$ has no Ext-injective objects.*

(b) *Each component of the AR-quiver of Λ lies entirely in $\mathcal{L} \setminus \mathcal{R}$, $\mathcal{L} \cap \mathcal{R}$ or $\mathcal{R} \setminus \mathcal{L}$.*

Proof: (a) This is in [HRS1].

(b) This is in [CS], but here we give a short direct version. Let X be in $\mathcal{R}$. Since X is not Ext-projective by (a), there is some Y in $\mathcal{R}$ with $\mathrm{Ext}^1(X, Y) \neq 0$.

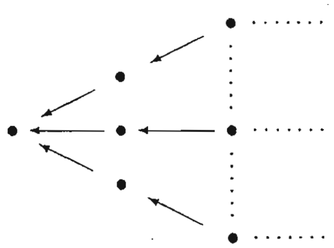
From the formula $D\,\mathrm{Ext}^1(X,Y) \simeq \overline{\mathrm{Hom}}(Y, D\,\mathrm{Tr}\,X)$, it follows that $\mathrm{Hom}(Y, D\,\mathrm{Tr}\,X) \neq 0$. Then $D\,\mathrm{Tr}\,X$ is not in the torsionfree class $\mathcal{L} \setminus \mathcal{R}$, and hence $D\,\mathrm{Tr}\,X$ is in $\mathcal{R}$. Since also $\mathrm{Tr}\,DX$ is in $\mathcal{R}$, we have that if one module from a component is in $\mathcal{R}$, then the whole component is. The proof for $\mathcal{L}$ is similar, and we are done. □

Further information on components is given in the following [CH].

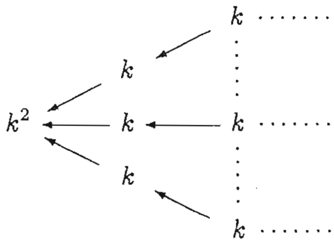
Theorem 3.10. *If Λ is quasitilted algebra, then there is a preprojective component.*

Tilted algebras can have many preprojective components. But it would be interesting to know if there is exactly one preprojective component when Λ is an indecomposable quasitilted algebra which is not tilted. For further information on components, see [CS]. One interesting open question is whether the regular components are ZA_∞ or tubes, when Λ is quasitilted and not tilted.

An interesting class of examples of quasitilted algebras is provided by the canonical algebras. They are of the form $\Lambda = \left(\begin{smallmatrix} k & 0 \\ M & S \end{smallmatrix}\right)$ where S is the path algebra over a field k of a quiver of the form



and M is an indecomposable S-module determined by a representation



where the images in k^2 of the composites along the arms are pairwise distinct. We write $\left(\begin{smallmatrix} k & 0 \\ M & S \end{smallmatrix}\right) = S[M]$, and it is called the one-point extension of S by M.

More generally it is of interest to investigate what it means for a module M that the one-point extension $S[M]$ is quasitilted. The most satisfying answer is in the tame case.

Theorem 3.11. *Let S be a tame hereditary path algebra, and let M be a regular S-module. Then $S[M]$ is quasitilted if and only if M is simple regular, that is, lies at the end of a tube in the AR-quiver.*

Note that some of the quasitilted algebras appearing in the above theorem are tilted and some not (see [R1],[HRS1]), so that examples of quasitilted algebras which are not tilted are provided. M is simple regular if and only if $D\operatorname{Tr}M$ is simple regular, so that in the tame case $S[D\operatorname{Tr}M]$ is quasitilted when $S[M]$ is. In the wild case we have the following.

Proposition 3.12. *Let S be an indecomposable wild hereditary algebra, and M indecomposable regular with $S[M]$ quasitilted. Then $S[D\operatorname{Tr}M]$ is quasitilted and there is some $i > 0$ such that $S[\operatorname{Tr}D^iM]$ is not quasitilted.*

This picks out a particular M in a $D\operatorname{Tr}$-orbit giving rise to quasitilted algebras, and we point out that in the case of the canonical algebra the corresponding M is the special choice.

The following property of M is intimately related with the question whether $S[M]$ is quasitilted.

Let M be an indecomposable regular module over a hereditary algebra S, and $\mathcal{X}$ a class of S-modules. Then M *dominates* $\mathcal{X}$ if for any right minimal map $g : M^i \to X$ with X in $\mathcal{X}$, we have that $\ker g$ is projective.

There is the following basic property.

Proposition 3.13. *Let M be an indecomposable regular S-module with S hereditary and* $\operatorname{End}(M) \simeq k$*, such that M dominates the injective modules. Then M dominates* $\operatorname{mod}\Lambda$.

We have the following.

Proposition 3.14. *Let M be an indecomposable regular S-module with S hereditary and* $\operatorname{End}(M) \simeq k$*. If M dominates the injective modules, then $\Lambda = S[M]$ is tilted.*

Proof: Since M dominates $\operatorname{mod}S$, it follows that if $h : M^i \to X$ is indecomposable with $i > 0$, then $\ker h$ is projective, which by [HRS1] means that the projective dimension of $h : M^i \to X$ as a $S[M]$-module is at most one. Hence only the simple injective object $k \to 0$ can have projective dimension two. Then the claim follows from [CS], since all but a finite number of indecomposable modules have projective dimension at most one. □

If M is indecomposable regular with $\operatorname{End}(M) \simeq k$ and M dominates $D\operatorname{Tr}I$ for each injective module I, $S[M]$ may not be tilted. For example the canonical algebras satisfy this property [HRS1]. But it would be interesting to know if this implies quasitilted.

We briefly mention that the concept of dominating is related to the concept of elementary modules as studied in [KL]. For details we refer to [HRS1].

Finally we want to indicate some recent results which we are unable to cover in detail. First recall that quasitilted algebras can be defined as endomorphism rings of "tilting objects" for hereditary abelian k-categories. Hence one approach to the understanding of quasitilted algebras goes via these categories. There are two main examples, namely module categories and coherent sheaves over noncommutative projective curves in the sense of Geigle and Lenzing [GL]. An open question is if there are any which are not derived equivalent to one of these two types. Under the additional assumption that the hereditary abelian k-category is noetherian, Lenzing has shown that there are no more examples [Le]. Also if one restricts to tame quasitilted algebras it was shown in [LS] that hereditary categories occurring are derived equivalent to the two known types. We point out that special classes of quasitilted algebras have been investigated in [LP], [LM], [M], and the tame ones have been classified in [Sk].

Artin algebras Λ being derived equivalent to a hereditary abelian k-category are called piecewise hereditary algebras. This generalizes the concept of quasitilted algebras. The precise connection to an iteration of the tilting process is worked out in [HRS2].

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EXCEPTIONAL OBJECTS IN HEREDITARY CATEGORIES

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ABSTRACT. Let k be a field and $\mathcal{A}$ a finite dimensional k-category which is a hereditary length category. We are going to show that the support algebra of any object of $\mathcal{A}$ without self-extension is a finite dimensional k-algebra. An object in $\mathcal{A}$ is said to be exceptional provided it is indecomposable and has no self-extensions. For an algebraically closed field k, Schofield has exhibited an algorithm for obtaining all exceptional objects starting from the simple ones. We will present a proof which works for arbitrary fields k.

Let $\mathcal{A}$ be an abelian category. The category $\mathcal{A}$ is said to be *hereditary* provided Ext^2 vanishes everywhere. Also, we recall that $\mathcal{A}$ is said to be a *length category* provided every object in $\mathcal{A}$ has finite length.

Let k be a field. We say that $\mathcal{A}$ is a *k-category* provided k operates centrally on all Hom-sets and such that the composition of maps is bilinear. Such a k-category is said to be *finite dimensional* provided the vector spaces $\mathrm{Hom}(X, Y)$ are finite dimensional, for all objects X, Y in $\mathcal{A}$.

Exceptional objects have been studied in various contexts. The terminology 'exceptional' was first used by Rudakov and his school [Ru] when dealing with vector bundles. The relevance of exceptional objects in the representation theory of finite dimensional hereditary k-algebras is well accepted; these exceptional modules are just the indecomposable partial tilting modules, they have also been called stones by Kerner [K1] and Schur modules by Unger [U]. We may refer to a recent survey of Kerner [K2] dealing with objects in hereditary length categories, or at least with representations of wild quivers.

The aim of this report is to focus attention to some interesting developments in the representation theory of finite dimensional hereditary algebras. This theory has an apparent combinatorial flavour; one of the reasons is the role the exceptional modules play. The existence of non-trivial finite dimensional modules without self-extensions should be considered as a feature which is peculiar

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to non-commutative representation theory. As we want to show, the existence of such modules seems also to be a kind of finiteness condition. We will present a proof of a very useful Theorem of Schofield [S2], for an arbitrary base field k. This result describes certain types of filtrations of an exceptional module X using as factors exceptional modules again; it is a kind of Jordan-Hölder theorem, but the classical unicity assertion is replaced by the assertion that there are precisely $s(X)-1$ essentially different kinds of filtrations, where $s(X)$ is the number of isomorphism classes of composition factors of X. Our presentation should be considered as a variation of the considerations by Crawley-Boevey [CB] dealing with a braid group operation on exceptional sequences, see also [R]. Along the way, we will focus attention to the so-called Bongartz complement of a sincere exceptional module.

1. Subfactors of objects without self-extensions

Let $\mathcal{A}$ be an abelian category and A an object in $\mathcal{A}$. Let $A'' \subseteq A' \subseteq A$ be a chain of subobjects. Then A'/A'' is said to be a *subfactor* of A. If $\mathcal{U}$ is a subcategory of $\mathcal{A}$, we denote by $\mathcal{I}(\mathcal{U})$ the class of all subfactors of objects in $\mathcal{U}$.

Recall that for any object A of $\mathcal{A}$, one denotes by $\operatorname{add} A$ the full subcategory given by all direct summands of finite direct sums of copies of A.

Lemma 1.1. *Let $\mathcal{A}$ be a hereditary abelian category, and $\mathcal{U}$ a subcategory which is closed under extensions. Then $\mathcal{I}(\mathcal{U})$ is closed under extensions.*

Proof. Let A, B be objects in $\mathcal{U}$. Let $A'' \subseteq A' \subseteq A$ and $B'' \subseteq B' \subseteq B$ be chains of subobjects. Thus, A'/A'' and B'/B'' are subfactors of objects in $\mathcal{U}$, and we consider an extension: assume that there is given an exact sequence

$$0 \to A'/A'' \to C \to B'/B'' \to 0.$$

We construct stepwise the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A'/A'' & \longrightarrow & C & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A/A'' & \longrightarrow & D & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & B/B'' & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

First, we form the induced exact sequence with respect to the inclusion $A'/A'' \to A/A''$ and obtain D with an inclusion map $C \to D$. The exact sequence with middle term D is induced from a third exact sequence with respect to the canonical epimorphism $A \to A/A'$; here we use that $\mathcal{A}$ is hereditary. In the diagram above, this third exact sequence has middle term E and there is an epimorphism $E \to D$. Again using that $\mathcal{A}$ is hereditary, there exists an exact sequence with middle term F and a monomorphism $E \to F$ which induces the third exact sequence with respect to the inclusion map $B'/B'' \to B/B''$. Finally, we form the induced sequence with respect to the canonical epimorphism $B \to B/B''$ and obtain an object G and an epimorphism $G \to F$. The maps

$$C \to D \leftarrow E \to F \leftarrow G$$

show that C is a subfactor of G. Since A, B belong to $\mathcal{U}$, and $\mathcal{U}$ is closed under extensions, the object G belongs to $\mathcal{U}$. This completes the proof.

Remark 1.2. The conclusion of Lemma 1.1 may be reformulated as follows: *$\mathcal{I}(\mathcal{U})$ is a Serre subcategory of $\mathcal{A}$.* Recall that a subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be a *Serre subcategory* provided for any exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$, the object A_2 belongs to $\mathcal{B}$ if and only if both A_1, A_3 belong to $\mathcal{B}$.

2. The support of an object without self-extension

Let $\mathcal{A}$ be a length category. For any object A in $\mathcal{A}$, we denote its isomorphism class by $[A]$. We denote by $\mathcal{S}(\mathcal{A})$ the 'set' of isomorphism classes of simple objects in $\mathcal{A}$ (it may not be a set, for set-theoretical reasons, thus we have used the quotation marks). Given two simple objects S, S' in $\mathcal{A}$, we draw an arrow $[S] \to [S']$ provided $\operatorname{Ext}^1(S, S') \neq 0$. In this way, $\mathcal{S}(\mathcal{A})$ becomes a 'quiver' (again, we use quotation marks, for set-theoretical reasons).

The support $\operatorname{supp} A$ of an object A in $\mathcal{A}$ is the set of isomorphism classes of composition factors of A, this is a 'subset' of $\mathcal{S}(\mathcal{A})$ (but since $\operatorname{supp} A$ is finite, we now deal with a 'subset' which really is a set). We consider $\operatorname{supp} A$ as a full 'subquiver' of $\mathcal{S}(\mathcal{A})$. If $\mathcal{S}'$ is a 'subset' of $\mathcal{S}$, we denote by $\mathcal{E}(\mathcal{S}')$ the class of objects of $\mathcal{A}$ with all composition factors belonging to $\mathcal{S}'$.

Proposition 2.1. *Let $\mathcal{A}$ be a hereditary length category. Let A be an object in $\mathcal{A}$ with $\operatorname{Ext}^1(A, A) = 0$. Then the support $\operatorname{supp} A$ of A is a directed quiver, and $\mathcal{E}(\operatorname{supp} A) = \mathcal{I}(\operatorname{add} A)$. If $\mathcal{A}$ is, in addition, a finite dimensional k-category, then the k-space $\operatorname{Ext}^1(S, S')$ is finite dimensional, for every pair S, S' of composition factors of A.*

Proof: Since $\operatorname{Ext}^1(A, A) = 0$, the subcategory $\operatorname{add} A$ of $\mathcal{A}$ is closed under extensions. According to Lemma 1.1, the class $\mathcal{I}(\operatorname{add} A)$ is closed under extensions. The composition factors of A belong to $\mathcal{I}(\operatorname{add} A)$, thus any object in

$\mathcal{E}(\operatorname{supp} A)$ belongs to $\mathcal{I}(\operatorname{add} A)$. Of course, conversely, the composition factors of subfactors of objects in $\operatorname{add} A$ belong to $\operatorname{supp} A$.

Let t be the Loewy length of A, this is the minimal length t of a filtration

$$0 = B_0 \subset B_1 \subset \cdots \subset B_t = A$$

of A with semisimple factors B_i/B_{i-1}. Note that any object in $\operatorname{add} A$, and therefore also any any subfactor of such an object has Loewy length at most t.

Now assume that there is an oriented cycle in the quiver $\operatorname{supp} A$, say $[S_0] \to [S_1] \to \cdots \to [S_s] = [S_0]$, with simple objects S_i. Since $\mathcal{A}$ is hereditary, one may construct serial objects U_n in $\mathcal{A}$ of arbitrarily large finite length n, such that the composition factors of U_n are of the form $S_0, \ldots, S_{s-1}$. In particular, U_n belongs to $\mathcal{I}(\operatorname{add} A)$. But the Loewy length of U_n is equal to its length n, since U_n is serial. It follows that $n \leq t$. This contradiction shows that there cannot be any oriented cycle in $\operatorname{supp} A$.

Let us assume now that $\mathcal{A}$ is, in addition, a finite dimensional k-category. Let us start with a composition series

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$$

of A, and let $S_i = A_i/A_{i-1}$. Let b be the k-dimension of the endomorphism ring of $\bigoplus_{i=1}^{n} S_i$. Clearly, b is a common bound for the k-dimension of $\operatorname{Hom}(A', A'')$, where A', A'' are subfactors of A. Let us fix $1 \leq i, j \leq n$. The embedding $S_i \subset A/A_{i-1}$ yields a surjection

$$\operatorname{Ext}^1(A/A_{i-1}, S_j) \to \operatorname{Ext}^1(S_i, S_j).$$

The exact sequence

$$0 \to A_{i-1} \to A \to A/A_{i-1} \to 0$$

yields an exact sequence

$$\operatorname{Hom}(A_{i-1}, S_j) \to \operatorname{Ext}^1(A/A_{i-1}, S_j) \to \operatorname{Ext}^1(A, S_j).$$

The epimorphism $A_j \to S_j$ yields a surjection

$$\operatorname{Ext}^1(A, A_j) \to \operatorname{Ext}^1(A, S_j),$$

and finally we consider the exact sequence

$$0 \to A_j \to A \to A/A_j \to 0;$$

it yields an exact sequence

$$\operatorname{Hom}(A, A/A_j) \to \operatorname{Ext}^1(A, A_j) \to \operatorname{Ext}^1(A, A).$$

Note that the last term is zero. Altogether we see that

$$\begin{aligned}\dim \operatorname{Ext}^1(S_i, S_j) &\le \dim \operatorname{Ext}^1(A/A_{i-1}, S_j)\\ &\le \dim \operatorname{Hom}(A_{i-1}, S_j) + \dim \operatorname{Ext}^1(A, S_j)\\ &\le \dim \operatorname{Hom}(A_{i-1}, S_j) + \dim \operatorname{Ext}^1(A, A_j)\\ &\le \dim \operatorname{Hom}(A_{i-1}, S_j) + \dim \operatorname{Hom}(A, A/A_j)\\ &\le 2b.\end{aligned}$$

This completes the proof.

Corollary 2.2. *Let $\mathcal{A}$ be a finite dimensional hereditary length k-category. Let A be an object without self-extensions with support* $\operatorname{supp} A$. *Then there exists a finite dimensional hereditary k-algebra Λ such that the category $\mathcal{E}(\operatorname{supp} A)$ is equivalent to the category of all Λ-modules of finite length. Under such an equivalence, A corresponds to a faithful Λ-module.*

If Λ is any k-algebra, a Λ-module X is said to be *sincere* provided every simple Λ-module occurs as a composition factor of X.

Corollary 2.3. *Let Λ be a hereditary k-algebra and X a finite dimensional Λ-module. Assume that X is sincere and has no self-extensions. Then Λ is finite dimensional and X is faithful.*

Proof: Apply the previous considerations to the category $\mathcal{A}$ of all finite dimensional Λ-modules.

It seems that finite dimensional modules without self-extensions have been considered before mainly for k-algebras Λ which are finite dimensional. For Λ hereditary, the corollary asserts that in essence one obtains in this way all such modules. The fact that for a finite dimensional hereditary k-algebra, a sincere module without self-extension is faithful, is well-known, see for example Kerner [K2], Lemma 8.3.

Remark 2.4. Let $\mathcal{A}$ be a hereditary length category with an exceptional object A whose support is $\mathcal{S}(\mathcal{A})$. If $\mathcal{A}$ is a finite dimensional k-category, then Corollary 2.2 shows that $\mathcal{A}$ has enough projective objects and enough injective objects. In general, this may not be the case: consider a field extension $k \subset K$ of infinite degree, let $\Lambda = \begin{bmatrix} k & 0 \\ K & K \end{bmatrix}$, and let $\mathcal{A}$ be the category of all (left) Λ-modules of finite length. The indecomposable projective Λ-module P of length 2 is exceptional and satisfies $\operatorname{supp} P = \mathcal{S}(\mathcal{A})$. The category $\mathcal{A}$ has enough projective objects, but not enough injective objects.

3. Schofield's Theorem

Let $\mathcal{A}$ be a finite dimensional k-category which is a hereditary length category. We are going to present a Theorem of Schofield which yields an inductive way for constructing all exceptional objects A in $\mathcal{A}$. The Theorem asserts that any exceptional object A is obtained as the middle term of a suitable exact sequence

$$0 \to U^u \to A \to V^v \to 0 \qquad (*)$$

where U, V are again exceptional objects and u, v are positive integers. More precisely, there is such an exact sequence where U, V are exceptional objects and where the objects U, V satisfy in addition the following conditions:

$$\mathrm{Hom}(U,V) = \mathrm{Hom}(V,U) = \mathrm{Ext}^1(U,V) = 0. \qquad (**)$$

A pair (V, U) of exceptional objects satisfying these conditions $(**)$ is called an *orthogonal exceptional pair* (the general notion of an exceptional pair will be recalled below). Given an orthogonal exceptional pair (V, U), we want to consider the full subcategory $\mathcal{E}(U, V)$ of all objects of $\mathcal{A}$ which have a filtration with factors of the form U and V. Note that for any object A in $\mathcal{E}(U, V)$, there exists an exact sequence of the form $(*)$ with non-negative integers u, v.

The reduction problem to be considered is the following: Given an exceptional object A, we want to find orthogonal exceptional pairs (V, U) such that A belongs to $\mathcal{E}(U, V)$, but A is not one of the two simple objects of $\mathcal{E}(U, V)$. One may ask for all possible pairs of this kind, and it is amazing that there exists an intrinsic characterization of the number of such pairs.

Theorem 3.1 (Schofield). *Let $\mathcal{A}$ be a finite dimensional k-category which is a hereditary length category. Let A be an exceptional object in $\mathcal{A}$. Then there are precisely $s(A) - 1$ orthogonal exceptional pairs (V_i, U_i) such that A belongs to $\mathcal{E}(U_i, V_i)$ and is not a simple object in $\mathcal{E}(U_i, V_i)$.*

Proof: We want to find exact sequences of the form $(*)$. Note that the objects U, V have to belong to $\mathcal{E}(\mathrm{supp}\, A)$, thus we may assume that $\mathcal{A}$ is equal to $\mathcal{E}(\mathrm{supp}\, A)$. This means that we may assume that $\mathcal{A}$ is the category of all finite length Λ-modules, where Λ is a finite dimensional hereditary k-algebra and that we consider a faithful exceptional Λ-module.

Thus, let Λ be a finite dimensional hereditary k-algebra and X a faithful exceptional Λ-module.

We will need some preliminary considerations. A pair (B, A) of exceptional objects in a hereditary abelian category $\mathcal{A}$ is said to be an *exceptional pair* provided we have $\mathrm{Hom}(A, B) = \mathrm{Ext}^1(A, B) = 0$.

Let (Y, X) be an exceptional pair of Λ-modules. We define $\mathcal{C}(X, Y)$ to be the closure of the full subcategory with objects X, Y under kernels, images, cokernels and extensions; of course, in case (Y, X) is in addition orthogonal, then

$\mathcal{C}(X,Y) = \mathcal{E}(X,Y)$. Let us recall the following facts: This subcategory $\mathcal{C}(X,Y)$ is an exact abelian subcategory, it is the smallest exact abelian subcategory of the category of all Λ-modules containing X, Y and being closed under extensions. It is of importance that $\mathcal{C}(X,Y)$ is equivalent to the category of all finite length modules over a finite dimensional hereditary k-algebra Θ with precisely 2 simple modules S, T; these modules S, T have no self-extensions and they satisfy $\mathrm{Ext}^1(S,T) = 0$. Under such an equivalence, the pair (Y, X) corresponds to an exceptional pair of Θ-modules. The proofs rely on the use of perpendicular categories as considered by Geigle-Lenzing [GL] and Schofield [S1], see Crawley-Boevey [CB] (the latter paper assumes that k is an algebraically closed field, but the relevant proofs needed here are valid in our more general setting).

The (finite dimensional) exceptional Θ-modules are well-known: they are just the preprojective and the preinjective Θ-modules. Also, the exceptional pairs of Θ-modules are easy to describe: For any exceptional Θ-module X, there is (up to isomorphism) a unique module Y such that (Y, X) is an exceptional pair of Θ-modules. Finally, if (Y, X) is an exceptional pair of Θ-modules, and $\mathrm{Hom}(Y, X) \neq 0$, then we must have $\mathrm{Ext}^1(Y, X) = 0$ (so that $X \oplus Y$ is a module without self-extensions). Note that the last assertion remains valid for arbitrary exceptional pairs of Λ-modules.

From now on, we fix a faithful exceptional Λ-module X. Let us stress the following conclusion: *the orthogonal exceptional pairs (V, U) with X in $\mathcal{E}(U, V)$ and X not simple in $\mathcal{E}(U, V)$ correspond bijectively to the exceptional pairs (Y, X) such that X is not simple in $\mathcal{C}(X, Y)$*; at least if the pairs in question are considered as pairs of isomorphism classes, not as pairs of modules. Namely, if the pair (V, U) is given, then there is (up to isomorphism) a unique Λ-module Y in $\mathcal{E}(U, V)$ such that (Y, X) is an exceptional pair in $\mathcal{E}(U, V)$ and therefore in $\mathcal{A}$. Also, we have $\mathcal{E}(U, V) = \mathcal{C}(X, Y)$. Conversely, if (Y, X) is an exceptional pair in $\mathcal{A}$, then $\mathcal{C}(X, Y)$ is a hereditary length category with precisely two simple objects, say U, V, and such that (V, U) is an (even orthogonal) exceptional sequence. Again we have $\mathcal{E}(U, V) = \mathcal{C}(X, Y)$. If we assume that X is not simple in $\mathcal{E}(U, V)$, then the pair (U, V) cannot be exceptional, thus not only the set $\{V, U\}$, but the pair (V, U) is uniquely determined by the pair (Y, X).

Thus, our aim is to classify all exceptional pairs (Y, X) such that X is not simple in $\mathcal{C}(X, Y)$. It will turn out that there is a constructive way of obtaining these pairs.

Lemma 3.2. *Let (Y, X) be an exceptional pair. Then the following assertions are equivalent:*

(i) *X is not simple in $\mathcal{C}(X, Y)$.*
(ii) *Y is not injective in $\mathcal{C}(X, Y)$.*
(iii) *Y is cogenerated by X.*

Proof: This follows from an easy inspection of all the exceptional sequences of Θ-modules. In order to see the implication (ii) $\Longrightarrow$ (iii), just use the almost split sequence in $\mathcal{C}(X,Y)$ starting with Y, its left hand map is a monomorphism of the form $Y \to X^n$, for some n.

Let X' be the universal extension of ${}_\Lambda\Lambda$ by copies of X, thus there is an exact sequence

$$0 \to {}_\Lambda\Lambda \to X' \to X^m \to 0$$

for some m, we have $\mathrm{Ext}^1(X, X') = 0$, and m is chosen minimal (or, equivalently, X does not occur as a direct summand of X'). This module X' is called the *Bongartz complement* for X. It is well-known (and easy to see) that $X \oplus X'$ is a tilting module. The Bongartz complement of a module without self-extension has been used before in many different situations, and a lot is known about its properties. For the convenience of the reader, we will include proofs of all the facts which are relevant for our consideration.

Lemma 3.3. *Let X be a faithful exceptional module. The Bongartz complement X' of X is cogenerated by X and therefore* $\mathrm{Hom}(X, X') = 0$.

Proof: Since X is faithful, there is a monomorphism $\alpha\colon {}_\Lambda\Lambda \to X^s$ for some s. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_\Lambda\Lambda & \longrightarrow & X' & \longrightarrow & X^m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} & & \| & & \\ 0 & \longrightarrow & X^s & \longrightarrow & X'' & \longrightarrow & X^m & \longrightarrow & 0 \end{array}.$$

The lower sequence splits, since X is exceptional, thus X'' is isomorphic to X^{s+m}. With α also α' is injective, thus X' is cogenerated by X.

Let us assume that there exists a non-zero homomorphism $\beta\colon X \to X'$. Since X' is cogenerated by X, we find $\beta'\colon X' \to X$ such that $\beta'\beta \neq 0$. This is a non-zero endomorphism of the exceptional module X, thus invertible. But this implies that β is a split monomorphism, impossible.

As a consequence, we obtain the following characterization of the indecomposable direct summands of the Bongartz complement of a faithful exceptional module:

Lemma 3.4. *Let X be a faithful exceptional module, let Y be indecomposable. The following assertions are equivalent:*

(i) *Y is a direct summand of the Bongartz complement of X.*
(ii) *(Y, X) is an exceptional pair and Y is cogenerated by X.*

Proof: Let X' be the Bongartz complement of X. First, let us assume that Y is a direct summand of X'. In particular, we have $\mathrm{Ext}^1(X, Y) = 0$. According to the previous Lemma, Y is cogenerated by X and $\mathrm{Hom}(X,Y)=0$. But this also means that (Y,X) is an exceptional pair.

For the proof of the converse, we first note the following: Let Z be a module cogenerated by X, say with a monomorphism $\gamma\colon Z\to X^t$, and let Z' be a module with $\mathrm{Ext}^1(X,Z')=0$. The long exact sequence for $\mathrm{Hom}(-,Z')$ yields an epimorphism $\mathrm{Ext}^1(X^t,Z')\to\mathrm{Ext}^1(Z,Z')$, thus we see that $\mathrm{Ext}^1(Z,Z')=0$.

Now, let us assume that (Y,X) is an exceptional pair and that Y is cogenerated by X. Since $\mathrm{Hom}(Y,X)\neq 0$, we have $\mathrm{Ext}^1(Y,X)=0$. The previous considerations yield $\mathrm{Ext}^1(X',Y)=0$ and $\mathrm{Ext}^1(Y,X')=0$, since both modules X',Y are cogenerated by X and since they satisfy $\mathrm{Ext}^1(X,X')=0$ and $\mathrm{Ext}^1(X,Y)=0$. It follows that $X\oplus X'\oplus Y$ is a tilting module. As a consequence, Y is isomorphic to a direct summand of $X\oplus X'$. Since $\mathrm{Hom}(X,Y)=0$, we see that Y is isomorphic to a direct summand of X'.

Proof of Schofield's Theorem: Since X is a faithful Λ-module, $s=s(X)$ is the number of simple Λ-modules. Let $Y_1,\dots,Y_{s-1}$ be pairwise non-isomorphic direct summands of the Bongartz complement X' of X (recall that a tilting module has precisely s isomorphism classes of indecomposable direct summands). Then, the pairs (Y_i,X) are exceptional with Y_i being cogenerated by X, thus X is not simple in the subcategory $\mathcal{C}(Y_i,X)$.

On the other hand, consider an exceptional pair (Y,X) with X not simple in $\mathcal{C}(X,Y)$. Then Y is cogenerated by X, thus Y is isomorphic to a direct summand of X', thus to one of the modules Y_i. This completes the proof.

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1-SEMI-QUASI-HOMOGENEOUS SINGULARITIES OF HYPERSURFACES IN CHARACTERISTIC 2

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ABSTRACT. In arbitrary characteristic different from 2, the singularities with semi-quasi-homogeneous equations characterized by the condition to have Saito-invariant 1 are the "classical" quasi-homogeneous ones, known over the field of complex numbers as simple elliptic singularities (Saito, [10]). Here we find them in characteristic 2 as well: In odd dimensions and for weights $\tilde{E}_6$ and $\tilde{E}_8$ non-quasi-homogeneous equations appear.

0. The problem

k denotes an algebraically closed field. Let X be a finite set of indeterminates x equipped with positive weights $w(x) \in \mathbb{Q}$ and $f \in k[[X]]$ be a formal power series consisting of monomials of weight ≥ 1 such that f_1 (:= sum of terms of total degree 1) defines an isolated singularity (i.e. the partial derivatives generate an ideal which is primary for the maximal ideal in $k[[X]]/(f)$). Then we associate to f the "Saito-invariant" $s := |X| - 2\sum_{x\in X} w(x)$. We say "$f$ is s-semi-quasi-homogeneous" (or short: "s-sqh") with respect to the given weights. For $f = f_1$, f is said to be "1-quasi-homogeneous". The case of $s < 1$ gives the rational double points (the simple singularities or, equivalently, the absolutely isolated Cohen Macaulay double points, cf. [3], [6], [4]). Here the "boundary case" of $s = 1$ is considered, which corresponds in the complex-analytic case to the simple elliptic singularities ([10]). Note however, that for *char* $k = 2$ not all of those singularities arise from dimension 2, so here they better will be referred to only as 1-semi-quasi-homogeneous. As for the simple singularities, the case of characteristic 2 is most complicated in the sense of stable equivalence for different dimensions. From the point of representations (considering the Auslander-Reiten quiver of maximal Cohen-Macaulay modules over the local

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ring of the singularity), the usual Knörrer-periodicity has to be replaced by Solberg's periodicity (taking dimensions *mod* 2), and the results of Kahn (cf. [5]) may apply at least to some of the singularities found here.

1. The quasi-homogeneous case

Write $X = \{X_0, \dots, X_n\}$ and $w(X_i) = w_i$. We always assume $w_i \leq \frac{1}{2}$; this is no loss of generality (cf. e.g. [6]). Let $k[X]_1$ denote the polynomials which are sum of monomials of weight 1 (set of "quasi-homogeneous polynomials" with respect to the given weights). Then we have the following

Cancellation property: Let $f, g \in k[X]_1$ define isolated singularities, and let q_1, $q_2 \in k[Y]$ be nondegenerate quadratic forms in a finite set Y of new variables of weight $\frac{1}{2}$. Suppose $f + q_1$ can be transformed into $g + q_2$ by an automorphism Φ of $k[X, Y]$ preserving the grading. Then there exists an automorphism Ψ of $k[X]$ which preserves the grading and such that $f = g \circ \Psi$.

This is a consequence of the following (cf. [6])

Proposition (Saito, Knop): Choose $f \in k[[X]]$ defining an isolated singularity.

(i) If $Y \subseteq X$, then one of the following is satisfied:
   (a) There exists $X^\alpha \in supp(f)$ such that $X^\alpha \in k[Y]$, or
   (b) There exists an injective map $\varphi : Y \hookrightarrow X - Y$ and a map $\psi : Y \to \mathbb{N}^Y$ such that $Y^{\psi(y)} \cdot y \cdot \varphi(y) \in supp(f)$ for every $y \in Y$.

(ii) Assume f is quasi-homogeneous of degree 1. Then up to an automorphism of $k[X]$ which preserves the grading, $f = f_1 + \sum_{x \in A} x\phi(x)$, where $A = \{x \in X,\ w(x) > \frac{1}{2}\}$ and $\phi : A \hookrightarrow X - A$ is an injection, $f_1 \in k[X - (A \cup \phi(A))]$. Now, choose all $w_i \leq \frac{1}{2}$ and denote $Q := \{x \in X |\ w(x) = \frac{1}{2}\}$, $R := X - Q = \{x \in X |\ w(x) < \frac{1}{2}\}$.
   Up to a graded automorphism[1], f is of the following form:
   (a) $f = f_1 + q$, $f_1 \in k[R]$, and $q \in k[Q]$ a nondegenerate quadratic form.
   (b) *char* $k = 2$, and there exists $x_0 \in Q$ such that $f = f_1 + f_2 \cdot x_0 + x_0^2 + q$, where $q \in k[Q - \{x_0\}]$ (q nondegenerate quadratic form), $f_i \in k[R]$ for $i = 1, 2$.

We deduce a

[1]tacitly assumed to be of degree 0

Proof of the cancellation property:
Let $f+q_1=(g+q_2)\circ\Phi$. In case of part (ii) (a) of the preceding proposition, we may assume $X=R$, i.e. $f,g\in(X_0,\dots,X_n)^3$, $w(X_i)<\frac{1}{2}$, thus $\Phi(X_i)\in k[X]$, and after a linear change of coordinates in Y, $\Phi(Y_i)=Y_i$.

Now let *char* $k=2$ and suppose f has the form (ii) of (b), $f+q_1=f_1+f_2X_0+X_0^2+q$, where $f_i\in k[X_1,\dots,X_n]$ and $q\in k[Y]$ is a nondegenerate quadratic form. We may assume $g+q_2=g_1+g_2X_0+X_0^2+q$, and also $g_i,f_i\in k[Y]$, $|Y|=m$ even and $q=Y_1Y_2+\dots+Y_{m-1}Y_m$ (classification of quadratic forms in characteristic 2). Then, if $f=g\circ\Phi$, Φ graded. We obtain $\Phi(R)\subseteq k[R]$, $R=\{X_1,\dots,X_n\}$, Φ induces a linear transformation in the variables $\{X_0\}\cup Y\ mod(X_1,\dots,X_n)^2$, fixing $X_0^2+q(Y)\ mod(X_1,\dots,X_n)^2$. Thus we may assume $\Phi(X_0)=X_0+\phi_0$, $\Phi(Y_i)=Y_i+\phi_i$, $\phi_i\in k[X]$ of weight $\frac{1}{2}$. But q is nondegenerate, thus $\phi_1=\dots=\phi_m=0$.

Definition: Choose $f\in k[[X]]$ and $g\in k[[X']]$.

(i) f,g are said to be right equivalent if $X=X'$ and there exists an automorphism Φ of $k[[X]]$ such that $f=g\circ\Phi$. In this case, we write $f\overset{r}{\sim}g$ (without loss of generality, Φ can be choosen homogeneous of degree 0 if $f\in k[X]_1$, $g\in k[X]_1$ for a fixed weight w).

(ii) Assume there exist nondegenerate quadratic forms $q\in k[Z]$, $q'\in k[Z']$ respectively in finite sets Z, resp. Z' of new variables such that $f+q\overset{r}{\sim}g+q'$. Then f,g are said to be stable-equivalent[2]. We write $f\overset{s}{\sim}g$. The polynomials $f+q$, $g+q'$ respectively will be referred to as "quadratic suspensions" of f,g respectively.

Thus, the above cancellation property says: If f, g (as above) have the same number of variables and $f\overset{s}{\sim}g$, then $f\overset{r}{\sim}g$

If $f\overset{s}{\sim}g$, then $s(f)=s(g)$, and always $0\le s(f)<|X|$. The classes of f having $s(f)<1$ are precisely the quasi-homogeneous forms of the simple singularities ADE (cf. [6], [4]); their behavior under the canonical local resolution is studied in [7].

For the 1-qh polynomials we have the following

Theorem: Let $f\in k[X]$ be a polynomial defining an isolated singularity such that f is quasi-homogeneous for some weight w with $s=1$.

Then w is (up to permutation) one of the weights

$$\tilde{E}_6=(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\dots,\frac{1}{2}),\ \tilde{E}_7=(\frac{1}{4},\frac{1}{4},\frac{1}{2},\dots,\frac{1}{2}),\ \tilde{E}_8=(\frac{1}{3},\frac{1}{6},\frac{1}{2},\dots,\frac{1}{2})$$

and f is stable-equivalent with one of the following polynomials ($t\in k$ denotes a parameter):

[2]Note, the condition implies that the total number of variables has to be the correct one.

Case A: $char(k) \neq 2$

$$\tilde{E}_6:\quad f = X_1(X_1 - X_0)(X_1 - tX_0) - X_0X_2^2, \quad t \neq 0,1$$
$$\tilde{E}_7:\quad f = X_0X_1(X_1 - X_0)(X_1 - tX_0), \quad t \neq 0,1$$
$$\tilde{E}_8:\quad f = X_0(X_0 - X_1^2)(X_0 - tX_1^2), \quad t \neq 0,1$$

Case B: $char(k) = 2$

1. n odd

$$\tilde{E}_6(0):\quad X_0^3 + X_1^2X_2 + X_1X_2^2 + X_3^2$$
$$\tilde{E}_6(t):\quad X_0^3 + tX_2^3 + X_1^2X_2 + X_0X_1X_2 + X_3^2, \qquad t \neq 0$$
$$\tilde{E}_7(t):\quad X_0X_1(X_1 + X_0)(X_1 + tX_0), \quad t \neq 0,\ 1$$
$$\tilde{E}_8(t):\quad X_0(X_0 + X_1^2)(X_0 + tX_1^2), \quad t \neq 0,\ 1$$

2. n even

$$\tilde{E}_6(0):\quad X_0^3 + X_1^2X_2 + X_1X_2^2$$
$$\tilde{E}_6(t):\quad X_0^3 + tX_2^3 + X_1^2X_2 + X_0X_1X_2, \qquad t \neq 0$$

$$\tilde{E}_{7,1}(t):\quad X_0^2 + X_0X_1^2 + X_1X_2^2(tX_1 + X_2)$$
$$\tilde{E}_{7,2}(t):\quad X_0^2 + X_0X_1X_2 + X_1X_2(tX_1 + X_2)^2, \qquad t \neq 0$$

$$\tilde{E}_8(t):\quad X_0^2 + X_0X_1X_2 + X_1(X_1 + X_2^2)(X_1 + tX_2^2), \qquad t \neq 0$$

Proof: To start with, we need the following

Lemma: With the previous notations, assume $s = |X| - 2\sum_{x \in X} w(x) = 1$, i.e.

$$\sum_{x \in R} w(x) \;=\; \frac{1}{2}(|R| - 1)$$

and such that there exists a polynomial $f \in k[X]_1$ with an isolated singularity. Then

(i) $|R| \neq 0, 1$

(ii) $S := \{x \in X \mid \frac{1}{3} < w(x) < \frac{1}{2}\} = \emptyset$

(iii) $|R| \leq 3$ with equality at most if $w_0 = w_1 = w_2 = \frac{1}{3}$ (up to permutation of indices of the X_i).

(iv) If $|R| = 2$, then $w_0 = w_1 = \frac{1}{4}$, or $w_0 = \frac{1}{3}$, $w_1 = \frac{1}{6}$ (again, indices may permute).

Proof of the Lemma: (i) is an obvious consequence of $s = 1$.
To show (ii), (iii), apply (i) in the above proposition: Choose maps φ, ψ with the property (b) and obtain:

$$\begin{aligned}
\frac{1}{2}(|R|-1) &= \sum_{x\in R} w(x) = \\
&= \sum_{x\in S} w(x) + \sum_{x\in S} w(\varphi(x)) + \sum_{x\in R-(S\cup\varphi(S))} w(x) \\
&= \sum_{x\in S} w(x) + \sum_{x\in S}(1 - w(x) - w(S^{\psi(x)})) + \sum_{x\in R-(S\cup\varphi(S))} w(x) \\
&= \sum_{x\in S}(1 - w(S^{\psi(x)})) + \sum_{x\in R-(S\cup\varphi(S))} w(x) \\
&\leq \frac{2}{3}|S| + \frac{1}{3}|R - S\cup\varphi(S)|
\end{aligned}$$

(note that $S^{\psi(x)}\cdot x\cdot\varphi(x) \in supp(f)$ for all x, i.e. $w(S^{\psi(x)})+w(x)+w(\varphi(x)) = 1$; also, $\varphi(S) \subseteq R$). Thus

$$\frac{1}{2}(|R|-1) \leq \frac{1}{3}(2|S| + |R - (S\cup\varphi(S))|) = \frac{1}{3}|R|.$$

To prove (iv), we may assume $w_0 + w_1 = \frac{1}{2}$, $w_i = \frac{1}{2}$ for $i > 1$, i.e. for $w_0 = w_1$ we are done. Assume $w_0 > w_1$, then $\frac{1}{4} < w_0 < \frac{1}{2}$. If $X_0^3 \notin supp(f)$, then no power of X_0 is in $supp(f)$, and (i) (a) in the Proposition implies (using $Y = X_0$) that one of the monomials $X_0^{\alpha+1}X_i$ ($\alpha \in \mathbb{N}$, $i \in \{1,\ldots,n\}$) is in $supp(f)$. This implies $w_0 = \frac{1}{2\alpha}$ or $w_0 = \frac{1}{2(\alpha+1)}$ (contradiction, since $\alpha \in \mathbb{N}$). Thus $X_0^3 \in supp(f)$, i.e. $w_0 = \frac{1}{3}$, $w_1 = \frac{1}{6}$.

Now, a detailed case by case analysis gives the

Proof of the Theorem:

Choose e.g. the case of $\tilde{E}_6$ in even dimension, i.e. here without loss of generality in dimension 2. Then in coordinates $(x0 : x1 : x2)$, the corresponding equation $f = 0$ defines a smooth curve C of degree 3 in the projective plane. We obtain the above normal form after a linear change of coordinates. In *char* $k = 2$ we have two cases: $\tilde{E}_6(0)$ if the elliptic curve is supersingular, $\tilde{E}_6(t)$, with $t \neq 0$ otherwise.

For the weights $\tilde{E}_7$, $\tilde{E}_8$, a geometric analysis of the relevant forms is necessary, giving different equations in even and odd dimensions for *char* $k = 2$.
We apply the proposition to obtain the list of equations; choose e.g. f of weight $\tilde{E}_7$, *char* $k = 2$:

We may assume $X = \{X_0, \ldots, X_n\}$ with

(a) $n = 1$, $w_0 = w_1 = \frac{1}{4}$, $f = f(X_0, X_1)$ homogeneous of degree 4 and defining an isolated singularity, i.e. f with 4 different zeroes on $\mathbb{P}^1$.
(b) $n = 2$, $w_0 = \frac{1}{2}$, $w_1 = w_2 = \frac{1}{4}$, $f = x_0^2 + gx_0 + h$, $g \in k[X_1, X_2]$ homogeneous of degree 2, $h \in k[X_1, X_2]$ homogeneous of degree 4.

If (b1) $g = 0$, then coordinates can be choosen such that $X_1X_2^3 \notin supp\, h$, thus $V(X_1, X_0^2 + g(X_1, X_2)) \subseteq sing(f)$, i.e. the singular locus has positive dimension. Now assume (b2) $g = X_1^2$, then $f = X_0^2 + X_0X_1^2 + h(X_1, X_2)$. Write $h(X_1, X_2) = \sum_{\nu=0}^{4} h_\nu X_1^\nu X_2^{4-\nu}$. Then f defines an isolated singularity if $h_1 \neq 0$; we may assume $h_1 = 1$. A coordinate transformation $X_0 := X_0 + aX_1^2 + bX_1X_2 + cX_2^2$ brings h into the form $h = X_1X_2^2(tX_1 + X_2)$.

The case (b3) $g = X_1X_2$ is done in a similar way.

Remark: Note that also in *char* $k = 2$, the equations for $\tilde{E}_6$ can be written in a form such that $\tilde{E}_6(0)$ and $\tilde{E}_6(t)$, $t \neq 0$ are in the same 1-parameter family: Take $n = 2$ and let $C(s)$ be the curve defined in the projective plane by

$$X_0^3 + X_1^3 + X_2^3 + sX_0X_1X_2 = 0$$

where $s \in k$. For $s^3 \neq 1$ this is an elliptic curve with absolute invariant $j = \frac{s^{12}}{(s^3+1)^3}$, and $\tilde{E}_6(0)$ is the cone over an elliptic curve with invariant 0, thus isomorphic to the cone over $C(0)$. For fixed $t \neq 0$, the equation $ts^{12} + s^9 + s^6 + s^3 + 1 = 0$ has 12 different solutions s. We obtain several $C(s)$ with invariant $j = \frac{1}{t}$.
Thus any 2-dimensional quasi-homogeneous singularity of type $\tilde{E}_6$ is obtained as cone over some $C(s)$.

Corollary: Let $f \in k[X]$ be quasi-homogeneous of some weight $w = w(f)$ and assume $s = s(f) \leq 1$. Then w is uniquely determined up to permutation in the class of quasi-homogeneous functions which are stable equivalent f. Especially, the number s is well defined on the equivalence class.

Remark: In the case considered here, w (up to permutation) and therefore $s(f)$ depends only on the complete local ring of the singularity. It is not known to the author, if this is generally so for $s(f) > 1$ (but it is always true for $k = \mathbb{C}$ by [10]).

2. Normal forms of semi-quasi-homogeneous functions

Now let $f = f_1 + f_{>1}$ be a formal power series which contains no monomials of weight < 1 with respect to the given weight w. Put $f_1 :=$ sum of terms of weight 1 in f and assume f_1 defines an isolated singularity.

f is said to be contact equivalent with a power series g, if the k-algebras $k[[X]]/(f)$ and $k[[X]]/(g)$ of formal power series are isomorphic.

The following result reduces the part $f_{>1}$ into a normal form without changing f_1 and the contact equivalence class of f. $T(f_1)$ denotes the "Tjurina-algebra",

$$T(f_1) := k[[X]]/(f_1, \frac{\partial f_1}{\partial X_0}, \dots, \frac{\partial f_1}{\partial X_n}).$$

We have $dim_k(T(f_1) < \infty$.

Theorem: Let $(\overline{e_1}, \dots, \overline{e_s})$ denote any maximal linear independent set of classes in $T(f_1)$ of monomials e_i having weight > 1 ("superdiagonal monomials"). Then $f = f_1 + f_{>1}$ is contact equivalent with $f_1 + c_1 e_1 + \cdots + c_s e_s, \quad c_i \in k$.

Proof: Let $w = (\frac{m_0}{d}, \dots, \frac{m_n}{d})$ with positive integers m_i, d. Denote $o_m(h)$ the total order of the initial term of a power series $h \in k[[X_0, \dots, X_n]]$ with respect to $(m_0, \dots, m_n)$.
If the classes of superdiagonal monomials $\{e_1, \dots, e_s\}$ form a basis of the subspace generated by all superdiagonal monomials in the Tjurina algebra $T(f_1)$, then the similar assertion is true for any fixed order d', i.e. let $\{e_{i_1}, \dots, e_{i_q}\}$ be the subset of monomials such that $o_m(e_{i_j}) = d'$, then this is a basis for the subspace in $T(f_1)$ generated by the classes of all monomials having $o_m = d'$ (f_1 is homogeneous).
Obviously, an inductive convergence argument gives the result, if we show the following

Lemma: Let (after some permutation) $e_1, \dots, e_r$ be the monomials of order $o_m(e_i) = d' > d$ in $\{e_1, \dots, e_s\}$. Then f is contact equivalent with a series

$$f_1 + f'_{>1} + \sum_{i=1}^{r} c_i e_i + h,$$

where $f'_{>1}$ is the sum of terms of order $o_m < d'$ in $f_{>1}$, $c_i \in k$ and $h \in k[[X_0, \dots, X_n]]$ has order $o_m(h) > d'$.
(Note that the case is included, where $\{e_1, \dots, e_r\}$ is the empty set.)

Proof of the Lemma: Choose $c_i \in k$ such that

$$g - \sum_{i=1}^{r} c_i e_i = q \cdot f_1 + \sum_{i=0}^{n} v_i \frac{\partial f_1}{\partial X_i}$$

for some q, $v_i \in k[[X_0, \dots, X_n]]$ and g the sum of monomials of order d' in $f_{>1}$.

Without loss of generality, q and v_i are quasi-homogeneous for $(m_0, \dots, m_n)$ of order

$$o_m(q) = d' - d =: \delta > 0$$
$$o_m(v_i) = d' - (d - m_i) = \delta + m_i > m_i,$$

respectively. We obtain

$$(*) \quad f_1 + f'_{>1} + \sum_{i=0}^{r} c_i e_i = (1-q)(f_1 + f_{>1}) + q f_{>1} - \sum_{i=0}^{n} v_i \frac{\partial f_1}{\partial X_i} + p,$$

where in the right hand term $o_m(qf_{>1}) > d'$, $v_i \frac{\partial f_1}{\partial X_i}$ is quasi-homogeneous with $o_m(v_i \frac{\partial f_1}{\partial X_i}) = d'$, and $o_m(p) > d'$.
Assume without loss of generality $m_0 \geq m_1 \geq \cdots \geq m_n$. Let $X_i := X'_i - v_i(X')$, then $o_m(v_i) > m_i = o_m(X_i)$ implies: The linear part of this coordinate transformation has a lower triangular matrix (a_{ij}) with $a_{ii} = 1$, and $a_{ij} \neq 0$ for $i > j$ is possible only if $m_i > m_j$. The above substitution sends

$$(**) \quad f_1(X) \mapsto \quad f_1(X') - \sum_{i=0}^{r} v_i(X') \frac{\partial f_1(X')}{\partial X'_i} + \text{ terms in } X' \text{ of order } o_m \ > d'$$

(if we take the same weights for the X').
By $(*)$, we have

$$(***) \quad (1 - q(X))f(X) \equiv f_1 + \sum_{i=0}^{n} v_i(X) \frac{\partial f_1}{\partial X_i} + (f'_{>1}(X) + \sum_{i=0}^{r} c_i e_i(X))$$

mod terms of order $o_m \ > d'$. If we apply $(**)$ and remember $o_m(v_i) > d' - d$, the substitution above transforms the right hand side of $(***)$ into

$$f_1(X') + (f_{>1}(X') + \sum_{i=0}^{r} c_i e_i(X')) + h(X'),$$

where $o_m(h) > d'$. This completes the proof.

Note that we do not need any assumption on *char* k. If *char* $k = 0$, by Euler's formula we have $f_1 \in (\frac{\partial f_1}{\partial X_0}, \ldots, \frac{\partial f_1}{\partial X_n})$, i.e. the Tjurina-algebra $T(f_1)$ coincides with the Milnor-algebra $M(f_1) = k[[X]]/(\frac{\partial f_1}{\partial X_0}, \ldots, \frac{\partial f_1}{\partial X_n})$, and in this case the result coincides with ([1], 12.6).

3. Results in the 1-semi-quasi-homogeneous case

Using a computer[3], from the theorem in section 2. we obtain easily:

semi-quasi-homogeneous singularities with $s = 1$ in characteristic 2			
type	Tjurina-number	maximal set of linearly independent superdiagonal monomials	total number
case 1: dimension $\equiv 1$ *mod* 2			
$\tilde{E}_6(0)$	16	$X_0X_1X_3,\ X_1X_2X_3,\ X_0X_2X_3,\ X_0X_1X_2X_3$	4
$\tilde{E}_6(t)$	16	$X_1^2X_3,\ X_2^2X_3,\ X_1X_2X_3,\ X_2^3X_3$	4
$\tilde{E}_7$	9	$\emptyset$	0
$\tilde{E}_8$	12	$X_0X_1^5$	1
case 2: dimension $\equiv 0$ *mod* 2			
$\tilde{E}_6(0)$	8	$\emptyset$	0
$\tilde{E}_6(t)$	8	$\emptyset$	0
$\tilde{E}_{7,1}(t)$	10	$\emptyset$	0
$\tilde{E}_{7,2}(t)$	10	$\emptyset$	0
$\tilde{E}_8(t)$	10	$\emptyset$	0

Thus e.g. for n odd, the 1-sqh singularities with first term $\tilde{E}_8$ (as in the theorem of section 1) are given by adding a constant multiple the monomial $X_0X_1^5$. If the coefficient is not zero, an easy coordinate transformation leads to the only non quasi-homogeneous 1-sqh singularity of that weight; it is given by the equation $X_0(X_0 + X_1^2)(X_0 + tX_1^2) + X_0X_1^5 = 0$ $(t \notin \{0,1\})$ with Tjurina number 11.

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SOME EXAMPLES OF RICKARD COMPLEXES

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Abstract. After a presentation of Broué's conjecture for principal blocks with an abelian defect group, we describe a Rickard complex for $GL_2(q)$ arising from the ℓ-adic cohomology of a Deligne-Lusztig variety, in accordance with the explicit form given by Broué to his conjecture in the case of Chevalley groups in non natural characteristic.

1. Overview of Broué's conjecture

Let G be a finite group and ℓ a prime number. Let P be a Sylow ℓ-subgroup of G and assume P is abelian. Let $H = N_G(P)$. Let $\mathcal{O}$ be the ring of integers of a finite unramified extension K of $\mathbf{Q}_\ell$, such that KG and KH are split. Let A and B be the principal blocks of G and H over $\mathcal{O}$. Let us denote by H° the group opposite to H. Similarly, B° denotes the algebra opposite to B. We put $\Delta P = \{(x, x^{-1}) | x \in P\} \leq G \times H^\circ$. The sign $\otimes$ means $\otimes_{\mathcal{O}}$. Finally, if M is an $\mathcal{O}$-module, we put $KM = K \otimes M$.

Conjecture 1.1. *The blocks A and B are Rickard equivalent. More precisely, there is a complex C of (left) $A \otimes B^\circ$-modules which are direct summands of relatively ΔP-projective permutation modules such that:*

$$C^* \otimes_A C \simeq B \text{ in } K^b(B \otimes B^\circ)$$
$$C \otimes_B C^* \simeq A \text{ in } K^b(A \otimes A^\circ).$$

For the sake of simplicity and for the lack of a final form of the conjecture in the general case, we have stated the conjecture for principal blocks only. The original statement of the conjecture [Br1] makes no assumption on C. That C should be of this special type (" splendid ") appeared in [Ri4].

The conjecture is known (to the author) to hold in the following cases:

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- P cyclic [Ri1, Li, Rou];
- $G = \mathbf{G}(\mathbf{F}_q)$ the group of rational points of a connected reductive algebraic group, when $\ell | q-1$ but ℓ does not divide the order of the Weyl group [Pu];
- $G = A_5$ and $\ell = 2$ [Ri4];
- $G = SL_2(8)$ and $\ell = 2$ [Rou];
- G is ℓ-solvable.

For Chevalley groups, when ℓ is not the natural characteristic of the group, there is a very precise conjecture of Broué giving a candidate for C, in terms of ℓ-adic cohomology of certain Deligne-Lusztig varieties [Br-Ma]. The aim of the second part is to present the simplest case of this conjecture.

2. A geometrical construction for $GL_2(q)$, $\ell|(q+1)$

Let q be a prime power. Consider the affine curve X with equation $(xy^q - x^q y)^{q-1} = -1$ over an algebraic closure $\bar{\mathbf{F}}_q$ of $\mathbf{F}_q$. The group $G = GL_2(\mathbf{F}_q)$[1] acts naturally on the affine plane over $\bar{\mathbf{F}}_q$ and this induces an action of G on X. There is also an action of the group of rational points $T \simeq \mathbf{F}_{q^2}^*$ of a Coxeter torus of $GL_2(\bar{\mathbf{F}}_q)$ by scalar multiplication (in the isomorphism above, F acts on T as $x \mapsto x^q$ on $\mathbf{F}_{q^2}^*$). Finally, the variety X is defined over $\mathbf{F}_q$, with corresponding Frobenius endomorphism F.

This variety is actually the Deligne-Lusztig variety associated to the non trivial element of the Weyl group[2] [De-Lu, 2.2].

Let $\ell | q+1$ be an odd prime and $\mathcal{O}$ be the ring of integers of a finite unramified extension K of $\mathbf{Q}_\ell$, such that KG and KH are split, where $H = N_G(T) = T \rtimes W$ and $|W| = 2$. Let P be the ℓ-Sylow subgroup of T.

Our object of study is the complex $\mathrm{R}\Gamma_c(X, \mathcal{O})$ ($\mathcal{O}$ is the constant ℓ-adic sheaf) giving rise to the compact support ℓ-adic cohomology: this is an object in the derived category of $(\mathcal{O}G) \otimes (\mathcal{O}T)^\circ$-modules. Actually, we will consider the finer invariant $C = \Lambda_c(X, \mathcal{O})$ in the *homotopy* category of $(\mathcal{O}G) \otimes (\mathcal{O}T)^\circ$-modules, as defined by J.Rickard [Ri3]. This is a complex of direct summands of permutation $\mathcal{O}(G \times T^\circ)$-modules. Note that there is an action of the Frobenius F on C, giving rise to a right action of $\mathcal{O}T \rtimes F$.

Let e be the sum of the ℓ-blocks with positive defect of $\mathcal{O}G$. Define $A = \mathcal{O}Ge$ and $B = \mathcal{O}H$.

Proposition 2.1. [3] *The action of $\mathcal{O}T \rtimes F$ on C factors through an action of an algebra isomorphic to B. The action of $\mathcal{O}G$ on C factors through an*

[1] In the talk, the case of $SL_2(q)$ had been considered, where the same methods apply.

[2] This is a Coxeter variety, *i.e.*, the variety associated to a Coxeter element of the Weyl group. These varieties should be studied by the author in a future paper.

[3] The proposition actually holds for $\mathcal{O}$ replaced by $\mathbf{Z}_\ell$, as suggested by K.W.Roggenkamp.

action of A : the complex C is then a complex of direct summands of relatively ΔP-projective permutation modules. We have

$$C^* \otimes_A C \simeq B \text{ in } K^b(B \otimes B^\circ) \text{ and}$$
$$C \otimes_B C^* \simeq A \text{ in } K^b(A \otimes A^\circ).$$

Proof. Since X is an affine curve, the cohomology groups $H_c^i(X, \mathcal{O})$ are zero for $i = 0$ and $i > 2$. Since X is in addition smooth, the cohomology groups $H_c^1(X, \mathcal{O})$ and $H_c^2(X, \mathcal{O})$ are free as $\mathcal{O}$-modules [SGA4$\frac{1}{2}$, Arcata, III.§3].

Since both G and T act freely on X, the complex C is perfect (*i.e.* isomorphic to a bounded complex of projective modules) as an object of $\mathcal{D}^b(\mathcal{O}G)$ and as an object of $\mathcal{D}^b(\mathcal{O}T)$ [De-Lu, (proof of) 3.5].

The representation of $G \times T^\circ$ on $H_c^2(X, \mathcal{O})$ is isomorphic to the permutation representation on the connected components of X. Its character is

$$\sum_{\alpha \in \mathrm{Irr}(\mathbf{F}_q^*)} \det_\alpha \otimes \alpha$$

where $\det_\alpha$ is the character $\alpha \circ \det$ of G. The Frobenius morphism F acts with the eigenvalue q on $H_c^2(X, \mathcal{O})$.

The character of the $KG \otimes (KT)^\circ$-module $H_c^1(X, K)$ is [Di-Mi2, 15.9]:

$$\sum_{\alpha \in \mathrm{Irr}(\mathbf{F}_q^*)} \mathrm{St}_\alpha \otimes \alpha + \sum_{\substack{\omega \in \mathrm{Irr}(\mathbf{F}_{q^2}^*)/W, \\ \omega^{q-1} \neq 1}} [q-1]_\omega \otimes (\omega + \omega^q)$$

where $\mathrm{St}_\alpha = \mathrm{St} \cdot \det_\alpha$, St is the Steinberg character of G and

$$\{[q-1]_\omega\}_{\omega \in \mathrm{Irr}(\mathbf{F}_{q^2}^*)/W,\ \omega^{q-1} \neq 1}$$

is the set of irreducible characters of G with degree $q-1$. The Frobenius F acts with the eigenvalue 1 on the G-isotypic component with character St_α and with eigenvalues $\sqrt{-q}$ and $-\sqrt{-q}$ on the component with character $[q-1]_\omega$ (this is a consequence of Lefschetz formula, [Di-Mi1, V.1.3]). From this description of the character of $H^*(C)$, it follows that $\mathcal{O}G$ acts on C through A.

Let $\sigma \in KT \rtimes F$ defined by

$$\sigma = \frac{i}{q-1}(2F - 1 - q) + \frac{i-1}{\sqrt{-q}} F$$

where $i = \frac{1}{q^2-1} \sum_{t \in T} t^{q-1}$ and where $\sqrt{-q} \in \mathcal{O}$ is chosen such that $\ell | 1 - \sqrt{-q}$. Note that σ is actually in $\mathcal{O}T \rtimes F$, since

$$\sigma = \frac{-1}{(q-1)^2} \sum_{t \in T} t^{q-1} - \frac{1}{\sqrt{-q}} F + \frac{1 - \sqrt{-q}}{\sqrt{-q}(q-1)^2(1+\sqrt{-q})} \sum_{t \in T} t^{q-1} F.$$

For $t \in T$, we have $\sigma t = t^q \sigma$ since $Ft = t^q F$. Now, we see that σ acts trivially on $H_c^2(X, K)$, with eigenvalue -1 on the G-isotypic components of $H_c^1(X, K)$

with character St_α and with eigenvalues 1 and -1 on the G-isotypic components with character $[q-1]_\omega$; in particular, σ^2 acts trivially on $H_c^*(X,K)$. Finally, the image in $\mathrm{End}_{K^b(KA)}(KC)$ of the sub-algebra of $KT \rtimes F$ generated by T and σ is isomorphic to KH. But it is clear that $\mathrm{End}_{K^b(KA)}(KC)$ is isomorphic to KH : this means that the image in $\mathrm{End}_{K^b(KA)}(KC)$ of the sub-algebra of $KT \rtimes F$ generated by T and σ is actually the image of $KT \rtimes F$. Hence, we have proven the analog of the proposition where scalars are extended to K.

For $\alpha \in \mathrm{Irr}(\mathbf{F}_q^*)$, let e_α be the sum of the blocks of G containing the characters $[q-1]_\omega$ with $\omega^{q+1} = \alpha$ and the characters $\det_\alpha$ and St_α. Let $C_\alpha = e_\alpha C$. Since $H^i(C)$ is free over $\mathcal{O}$ and C is perfect in $\mathcal{D}^b(\mathcal{O}G)$, it is isomorphic to a complex of projective modules $0 \to C^1 \xrightarrow{\varphi} C^2 \to 0$ as an $\mathcal{O}G$-module. Let us choose C^1 such that $\mathrm{Im}\,\varphi$ has no projective direct summand. Put $C_\alpha^i = e_\alpha C^i$. Then, C_α^2 is a projective cover of $\det_\alpha$, since $H^2(C_\alpha) \simeq \det_\alpha$. Now, C_α splits as

$$0 \to C'^1_\alpha \to C^2_\alpha \to 0 \oplus 0 \to C"^1_\alpha \to 0 \to 0$$

where $C'^1_\alpha \to C^2_\alpha$ is the beginning of a projective resolution of $\det_\alpha$. From this description and from the knowledge of the characters of $C'^1_\alpha, C"^1_\alpha$ and C^2_α, it follows that C is a tilting complex for A.

Let B' be the image of the sub-algebra of $\mathcal{O}T \rtimes F$ generated by T and σ in $\mathrm{End}_{K^b(A)}(C)$. The algebra B' is isomorphic to $\mathcal{O}H$ and C is perfect in $\mathcal{D}^b(B')$, since it is perfect in $\mathcal{D}^b(\mathcal{O}T)$.

A proof similar to the one above shows that C is a tilting complex for B'. Now, by [Br2, théorème 2.3], this implies that C is a two-sided tilting complex for $A \otimes B^\circ$, *i.e.*, the isomorphisms of the proposition hold in the derived categories and a priori not in the homotopy categories. Note that we have obtained that B' is the whole of $\mathrm{End}_{K^b(A)}(C)$, hence B' is the image of $\mathcal{O}T \rtimes F$ in $\mathrm{End}_{K^b(A)}(C)$.

If S is a non trivial ℓ-subgroup of $G \times T^\circ$ which is not conjugate to ΔP, then S acts freely on X, hence by [Ri3, Corollary 3.3], C is a complex of direct summands of relatively ΔP-projective permutation modules. If S is a non trivial ℓ-subgroup of $G \times T^\circ$, then the fixed points set X^S has dimension zero, hence $\Lambda_c(X^S, \mathcal{O})$ is concentrated in degree 0. Hence, by [Ri3, Theorem 4.2], C is homotopic to a bounded complex of modules which are all projective $A \otimes B^\circ$-modules, except C^0 ; this implies that the isomorphisms of the proposition hold indeed in the homotopy category [Ri2, (proof of) Corollary 5.5]. □

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DOUBLING A PATH ALGEBRA, OR: HOW TO EXTEND INDECOMPOSABLE MODULES TO SIMPLE MODULES

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Introduction

For a field $\mathfrak{k}$ of characteristic 0 and a representation-finite quiver $\vec{\Delta}$, it has been observed [8] that the indecomposable $\mathfrak{k}$-linear $\vec{\Delta}$ - representations are independent of the chosen orientation. In fact, they can be associated [6] to the positive roots of the corresponding semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}(\Delta)$, where Δ denotes the unoriented graph underlying $\vec{\Delta}$. Thus if we consider the graph $\overleftrightarrow{\Delta}$ with pairs of arrows in opposite direction for every edge of Δ, we may ask whether there exists a $\overleftrightarrow{\Delta}$-representation which induces the indecomposables for each orientation $\vec{\Delta}$ of Δ. Of course, for a reasonable solution, one should expect that the maps representing the arrows of $\vec{\Delta}$ are connected with the opposite maps by a suitable relation, and the resulting $\overleftrightarrow{\Delta}$-representation should be more or less unique.

Our first aim in this paper is to show that this problem is indeed solvable (Theorem 1.1) and admits a unique solution depending on a fixed element α in the root space $\mathfrak{k}\Delta_0$ which does not ly on a wall of the Weyl chambers. (For arbitrary $\alpha \in \mathfrak{k}\Delta_0$, the $\overleftrightarrow{\Delta}$-representation is still unique, but for some orientations $\vec{\Delta}$, the induced representation might decompose.) The mentioned relation (see (1) below) between the maps representing $\vec{\Delta}$ and the opposite maps is well-known in the particular case $\alpha = 0$: Then it turns into the defining relation of the preprojective algebra [10, 7] of the graph Δ. Our construction in §2 therefore leads to a semisimple deformation A of the preprojective algebra, which, as we shall see, can be regarded as a "double" of the path algebra $\mathfrak{k}\vec{\Delta}$. We shall prove that the indecomposable $\mathfrak{k}\vec{\Delta}$-representations correspond to the simple A-modules.

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There are quite different contexts where quivers of the form $\overset{\leftrightarrow}{\Delta}$ and relations similar to (1) have occurred. We thank the referee for directing our attention to some of them. McKay [15] observed that for a binary polyhedral group G, if the two-dimensional irreducible representation R (over $\mathbb{C}$) is tensored with all irreducibles $R_1, \ldots, R_m$, say $R \otimes R_i = \coprod R_j^{(a_{ij})}$, the oriented graph with adjacency matrix (a_{ij}) is of the form $\overset{\leftrightarrow}{\Delta}$, with an extended Dynkin diagram Δ. An explanation is provided, among others, by M. Auslander [3] (see also [4]). He establishes a one-to-one correspondence between $R_1, \ldots, R_m$ and the indecomposable projective modules over the twisted group ring $S[G]$, where $S = \mathbb{C}[[x,y]]$, and shows that the McKay quiver $\overset{\leftrightarrow}{\Delta}$ coincides with the Gabriel quiver of $S[G]$. Now if $\overset{\leftrightarrow}{\Delta}$ is regarded as a translation quiver with the identity as translation, the relations in $\overset{\leftrightarrow}{\Delta}$ given by the ring structure of $S[G]$ are just the mesh relations, that is, the relations (1) with $\alpha = 0$.

On the other hand, the inhomogenous equations (1) are connected with the resolution of singularities, and with minimum action solutions of SU_2 Yang-Mills fields, also called instantons or pseudo-particles. Atiyah and Ward [2] have shown that self-dual instantons in $\mathbb{R}^4$ correspond to certain two-dimensional algebraic vector bundles over $P_3(\mathbb{C})$, and in [1] these bundles have been constructed in terms of pure linear algebra. A quadratic equation between matrices, similar to (1), results as a defining relation.

Every binary polyhedral group G, viewed as a subgroup of $SU_2(\mathbb{C})$, gives rise to a quotient singularity $\mathbb{C}^2/G$ with intersection matrix of Dynkin type. If X_G denotes the 4-manifold underlying the minimal resolution of $\mathbb{C}^2/G$, Peter Kronheimer has shown [11] that X_G admits the structure of an ALE hyper-Kähler 4-manifold, that is, a self-dual gravitational instanton. Here, ALE (= asymptotically locally Euclidean) signifies that at infinity, X_G resembles $\mathbb{R}^4/G$, and the Riemannian metric is Euclidean up to $O(r^{-4})$. "Hyper-Kähler" means that X_G is equipped with three covariant-constant complex structures I, J, K connected by the relations of quaternion units. Moreover, he proved that every ALE hyper-Kähler 4-manifold is diffeomorphic to some X_G. In his construction of X_G, Kronheimer starts with the manifold $M = (R \otimes \mathrm{End}\mathbb{C}G)^G$, on which the group F of G-invariant unitary transformations of $\mathbb{C}G$ operates. Then X_G is obtained as a hyper-Kähler quotient M'/F, where $M' \subset M$ is defined by a quadratic relation similar to (1). The left-hand side of this relation plays the rôle of a (hyper-Kähler) moment map. In a similar way, relation (1) enters into the definition of certain varieties of quivers considered by Lusztig ([12], §8; [13], §12) and Nakajima's quiver varieties [16, 17].

Our subsequent article will be self-contained. No use is made of the above mentioned connections.

1. Extensions of indecomposables

Let $\mathfrak{k}$ be a field of characteristic 0 and $\vec{\Delta}$ a finite oriented graph with vertex set $\Delta_0 = \{1, \dots, n\}$. A *representation* (V, f) of $\vec{\Delta}$ over $\mathfrak{k}$ is defined as a family $V = (V_1, \dots, V_n)$ of finite dimensional $\mathfrak{k}$-vector spaces and a family f of $\mathfrak{k}$-linear maps $f_{ji} : V_i \to V_j$ for each arrow $i \to j$ in $\vec{\Delta}$. Let $\mathfrak{k}\vec{\Delta}$ denote the *path algebra* of $\vec{\Delta}$,i.e. the $\mathfrak{k}$-algebra with paths $p : i_0 \to i_1 \to \dots \to i_r$ in $\vec{\Delta}$ as basis such that, for another path $q : j_0 \to \dots \to j_s$ in $\vec{\Delta}$, the product qp is given by the composition $i_0 \to \dots \to i_r \to j_1 \to \dots \to j_s$ if $i_r = j_0$, and $qp = 0$ otherwise. Then in an obvious way, each $\vec{\Delta}$-representation (V, f) can be regarded as a finitely generated $\mathfrak{k}\vec{\Delta}$-module. The vector $\mathbf{dim}(V, f) = (d_1, \dots, d_n) \in \mathbb{N}^n$ with $d_i = \dim_{\mathfrak{k}} V_i$ is called the *dimension vector* of (V, f).

Gabriel's theorem [8] states that the number of (isomorphism classes of) indecomposable $\vec{\Delta}$-representations (i.e. $\mathfrak{k}\vec{\Delta}$-modules) is finite if and only if the unoriented graph Δ underlying $\vec{\Delta}$ is a disjoint union of Dynkin diagrams. Moreover, the set of dimension vectors of indecomposable $\vec{\Delta}$-representations coincides with the set Φ^+ of positive roots of the semisimple Lie algebra $\mathfrak{g}$ of type Δ. Thus for a given root $\mathbf{d} \in \Phi^+$, there is an indecomposable representation with dimension vector $\mathbf{d}$ for each orientation $\vec{\Delta}$ of Δ.

Now let Δ be a Dynkin diagram. If $\overleftrightarrow{\Delta}$ denotes the oriented graph with arrows $i \to j$ and $j \to i$ for each edge $i - j$ in Δ, the question arises for a given dimension vector $\mathbf{d} \in \Phi^+$, whether there exists a $\overleftrightarrow{\Delta}$-representation which induces the indecomposable $\vec{\Delta}$-representations of all orientations via the natural embedding $\mathfrak{k}\vec{\Delta} \hookrightarrow \mathfrak{k}\overleftrightarrow{\Delta}$.

Our first result (Theorem 1.1) will give a solution to this problem, including an explicit construction of the $\mathfrak{k}\overleftrightarrow{\Delta}$-modules in question. Furthermore, we shall prove that these $\mathfrak{k}\overleftrightarrow{\Delta}$-modules can be regarded as simple modules over a semisimple algebra associated with Δ. Let us define a Δ-*representation* of *type* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathfrak{k}^n$ as a $\overleftrightarrow{\Delta}$-representation (V, f) which satisfies for each $i \in \Delta_0$ the relation

$$\sum_j f_{ij} f_{ji} = \alpha_i \cdot 1_{V_i} \tag{1}$$

where j runs over the *vicinity* $\mathrm{V}(i)$ of i, that is, the set of vertices j adjacent to i in Δ. If $\mathbf{d}$ is the dimension vector of (V, f), the following relation necessarily holds for the α_i:

$$\sum \pm d_i \alpha_i = 0. \tag{2}$$

Here, the + and − signs are chosen according to any fixed partition $\Delta_0 = \Delta_0^+ \cup \Delta_0^-$ such that adjacent vertices belong to different signs. (Since Δ is a tree, there are only two such partitions!) In fact, if we take the trace on both sides of (1), we get

$$\sum_{j \in V(i)} d_{ij} = \alpha_i d_i$$

where $d_{ij} := \operatorname{tr}(f_{ij} f_{ji}) = d_{ji}$. Thus (2) immediately follows. For a fixed partition $\Delta_0 = \Delta_0^+ \cup \Delta_0^-$, let us define $\Phi_{\mathbf{d}} \in \mathfrak{k}[x_1, \ldots, x_n]$ by

$$\Phi_{\mathbf{d}} = \sum \pm d_i x_i = \sum_{i \in \Delta_0^+} d_i x_i - \sum_{i \in \Delta_0^-} d_i x_i. \tag{3}$$

Theorem 1.1. *Let Δ be a Dynkin diagram and $\alpha \in \mathfrak{k}^n$ with $\Phi_{\mathbf{d}}(\alpha) = 0$. Then every indecomposable $\vec{\Delta}$-representation (V, f) with dimension vector $\mathbf{d}$ has a unique extension to a Δ-representation $(\tilde{V}, \tilde{f})$ of type α. Moreover, the $\tilde{f}_{ij}$ depend polynomially on α.*

As a consequence, we get the solution to the above question:

Corollary 1.2. *Let (V, f) be a Δ-representation of dimension vector $\mathbf{d} \in \Phi^+$ and type α such that $\sum \pm d'_i \alpha_i \neq 0$ for all $\mathbf{d}' \neq \mathbf{d}$ in Φ^+. Then the induced $\vec{\Delta}$-representations of all orientations are indecomposable.*

Note that this choice of α is possible since $\mathfrak{k}$ is infinite. The corollary follows immediately by a proposition which we shall prove together with Theorem 1.1:

Proposition 1.3. *Let $\vec{\Delta}$ be an orientation of Δ and $\tilde{M}$ a Δ-representation which extends some $\vec{\Delta}$-representation M. Then $\tilde{M}$ has a subrepresentation with underlying $\vec{\Delta}$-representation S such that S is an indecomposable direct summand of M.*

2. The double of a path algebra

Our second purpose of this paper is to show that by Theorem 1.1, the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ of a semisimple Lie algebra $\mathfrak{g}$ has an analogue for associative algebras. Namely, the nilpotent Lie algebra $\mathfrak{g}^+$ corresponds to our path algebra $A^+ = \mathfrak{k}\vec{\Delta}$, the negative part $\mathfrak{g}^-$ to $A^- = (\mathfrak{k}\vec{\Delta})^{op}$, and the abelian part $\mathfrak{g}^0$ is replaced by some commutative algebra A^0 generated by n elements $\alpha_1, \ldots, \alpha_n$. The semisimple Lie algebra $\mathfrak{g}$ corresponds to a semisimple algebra A containing an order Λ_0 generated by A^+, A^-, and A^0.

Firstly, the Δ-representations over some extension field of $\mathfrak{k}$ can be interpreted as modules over the $\mathfrak{k}$-algebra

$$\Lambda = R_* \overset{\leftrightarrow}{\Delta} / I \tag{4}$$

where R_* is the polynomial ring $\mathfrak{k}[x_1, \dots, x_n]$ and I the principal ideal generated by

$$\sum e_{ij}e_{ji} - \sum x_i e_i. \tag{5}$$

Here e_{ij} denotes the path $j \to i$, and e_i is the primitive idempotent (empty path) corresponding to the vertex $i \in \Delta_0$. The first sum in (5) is to be extended over the edges i—j, the second over the vertices i in Δ. The residue class of x_i modulo I will be denoted by α_i. Thus $\alpha_1, \dots, \alpha_n \in \Lambda$ generate a subring $R = \mathfrak{k}[\alpha_1, \dots, \alpha_n]$ in the center of Λ, and Λ becomes an R-algebra.

Recall that an ideal P in a ring Γ is said to be *prime* if for any two ideals I, J in Γ, the inclusion $P \supset I \cdot J$ implies $P \supset I$ or $P \supset J$. An element $r \in \Gamma$ is called *regular* if the left and right multiplication by r is injective. A subring Γ of $\tilde{\Gamma}$ is said to be an *order* in $\tilde{\Gamma}$ if each regular $r \in \Gamma$ is invertible in $\tilde{\Gamma}$, and every element of $\tilde{\Gamma}$ is of the form $r^{-1}a$, and also of the form ar^{-1}, with $r, a \in \Gamma$ and r regular. The intersection $\mathrm{N}(\Gamma)$ of all prime ideals in Γ is called the *prime radical* ([19], chap. XV), and Γ is said to be *semiprime* if $\mathrm{N}(\Gamma) = 0$.

Now we define $\Lambda_0 := \Lambda/\mathrm{N}(\Lambda)$. The natural homomorphism $\Lambda \twoheadrightarrow \Lambda_0$ maps R onto a central subring $R_0 = \mathfrak{k}[\bar{\alpha}_1, \dots, \bar{\alpha}_n]$ of Λ_0, generated by the residue classes of the α_i modulo $\mathrm{N}(\Lambda)$. Let $\Phi \in R_*$ denote the product of all $\Phi_{\mathrm{d}}, d \in \Phi^+$. By virtue of (5), the defining relations ($e_i^2 = e_i$, $e_i e_{ij} = e_{ij}$, etc.) of the e_{ij} which encode the graph structure of Δ, are turned into a single relation $\Phi(\bar{\alpha}_1, \dots, \bar{\alpha}_n) = 0$ in R_0 encoding the root system of Δ:

Theorem 2.1. *The kernel of the natural map $\rho : R_* \twoheadrightarrow R_0$ is the principal ideal (Φ). If M is any Δ-representation such that for some orientation $\vec{\Delta}$ of Δ, the underlying $\mathfrak{k}\vec{\Delta}$-module is indecomposable, then $\mathrm{N}(\Lambda)M = 0$, i.e. M can be regarded as a Λ_0-module.*

For each orientation $\vec{\Delta}$ of Δ, the $\mathfrak{k}\vec{\Delta}$-module $\mathfrak{k}\vec{\Delta}$ extends to a Δ-representation M of type $(0, \dots, 0)$. Thus Theorem 2.1 implies $\mathrm{N}(\Lambda)M = 0$, whence $\mathfrak{k}\vec{\Delta} \cap \mathrm{N}(\Lambda) = 0$ in Λ. Therefore, the natural ring homomorphism

$$\mathfrak{k}\vec{\Delta} \hookrightarrow \Lambda \twoheadrightarrow \Lambda_0 \tag{6}$$

is injective, i.e. $\mathfrak{k}\vec{\Delta}$ and $(\mathfrak{k}\vec{\Delta})^{op}$ can be regarded as subrings of Λ_0, and Λ_0 is generated by these subrings together with R_0.

Since for each $\mathrm{d} \in \Phi^+$, there is an integral domain $R_{\mathrm{d}} := R_*/(\Phi_{\mathrm{d}})$ isomorphic to $\mathfrak{k}[x_1, \dots, x_{n-1}]$, with quotient field K_{d}, say, Theorem 2.1 yields a natural embedding

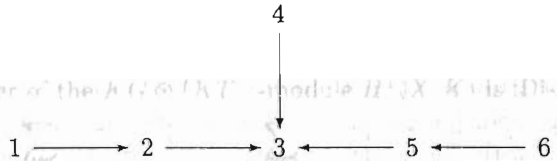
$$R_0 \hookrightarrow \prod_{\mathrm{d} \in \Phi^+} K_{\mathrm{d}} =: K, \tag{7}$$

and K is the classical quotient ring [19] of R_0. Consequently, there is a natural homomorphism

$$\Lambda \longrightarrow K \oplus_R \Lambda =: A. \tag{8}$$

Theorem 2.2. *The ring A is semisimple, the kernel of (8) is $\mathrm{N}(\Lambda)$, and thus Λ_0 is an order in A. The blocks of A correspond to the positive roots $\mathrm{d} \in \Phi^+$, and are matrix algebras $\mathrm{M}_d(K_{\mathrm{d}})$ with $d = d_1 + \ldots + d_n$.*

This theorem leads to a fairly precise description of the structure of Λ_0. In particular, each indecomposable $\vec{\Delta}$-representation $\vec{M}$ (for some orientation $\vec{\Delta}$ of Δ) extends to a Λ_0-representation $\vec{E}$ via Theorem 1.1, and for fixed $\mathrm{d} = \mathbf{dim}\vec{M}$ and all orientations $\vec{\Delta}$, the Λ_0-lattices $\vec{E}$ belong to a unique simple A-module S_{d}. (Such lattices over an order Λ_0 are said to be *irreducible.*) The reason that $\vec{M}$ extends to a Λ_0-lattice $\vec{E}$ depends on the fact that the extension problem in Theorem 1.1 admits an "integral" solution. For example, let $\vec{\Delta}$ be the following orientation of $\mathbb{E}_6$:



and (V, f) the indecomposable $\vec{\Delta}$-representation with

$$f_{21} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \; f_{32} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} ; \; f_{34} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; \; f_{35} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} ; \; f_{56} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the extension to a Δ-representation of type α is given by

$$f_{12} = (\alpha_1 \; \beta); \; f_{23} = \begin{pmatrix} \alpha_2 - \alpha_1 & \delta & -\beta \\ 0 & \delta & \alpha_2 \end{pmatrix}; \; f_{43} = (-\gamma \; -\delta \; \beta); \; f_{53} = \begin{pmatrix} \gamma & \alpha_5 - \alpha_6 & -\beta \\ \gamma & 0 & \alpha_5 \end{pmatrix};$$
$$f_{65} = (\alpha_6 \; \beta)$$

where $\beta = \alpha_3 - \alpha_2 - \alpha_5$, $\gamma = \alpha_2 - \alpha_1 - \alpha_3$, and $\delta = \alpha_5 - \alpha_3 - \alpha_6$. The entries of all matrices f_{ij} are integral in the $\alpha_1, \ldots, \alpha_6$. In general, this follows by means of

3. Reflection functors for Δ-representations

Reflection functors for Δ-representations, which, in virtue of Theorem 1.1, can be viewed as extensions of the functors $F_i^{\pm}$ of Bernstein, Gelfand, and Ponomarev [6] to Δ-representations.

Firstly, for each vertex $i \in \Delta_0$, define a linear map $\sigma_i \in \mathrm{Aut}\,\mathfrak{k}^n$ by $\sigma_i(x_1, \ldots, x_n) = (x_1', \ldots, x_n')$, where

$$x_j' = \begin{cases} -x_i & \text{for } j = i \\ x_j - x_i & \text{for } j \in \mathrm{V}(i) \\ x_j & \text{otherwise.} \end{cases} \tag{9}$$

For a given Δ-representation (V, f) of type α, and $i \in \Delta_0$, let us define a Δ-representation $\sigma_i^+(V, f)$ of type $\sigma_i(\alpha)$ as follows. For $j \in \mathrm{V}(i)$, the maps $f_{ij} : V_j \to V_i$ and $f_{ji} : V_i \to V_j$ give rise to $\mathfrak{k}$-linear maps $f_{i0} : \bigoplus V_j \to V_i$ and $f_{0i} : V_i \to \bigoplus V_j$ such that

$$f_{i0} f_{0i} = \alpha_i \cdot 1_{V_i}. \tag{10}$$

If $f'_{0i} : V'_i \hookrightarrow \bigoplus V_j$ is the kernel of f_{i0}, then $f_{i0}(f_{0i} f_{i0} - \alpha_i \cdot 1_{\bigoplus V_j}) = 0$, whence there exists a unique $f'_{i0} : \bigoplus V_j \to V'_i$ such that

$$f_{0i} f_{i0} - \alpha_i \cdot 1_{\bigoplus V_j} = f'_{0i} f'_{i0}. \tag{11}$$

Now the f'_{0i} and f'_{i0} decompose into $\mathfrak{k}$-linear maps $f'_{ji} : V'_i \to V_j$ and $f'_{ij} : V_j \to V'_i$, which replace the f_{ji} and f_{ij} in order to obtain $\sigma_i^+(V, f)$. It is easily verified that (1) is satisfied for each vertex $j \in \Delta_0$ if α is replaced by $\sigma_i(\alpha)$. The dimension vector $\mathbf{d}$ is changed into $\sigma_i(\mathbf{d}) = (d'_1, \ldots, d'_n)$, where $d'_i = -d_i + \sum_{j \in \mathrm{V}(i)} d_j$ and $d'_j = d_j$ for $j \neq i$.

If σ_i^+ is restricted to $\vec{\Delta}$-representations, we obtain a functor which is applicable without any condition on $\vec{\Delta}$. If i is a *sink*, i.e. no arrows start at i, this functor coincides with the "image functor" F_i^+ in [6]. Dually to σ_i^+, we define σ_i^- by taking the cokernel $f'_{i0} : \bigoplus V_j \twoheadrightarrow V'_i$ of f_{0i}, etc.

Remark. Since these functors are well defined even if $\mathfrak{k}$ is replaced by an integral domain, they can be applied to the irreducible representations $\vec{E}$ of Λ_0 considered above. However, since $\sigma_i^{\pm}$ is universally applicable, even if i is neither a sink nor a source, there exist "non-oriented" irreducibles of Λ_0, i. e. those irreducible Λ_0-lattices which do not arise by extension of some indecomposable $\vec{\Delta}$-representation via Theorem 1.1.

Now let (V, f) be a Δ-representation of type α and $i \in \Delta_0$. If $\alpha_i \neq 0$, then (10) and (11) state that the maps f_{0i}, f'_{0i} and $\frac{1}{\alpha_i} f_{i0}, -\frac{1}{\alpha_i} f'_{i0}$ form a biproduct diagram [14]

$$V_i \rightleftarrows \bigoplus_{j \in \mathrm{V}(i)} V_j \leftrightarrows V'_i, \tag{12}$$

that is, we have a natural isomorphism $\bigoplus V_j \cong V_i \oplus V'_i$. Hence σ_i^+ and σ_i^- coincide on (V, f). In this case, we simply write σ_i instead of $\sigma_i^{\pm}$. If $\alpha_i = 0$, we put $\sigma_i = 1$, the identity functor. Via these σ_i, the Weyl group of Δ operates on the category of Δ-representations.

4. Proofs

For the proof of the existence part of Theorem 1.1, these reflection functors σ_i suffice. For the uniqueness part, however, we have to make use of σ_i^+ and σ_i^-:

Proof of Theorem 1.1. If (V,f) is simple, then $\Phi_{\mathbf{d}}(\alpha) = \pm\alpha_i$ for some $i \in \Delta_0$, and the assertion is trivial. Otherwise, let i be a sink of $\vec{\Delta}$. Then $F_i^- F_i^+(V,f) = (V,f)$, where $F_i^{\pm}$ denote the classical reflection functors [6] for $\vec{\Delta}$-representations. As $\Phi_{\mathbf{d}}(\alpha) = 0$ implies $\Phi_{\mathbf{d}'}(\alpha') = 0$ for $\mathbf{d}' = \sigma_i(\mathbf{d})$ and $\alpha' = \sigma_i(\alpha)$, we may assume by induction that $F_i^+(V,f)$ is extendable to a Δ-representation (W,g) of type $\sigma_i(\alpha)$. Then (V,f) extends to the Δ-representation $\sigma_i^-(W,g)$ of type $\sigma_i\sigma_i(\alpha) = \alpha$. To prove the uniqueness of this Δ-representation, let $(\tilde{V},\tilde{f})$ be any such extension. Then $\sigma_i^+(\tilde{V},\tilde{f})$ extends $F_i^+(V,f)$, whence $\sigma_i^+(\tilde{V},\tilde{f}) = (W,g)$. Therefore, we get $(\tilde{V},\tilde{f}) = \sigma_i^-\sigma_i^+(\tilde{V},\tilde{f}) = \sigma_i^-(W,g)$. By the construction of $\sigma_i^{\pm}$, an inductive argument also proves the integrality property of $\tilde{f}$. □

Proof of Proposition 1.3. If $i \in \Delta_0$ is a source of $\vec{\Delta}$, we may apply σ_i^- to $\tilde{M}$ which extends the application of F_i^- to M. Then $\sigma_i^+\sigma_i^-\tilde{M} = \tilde{M}$, whence by induction, we may assume that M has a simple direct summand S concentrated at a source i. Thus $\alpha_i = 0$, and S extends to a subrepresentation $\tilde{S}$ of $\tilde{M}$. □

Next we shall focus our attention to Theorems 2.1 and 2.2. If α is specialized to $(0,\ldots,0)$, then Λ turns into the preprojective algebra $\Pi(\Delta)$ of Δ [10, 7, 5]. For $k \in \mathbb{N}$, define Λ_k as the R-submodule of Λ generated by the paths of length $\leq k$. This gives a filtration

$$R \subset \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda \tag{1}$$

of Λ, i.e. $\Lambda_i\Lambda_j \subset \Lambda_{i+j}$ for $i,j \in \mathbb{N}$. With $\Lambda_{-1} := 0$, we can form the associated graded R-algebra $\coprod(\Lambda_i/\Lambda_{i-1})$, wherein the defining relation (5) simplifies to $\sum \bar{e}_{ij}\bar{e}_{ji} = 0$, if $\bar{e}_{ij}$ denotes the residue class of e_{ij} in Λ_1/Λ_0. Hence, we have a natural epimorphism of R-algebras:

$$R \otimes_{\mathfrak{k}} \Pi(\Delta) \twoheadrightarrow \coprod_{i\in\mathbb{N}}(\Lambda_i/\Lambda_{i-1}). \tag{2}$$

Lemma 4.1. *Λ is finitely generated as an R-module.*

Proof. By virtue of (14), this follows from the well-known fact that $\Pi(\Delta)$ is finite dimensional. For the convenience of the reader, let us give a proof (cf. [9, 18]) which also sheds some light upon the structure of Λ via (14). It suffices to show that for large $m \in \mathbb{N}$, any path of length $\geq m$ becomes zero in $\Pi(\Delta)$. Let $i \in \Delta_0$ be a fixed vertex. We choose the unique orientation $\vec{\Delta}$ of Δ such that for each $j \in \Delta_0$, there is a path from i to j. Define $\mathbb{Z}\vec{\Delta}$ as the oriented graph with vertex set $\Delta_0 \times \mathbb{Z}$ and arrows $(i,l) \to (j,l) \to (i,l+1)$ for each arrow $i \to j$ in $\vec{\Delta}$ and $l \in \mathbb{Z}$. (Note that, as an abstract oriented graph, $\mathbb{Z}\vec{\Delta}$ does not depend on the orientation of $\vec{\Delta}$.) The numbering of $\mathbb{Z}\vec{\Delta}$ is given by the embedding $\vec{\Delta} \hookrightarrow \mathbb{Z}\vec{\Delta}$ with $i \mapsto (i,0)$ for $i \in \Delta_0$. Now let (V, f) be i-th projective $\vec{\Delta}$-representation, i.e. $V_j = \mathfrak{k}$ for all $j \in \Delta_0$ and $f_{jk} = 1$ for each arrow $k \to j$ in $\vec{\Delta}$. There is a natural way to extend (V, f) to a $\mathbb{Z}\vec{\Delta}$-representation. By induction, suppose that (V, f) is already extended to the full subgraph Γ of $\mathbb{Z}\vec{\Delta}$, and let (i,l) be a source in Γ such that $(j,m) \in \Gamma$ if there is an arrow $(i,l) \to (j,m)$ in $\mathbb{Z}\vec{\Delta}$, and $(i, l+1) \notin \Gamma$. We apply the classical reflection functor $F^-_{(i,l)}$ to (V, f). This gives a representation (V', f') where (i,l) is a sink. Define $V_{(i,l+1)} := V'_{(i,l)}$, $f_{(i,l+1)(k,l)} := f'_{(i,l)(k,l)}$ and $f_{(i,l+1)(j,l+1)} := f'_{(i,l)(j,l+1)}$ for arrows $j \to i \to k$ in $\vec{\Delta}$. Thus we obtain a $\mathbb{Z}\vec{\Delta}$-representation (V, f). If (W, g) is a non-simple indecomposable representation of some orientation of Δ, and j is a source for this orientation, then $W_j \to \bigoplus_{k \in \mathrm{V}(j)} W_k$ is always injective. Therefore, it is easily seen that a path in $\Pi(\Delta)$ starting in i vanishes if and only if the corresponding map $V_{(i,0)} \to V_{(j,l)}$ in (V, f) is zero. But since $\vec{\Delta}$ is representation-finite, $V_{(j,m)} = 0$ for sufficiently large m. □

Lemma 4.2. *Every prime ideal P of Λ contains some $\Phi_{\mathbf{d}}(\alpha)$ with $\mathbf{d} \in \Phi^+$.*

Proof. Since Λ is noetherian, Goldie's first theorem ([19], chap. II, Prop. 2.6) implies that Λ/P is an order in a simple ring B. The ideal $\mathfrak{p} = P \cap R$ of R is prime, whence B contains the quotient field F of $R+P/P \cong R/\mathfrak{p}$. By Lemma 4.1, Λ/P and F generate a finite dimensional F-algebra in B which therefore coincides with B. Consequently, B gives rise to a Δ-representation over F. By Proposition 1.3, at least one relation $\Phi_{\mathbf{d}}(\alpha) = 0$ must hold in F, whence $\Phi_{\mathbf{d}}(\alpha) \in \mathfrak{p} \subset P$. □

Corollary 4.3. *There is a positive integer s with $\Phi(\alpha)^s = 0$ in Λ.*

Proof. By Lemma 4.2, $\Phi(\alpha)$ lies in the prime radical $\mathrm{N}(\Lambda)$ of Λ which is a nil ideal ([19], chap. XV, Prop. 1.2). □

Next we consider the natural epimorphism

$$\rho : \mathfrak{k}[x_1, \ldots, x_n] \twoheadrightarrow \mathfrak{k}[\alpha_1, \ldots, \alpha_n] = R \tag{15}$$

with $\rho(x_i) = \alpha_i$.

Lemma 4.4. *The kernel of ρ is contained in the principal ideal (Φ).*

Proof. Let $\mathbf{d} \in \Phi^+$ be given. Take any orientation $\vec{\Delta}$ of Δ. By Theorem 1.1, the indecomposable $\mathfrak{k}\vec{\Delta}$-module with dimension vector $\mathbf{d}$ admits a unique extension to a Δ-representation M of type $(\bar{x}_1, \ldots, \bar{x}_n)$ over the quotient field $K_{\mathbf{d}}$ of $R_{\mathbf{d}} = \mathfrak{k}[x_1, \ldots, x_n]/(\Phi_{\mathbf{d}}) = \mathfrak{k}[\bar{x}_1, \ldots, \bar{x}_n] \cong \mathfrak{k}[x_1, \ldots, x_{n-1}]$, where $\bar{x}_i$ is the residue class of x_i modulo $\Phi_{\mathbf{d}}$. Hence, if $f \in \mathrm{Ker}\rho$, then $\bar{f}M = 0$ for the corresponding $\bar{f} \in R_{\mathrm{d}}$. Thus $\bar{f} = 0$, i.e. $f \in (\Phi_{\mathrm{d}})$. Since this holds for each $\mathrm{d} \in \Phi^+$, we obtain $\operatorname{Ker} \rho \subset \bigcap(\Phi_{\mathrm{d}}) = (\Phi)$. □

For any $\mathrm{d} \in \Phi^+$, the preceding lemma implies that $p := \Phi_{\mathrm{d}}(\alpha)$ is a prime element of R. For $A_{\mathrm{d}} := R_{(p)} \otimes_R \Lambda$, we have:

Lemma 4.5. $p \cdot A_{\mathrm{d}} = 0$.

Proof. Without loss of generality, we may suppose that $|\mathrm{V}(n)| \leq 1$, and $d_n = 1$. Let $\pi : R \to R_{(p)}$ be the natural homomorphism, and $\alpha_i' = \pi(\alpha_i)$ for $i \in \{1, ..., n-1\}$. The residue class field $R_{(p)}/pR_{(p)}$ is isomorphic to the quotient field K_{d} of $R_{\mathrm{d}} = R/pR$ which is isomorphic to the function field $\mathfrak{k}(x_1, \ldots, x_{n-1})$. Hence, K_{d} is isomorphic to the subfield $F := \mathfrak{k}(\alpha_1', \ldots, \alpha_{n-1}')$ of $R_{(p)}$, and $R_{(p)} = F \oplus pR_{(p)}$. Moreover, there is a unique $\alpha_n' \in F$ such that

$$\pi(\alpha_n) = \alpha_n' + \pi(p)\,; \quad \Phi_{\mathrm{d}}(\alpha_1', \ldots, \alpha_n') = 0 \tag{16}$$

holds in $R_{(p)}$. Next we consider the exact sequence

$$pA_{\mathrm{d}}/p^2A_{\mathrm{d}} \hookrightarrow A_{\mathrm{d}}/p^2A_{\mathrm{d}} \twoheadrightarrow A_{\mathrm{d}}/pA_{\mathrm{d}}. \tag{17}$$

For an arbitrary orientation $\vec{\Delta}$ of Δ, let X be the indecomposable $F\vec{\Delta}$-module. Since $pA_{\mathrm{d}}/p^2A_{\mathrm{d}}$ and $A_{\mathrm{d}}/pA_{\mathrm{d}}$ are Δ-representations of type $(\alpha_1', \ldots, \alpha_n')$ over F, the corresponding $F\vec{\Delta}$-modules are isomorphic to powers of X. Hence, $\operatorname{Ext}(X, X) = 0$ implies that the sequence (17) of $F\vec{\Delta}$-modules splits. Regarding $A_{\mathrm{d}}/p^2A_{\mathrm{d}}$ as a Δ-representation (V, f) over the ring $R_{(p)}/p^2R_{(p)}$, the $F\vec{\Delta}$-module (V, f) is thus of the form $(C, f') \oplus (pC, f'')$, that is, for each vertex $i \in \Delta_0$, we have $V_i = C_i \oplus pC_i$, and for each arrow $i \to j$ in $\vec{\Delta}$,

$$f_{ji} = \begin{pmatrix} f_{ji}' & 0 \\ 0 & f_{ji}'' \end{pmatrix}\,; \quad f_{ij} = \begin{pmatrix} f_{ij}' & 0 \\ h_{ij} & f_{ij}'' \end{pmatrix}.$$

Furthermore, the multiplication by p gives rise to an epimorphism $(C, f') \twoheadrightarrow (pC, f'')$ of $F\vec{\Delta}$-modules which splits by virtue of $\operatorname{Ext}(X, X) = 0$. Therefore, p can be regarded as the natural projection in a decomposition $(C, f') = (pC, f'') \oplus (D, g)$. Thus p induces an endomorphism p' of (C, f'), and the h_{ij} induce F-linear maps $h_{ij}' : C_j \to pC_i \hookrightarrow C_i$ which extend the $F\vec{\Delta}$-module (C, f') to a Δ-representation $(C, \tilde{f})$ of type $(0, \ldots, 0, p')$. For any edge i—j in Δ, let d_{ij} be the trace $\operatorname{tr}(\tilde{f}_{ij}\tilde{f}_{ji})$. Then $d_{ij} = d_{ji}$, and for each vertex $i \in \Delta_0$,

$$\sum_{j \in \mathrm{V}(i)} d_{ij} = \begin{cases} 0 & \text{if } i \neq n \\ \operatorname{tr} p' |_{C_n} & \text{if } i = n. \end{cases}$$

But this implies $\operatorname{tr} p' |_{C_n} = 0$, whence $pC = 0$ and thus $pA_{\mathrm{d}}/p^2A_{\mathrm{d}} = 0$. By Nakayama's lemma, we conclude $pA_{\mathrm{d}} = 0$. □

Using the notation of (7), we thus obtain that A_{d} is a finite dimensional K_{d}-algebra. Since the relation $\Phi_{\mathrm{d}}(\alpha) = 0$ in K_{d} holds for no positive root other than d, Proposition 1.3 and a similar argument as in the preceding proof, together with the uniqueness part of Theorem 1.1 imply that the Δ-representation A_{d} decomposes into simple Δ-representations of a single type α. Hence, the K_{d}-algebra A_{d} is simple. Consider the natural map

$$\pi_{\mathrm{d}} : \Lambda \longrightarrow A_{\mathrm{d}}. \tag{18}$$

The image of π_{d} is an order in A_{d}. Thus, by Goldie's theorem ([19], chap. II, Prop. 2.6), the kernel P_{d} of π_{d} is a prime ideal of Λ:

$$P_{\mathrm{d}} = \{a \in \Lambda \mid \exists r \in R \setminus R\Phi_{\mathrm{d}}(\alpha) : ra = 0\}. \tag{19}$$

Lemma 4.6. $P_{\mathrm{d}} \cap R = R\Phi_{\mathrm{d}}(\alpha)$.

Proof. Since $\Phi_{\mathrm{d}}(\alpha) \in R$ is prime, the inclusion "$\subset$" holds. Conversely, suppose $\Phi_{\mathrm{d}}(\alpha) \notin P_{\mathrm{d}}$. By Lemma 4.2, there would exist some $\mathrm{d}' \in \Phi^+$ with $\mathrm{d}' \neq \mathrm{d}$ and $\Phi_{\mathrm{d}'}(\alpha) \in P_{\mathrm{d}}$. Consider the epimorphism (15). By (19), we infer $\mathrm{Ker}\rho \not\subset (\Phi_{\mathrm{d}})$, which contradicts Lemma 4.4. □

Lemma 4.7. *Let M be a Δ-representation of type $\alpha \in F^n$ over an extension field F of $\mathfrak{k}$ such that, for some orientation $\vec{\Delta}$ of Δ, the underlying $\mathfrak{k}\vec{\Delta}$-module is indecomposable. If $N := \bigcap_{\mathrm{d}\in\Phi^+} P_{\mathrm{d}}$, then $NM = 0$.*

Proof. There is a unique $\mathfrak{k}$-algebra-homomorphism $\tau : \mathfrak{k}[x_1, \ldots, x_n] \to F$ with $\tau(x_i) = \alpha_i$. The kernel $\mathfrak{p}$ of τ is a prime ideal of $\mathfrak{k}[x_1, \ldots, x_n]$, and $\Phi_{\mathrm{d}} \in \mathfrak{p}$ holds for the dimension vector $\mathrm{d} \in \Phi^+$ of M. By Theorem 1.1, there is a unique Δ-representation M_{d} of type $(\bar{x}_1, \ldots, \bar{x}_n)$ over the quotient field K_{d} of $R_{\mathrm{d}} = \mathfrak{k}[x_1, \ldots, x_n]/(\Phi_{\mathrm{d}})$, where $\bar{x}_i = x_i + (\Phi_{\mathrm{d}})$, and there is an R_{d}-lattice E in M_{d} which is a Δ-representation over the ring R_{d} such that $M \cong F \otimes_{R_{\mathrm{d}}} E$. Since M_{d} is an A_{d}-module, we have $NM_{\mathrm{d}} \subset P_{\mathrm{d}}M_{\mathrm{d}} = 0$ and thus $NM = 0$. □

Lemma 4.8. $\mathrm{N}(\Lambda) = \bigcap_{\mathrm{d}\in\Phi^+} P_{\mathrm{d}}$ *and* $R \cap \mathrm{N}(\Lambda) = R \cdot \Phi(\alpha)$.

Proof. Let N be as in Lemma 4.7 and P a prime ideal in Λ. As in the proof of Lemma 4.2, Λ/P is an order in a simple F-algebra $B = S^d$ with a simple Δ-representation S over F. Choose any orientation $\vec{\Delta}$ of Δ. By Proposition 1.3, S is indecomposable as a $\mathfrak{k}\vec{\Delta}$-module. Hence, Lemma 4.7 implies $N(\Lambda/P) \subset NB = 0$. Consequently, $N \subset P$ and thus $\mathrm{N}(\Lambda) = N$. The second equation follows by Lemma 4.6 and Lemma 4.4. □

Proof of Theorem 2.1 and 2.2. Theorem 2.1 follows by Lemma 4.7 and 4.8. Since $A_{\mathrm{d}} \cong K_{\mathrm{d}} \otimes_R \Lambda$ by Lemma 4.5, Theorem 2.2 follows immediately by (18), (19), and Lemma 4.8. The endomorphism ring of the simple A_{d}-modules is K_{d} since this holds for the underlying indecomposable $K_{\mathrm{d}}\vec{\Delta}$-modules. □

5. An open question

We have shown that the element $\Phi(\alpha) = \prod \Phi_{\mathbf{d}}(\alpha)$ of Λ is nilpotent. On the other hand, Lemma 4.2 and (19) imply that there is a polynomial $r \in \mathfrak{k}[x_1, \ldots, x_n] \setminus (\Phi)$ with $r(\alpha)\Phi(\alpha) = 0$. Since $\mathfrak{k}[x_1, \ldots, x_n]$ is not a principal ideal domain for $n \geq 2$, we cannot conclude $\Phi(\alpha) = 0$. A direct calculation in Λ, however, shows that $\Phi(\alpha) = 0$ at least for $\Delta = \mathbb{A}_n$ and $\mathbb{D}_4$. If $\Phi(\alpha) = 0$ holds, then the distinction between R and R_0 can be dropped. A further simplification would arise if Λ is semiprime: then Λ_0 would coincide with Λ. By Lemma 4.8, this would also imply $\Phi(\alpha) = 0$.

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ON THE CLASSIFICATION OF FINITE DIMENSIONAL HOPF ALGEBRAS

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ABSTRACT. The initial part of this note presents some results on the classification of finite dimensional Hopf algebras. In the second part we shall "prove" that there are a finite number (up to an isomorphism) of semisimple and cosemisimple Hopf algebras of a given dimension.

INTRODUCTION

Throughout this paper we shall work over an algebraically closed field of characteristic 0. Recently there is some progress in the classification of finite dimensional Hopf algebras over k. The main aim of this note is to survey some of the work in this area. We shall start by describing the types of Hopf algebras of "small" dimension. Then, we shall point out the basic ideas of the proof of Zhu's result: a Hopf algebra of prime dimension is a group algebra of a cyclic group. We shall continue presenting some of Masuoka's results on the classification of semisimple Hopf algebras, and we shall end the first section by studying the types of pointed Hopf algebras. The starting point of the second section is a very old conjecture, mentioned by Kaplansky in his book [K], and which asserts that the set of types of Hopf algebras of a given dimension is finite. Actually, we shall show that this conjecture has a positive answer if we consider the case of the semisimple and cosemisimple Hopf algebras.

Preliminaries

Let k be a field (not necessarily algebraically closed and of characteristic 0). By a coalgebra C we mean a vector space over k with a coproduct map $\Delta : C \to C \otimes C$ and a counit $\varepsilon : k \to C$ (the tensor product $\otimes$ means $\otimes_k$). These obey axioms dual to the axioms of an algebra. A vector space which is both a coalgebra and an algebra such that the coalgebra structure maps are algebra morphisms is called a bialgebra. By a Hopf algebra H we shall mean a bialgebra with an antipode, that is with a k linear map $S : H \to H$ such that

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$\sum_i S(h_i')h_i'' = \sum_i h_i'S(h_i'') = \varepsilon(h)1_H$, for all h in H, where $\Delta(h) = \sum_i h_i' \otimes h_i''$. If H is a finite dimensional Hopf algebra then the dual space H^* of H has a natural structure of Hopf algebra. The Hopf algebra H will be called cosemisimple if H^* is semisimple, or equivalently, H is the direct sum of its simple subcoalgebras.

We recall that an element $g \neq 0$ is called a *group-like element* if $\Delta(g) = g \otimes g$. By definition, $x \in H$ is an *g, h-primitive element* if $\Delta(x) = x \otimes g + h \otimes x$, where g, h are two group-like elements. In the particular case when $g = h = 1$ we say that x is a *primitive element.* We denote by $G(H)$, $P(H)$ and $P_{g,h}(H)$ respectively the sets of group-like elements, of primitive elements and of g, h-primitive elements of H. A Hopf algebra H is called *pointed* if all its simple subcoalgebras are of dimension one.

1. Hopf algebras of dimension $p^\alpha q^\beta$

In this section we shall survey some results on the classification of Hopf algebras of dimensions $p^\alpha q^\beta$, where α and β are "small" natural numbers.

The start on investigating Hopf algebras (actually bialgebras) of dimension 2 or 3 was made by I. Kaplansky. He proved, by a case by case examination, that such a Hopf algebra must be isomorphic to the group algebra of the cyclic group with 2, respectively 3 elements.

Studying Hopf algebras of dimension 4 we are led to the first nontrivial result. D. Radford [R] proved that there is only one (up to an isomorphism) Hopf algebra of dimension 4, which is neither commutative nor cocommutative. So, if k is an algebraically closed field of characteristic 0, and H is a Hopf algebra of dimension 4 over k, then H is isomorphic with the group algebra of a group of order 4, or H is isomorphic with Radford's example. Indeed, if H is any commutative Hopf algebra over an algebraically closed field of characteristic 0 then, by a result of Sweedler, H is reduced so it is semisimple. Hence, the dual H^* of H is a pointed (H^* is cocommutative) and cosemisimple Hopf algebra. In conclusion, $H^* = \operatorname{corad}(H^*) = k[G]$, where G is a group of order $\dim(H)$. Moreover, if G is a commutative group, then H is cocommutative and cosemisimple (any semisimple Hopf algebra over a field of characteristic 0 is cosemisimple too), thus H is isomorphic with the group algebra of the Pontryagin dual of G. In particular, if H is a 4-dimensional commutative Hopf algebra, it is a group algebra. If H is cocommutative then H^* is commutative, so H is the group algebra of a group of order 4.

Regarding Hopf algebras of dimensions 5, Kaplansky conjectured that such a Hopf algebra is the group algebra of the cyclic group of order 5. The main obstacle for proving the conjecture was to show that H is commutative (see the above paragraph). But, if this is not the case, then the underlying algebra of H should be the direct product of k and two by two matrices over k, which is not possible, cf. [K, p. 44].

The results on Hopf algebras of dimension 2, 3 and 5 which we have discussed above led Kaplansky to the following conjecture [K, 1975]:

Conjecture 1.1. *Let H be a Hopf algebra over an algebraically closed field of characteristic 0. If* $\dim(H) = p$, *where p is a prime number, then H is the group algebra of the cyclic group of order p.*

The conjecture was solved affirmatively by Y. Zhu in 1992 [Z]. Let us sketch the main steps of Zhu's proof.

1) It suffices to show that H contains a group like element g, $g \neq 1$. Indeed, if G is the set of group-like elements of H, then $k[G]$ is a Hopf subalgebra of H, so by a well-known theorem due to W. Nichols, the dimension of any Hopf subalgebra of H divides the dimension of H. In particular, the order of G is 1 or p hence, if $G \neq \{1\}$ it results $H = k[G]$.

2) By a very easy argument, it follows that H is semisimple. The main step is to prove that the theory of characters of semisimple Hopf algebras is quite similar to the theory of characters of finite groups. More precisely, we have the following theorem which goes back to G. I. Kats:

Theorem 1.2. *If k is an algebraically closed field of characteristic 0 and H is a semisimple Hopf algebra over k, let $C(H)$ be the character ring of H. Then $C(H)$ is semisimple and if $(E_{\alpha ij})_{\alpha ij}$, is the canonical basis on $C(H) \otimes_{\mathbb{Q}} k \simeq \mathcal{M}_1 \times \dots \times \mathcal{M}_s$, where $\mathcal{M}_1, \dots, \mathcal{M}_s$ are full matrix algebras over k, then each $tr(E_{\alpha ij})$ is an integer number which divides* $\dim(H)$. *(Here $tr(f)$ denotes the trace of the linear operator on H^* sending g to fg).*

3) By the second step we can see easily that if $\dim(H) = p$ then $C(H) \otimes_{\mathbb{Q}} k \simeq H^*$, that is H is a direct product of copies of k so the theorem is proved by the first part of the proof.

We shall end this section with some results on the classification of two very important classes of Hopf algebras, namely the semisimple and pointed ones. First of all, a definition:

Definition 1.3. *Let H be a Hopf algebra. A Hopf subalgebra K of H is called normal if HK^+ is a two-sided ideal of K, where K^+ is the augmentation ideal of K.*

Actually, if HK^+ is a two-sided ideal of K it is a Hopf ideal, so we can construct the quotient Hopf algebra $\overline{H} = H/HK^+$. We shall say that H is an extension of $\overline{H}$ by K. The significance of extensions of Hopf algebras is emphasized by the following theorem, which is due to A. Masuoka [M1]:

Theorem 1.4. *If H is a semisimple Hopf algebra of dimension p^α, where p is a prime number, then there is a central group-like element of order p in H. In particular, H is an extension of the group algebra of the cyclic group of order p by a Hopf algebra K of dimension $p^{\alpha-1}$.*

Masuoka, by using the above theorem and Zhu's result, has obtained the classification of semisimple Hopf algebras of dimension $2p$, p^2 and p^3.

Theorem 1.5. *a)* [M2] *If H is a semisimple Hopf algebra of dimension $2p$, $p \neq 2$, then H is isomorphic to precisely one of the following three Hopf algebras: $k[\mathbf{C_{2p}}]$, $k[\mathbf{D_{2p}}]$, $k[\mathbf{D_{2p}}]^*$, where $\mathbf{C_{2p}}$, $\mathbf{D_{2p}}$ are the cyclic group and respectively the dihedral group of order $2p$.*

b) [M1] *If H is a semisimple Hopf algebra of dimension p^2 then H is the group algebra of a group of order p^2.*

c) [M3] *There are $p+1$ semisimple Hopf algebras of dimension p^3 which are neither commutative nor cocommutative besides trivial 7 ones (that is group algebras and their dual).*

We now consider the case of pointed Hopf algebras. The technical result necessary to study them is given in the next theorem

Theorem 1.6. [S1] *Let H be a pointed Hopf algebra. If H is not semisimple then there exist two natural numbers m, n, with $m \neq 1$ and m divides n, an m^{th} primitive root of 1 (denoted by ω) and two elements $g, x \in H$ such that*

a) $gx = \omega xg$.

b) g *is a group-like element of order* n.

c) $x \in P_{g,1}(H)$ *and* x^m *is either* 0 *or* $g^m - 1$.

Let n be a natural number and let ω be a primitive n^{th}-root of 1. We recall that, by definition, $H_{n^2,\omega}$ is the Hopf algebra generated as an algebra by two elements g and x satisfying the following relations

$$\begin{array}{lll} g^n = 1, & x^n = 0, & gx = \omega xg, \\ \Delta(g) = g \otimes g, & \Delta(x) = x \otimes g + 1 \otimes x, & \\ \varepsilon(g) = 1, & \varepsilon(x) = 0. & \end{array}$$

Using the preceding theorem, it is now easy to describe the types of pointed p^2-dimensional Hopf algebras over an algebraically closed field of characteristic 0, and to prove that any pointed Hcpf algebra of dimension pq is semisimple (q is a prime number, $q \neq p$).

Theorem 1.7. *If p is a prime natural number and H is a pointed Hopf algebra of dimension p^2, then $H \simeq k[G]$ or $H \simeq H_{p^2,\omega}$, where G is a group with p^2 elements and ω is a certain primitive n^{th}-root of 1.*

Theorem 1.8. *Let p and q be two different prime numbers. If H is a pointed Hopf algebra of dimension pq then H is semisimple.*

As an application of the last theorem we have obtained the complete classification of Hopf algebras of dimension 6.

Corollary 1.9. *Let H be a Hopf algebra of dimension 6. Then H is isomorphic to $k[\mathbf{C}_6]$, $k[\mathbf{S}_3]$ or $k[\mathbf{S}_3]^*$, where $\mathbf{C}_6$ and $\mathbf{S}_3$ are respectively the cyclic group with 6 elements and the symmetric group with 6 elements.*

2. On Kaplansky's conjecture

Let k be an algebraically closed field of *any* characteristic. We shall "prove" that the classification problem of finite dimensional Hopf algebras over k is not hopeless, at least in the case of semisimple and cosemisimple Hopf algebras. Indeed, we have

Theorem 2.1. [S2] *There are only a finite number (up to an isomorphism) of semisimple and cosemisimple Hopf algebras of a given dimension n.*

"Proof" The set of Hopf bialgebra structures which can be defined on a vector space of dimension n is an affine algebraic variety, which we shall denote by $\mathcal{B}(V)$. The basic properties of $\mathcal{B}(V)$ are:

1) $GL(V)$ acts on $\mathcal{B}(V)$, such that the orbit of a bialgebra A contains all structures isomorphic to A, so the set of types of bialgebras on V is in a one-to-one correspondence with the set of orbits.

2) Let A be a bialgebra, $A \in \mathcal{B}(V)$. Studying the tangent spaces $T_A(\overline{\mathcal{O}}_A)$ and $T_A(\mathcal{B}(V))$, where $\overline{\mathcal{O}}_A$ is the closure of the orbit through a bialgebra A, we are led in a natural way to a certain cohomology theory of the bialgebra A. If the cohomology of A vanishes we prove that the orbit of A is open. Therefore, the number of orbits which correspond to bialgebras having trivial cohomology is finite.

We end the proof of our result by showing that the cohomology of a semisimple Hopf algebra A vanishes if A is semisimple and cosemisimple.

Corollary 2.2. *If k is an algebraically closed field and* $\operatorname{char}(k) = 0$, *then the set of types of n-dimensional semisimple Hopf algebras is finite.*

Corollary 2.3. *If $char(k) = 0$ and n is a given natural number then there exists a semisimple Hopf algebra H_n, such that any semisimple Hopf algebra of dimension n can be embedded in H_n.*

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COHOMOLOGY AND NEAR-RINGS

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ABSTRACT. We do not know whether the near-rings can be used decisively for studying cohomology of non-abelian groups; probably not. But some sets of mappings playing a good part in this theory (like the set of pseudo-homomorphisms from G to G) are near-rings. We just consider such near-rings and their properties "related to cohomology".

1. GENERALITIES ON NEAR-RINGS.

In 1958, A. Fröhlich [7] tried to build a non-abelian homological algebra by using the near-ring $E(G)$ generated (in the right near-ring of all mappings from G to G) by the endomorphisms of a non-commutative group G. In this way, he obtained distributively generated near-rings which have been intensively studied for some years.

But Fröhlich's idea has not been used for the future constructions in non-abelian cohomology, mostly since it seemed to be too narrow, and Fröhlich himself did not visited it again.

In the cohomology theory for the non-abelian case, in spite of some similarities, there are naturally fundamental differences; so, the sum of two endomorphisms is no longer an endomorphism and this implies that the derived functors fail to be additive, but they preserve the null mappings.

However there are some possibilities of using near-rings of mappings on a group (which include $E(G)$) for studying non-abelian cohomology. Lockhart [14] has tried it and we followed his way to go a little bit further (see [23, 24]).

Let us recall first some definitions in the theory of near-rings (one can see more in Pilz's book [20]).

Definition 1.1. *A* **right near-ring** *is a triple* $< N, +, \cdot >$, *where* $< N, + >$ *is a group,* $< N, \cdot >$ *is a semigroup and, for all* $x, y, z \in N$, $(x+y)\cdot z = x\cdot z + y\cdot z$.

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The most important example is the set of all mappings preserving the neutral element on a group $< G, + >$, - let us denote it by $Ens_0(G)$, - with respect to the induced addition and the mapping composition.

$D(N) := \{d \in N | d \cdot (x + y) = d \cdot x + d \cdot y$, for all $x, y \in N\}$ is the set of all distributive elements of N.

If the right near-ring N is additively generated by $D(N)$, then it is called a **distributively generated near-ring.**

$D(Ens_0(G)) = End(G)$. Generally, $End(G)$ does not generate $Ens_0(G)$, but a subnearring of it, denoted $E(G)$.

Definition 1.2. *Let $< G, + >$ be a group and N be a right near-ring. G is called an N-***group,** *if there exists an operation : $N \times G \to G, (n, g) \mapsto n \cdot g$, such that for all $n, n' \in N$ and $g \in G$, the following conditions are fulfilled:*

(i) $(n + n') \cdot g = n \cdot g + n' \cdot g$;

(ii) $(n \cdot n') \cdot g = n \cdot (n' \cdot g)$;

(iii) $n \cdot 0 = 0$ (if the same condition is satisfied by the near-ring N which is then called **0-symmetric***).*

Definition 1.3. *A non-empty subset A of the right near-ring N is called an* **ideal** *of N, denote it by $A \triangleleft N$, if:*

(i) A is a normal subgroup in $< N, + >$;

(ii) $a \cdot x \in A$, for all $a \in A$ and $x \in N$;

(iii) $x \cdot (a + y) - x \cdot y \in A$, for all $a \in A$ and $x, y \in N$.

((i) and (ii) define a **right ideal,** *while (i) and (iii) define a* **left ideal** *of N.)*

In the same way, if we consider two groups (with additive notation) G and M, $Ens_0(G, M)$, the set of mappings $\varphi : G \to M$ such that $\varphi(0) = 0$, is a group, while $Hom(G, M)$, its subset formed by the homomorphisms from G to M, is not a group, but it generates a subgroup, $\mathcal{H}(G, M)$, of $Ens_0(G, M)$.

In this case, $E(G)$ acts on the right hand on $\mathcal{H}(G, M)$, and $End(M)$ has a left action on it (Lemma 2.2, [23]).

In the case M is an abelian group, $Hom(G, M)$ is a group and, if we denote by $[G, G]$ the commutator of G, and by $\pi : G \to G/[G, G]$ the mapping of abelianising on G, then

$$Hom(G, M) \simeq Hom(G/[G, G], M).$$

By considering two endomorphisms $\varphi, \psi \in End(G)$ and their sum, let us note that its behavior on a sum of elements in G is:

$$(\varphi + \psi)(x + y) = (\varphi + \psi)(x) + (\varphi + \psi)(y) +$$

$$+ \{-\psi(y) + [\varphi(y), \psi(x)] + \psi(y)\},$$

where $[a, b] := -a - b + a + b$, for $a, b \in G$.

But $c + [a,b] - c \in [G,G]$, for all $a,b,c \in G$, since $[G,G]$ is a characteristic subgroup of G.

This led us to the idea of considering **pseudo-homomorphisms** from G to M :

$$\varphi : G \to M, \text{ with } \varphi(x+y) = \varphi(x) + \varphi(y) + d_\varphi(x,y) \text{ and } \varphi(0) = 0$$

for $x, y \in G$, where $d_\varphi : G \times G \to [M,M]$.

It is clear that, for $x,y \in G$, $d_\varphi(0,y) = d_\varphi(x,0) = 0$. If $d_\varphi(x,y) = 0$, for all $x,y \in G$, then $\varphi \in Hom(G,M)$, therefore each homomorphism is a pseudo-homomorphism. We note that *the sum of two homomorphisms from G to M is not a homomorphism, but a pseudo-homomorphism.*

The set of all pseudo-homomorphisms from G to M, $P(G,M)$, is a group, while the set $P(G,G)$ is a right near-ring studied in [23] and [24].

Let us recall some results concerning pseudo-homomorphisms:

Proposition 1.4. *Let G and M be two groups, $P(G,M)$ (resp. $P(G,G)$) be the set of pseudo- homomorphisms from G to M (resp. from G to G). Then:*

(i) $< P(G,M), + >$ is a group and $\mathcal{H}(G,M) \leq P(G,M)$;

(ii) $< P(G,G), +, \circ >$ is a near-ring included in $Ens_0(G)$ and including $E(G)$;

(iii) There exists an epimorphism of near-rings:

$\Psi : P(G,G) \to End(G/[G,G])$, defined by $\Psi(\theta)(g + [G,G]) = \theta(g) + [G,G]$, for all $\theta \in P(G,G]$ and $g \in G$;

(iv) If $K := Ker\Psi$, then the sequence of near-rings:

$$0 \to K \to P(G,G) \xrightarrow{\Psi} End(G/[G,G])$$

is exact. (See [23], Lemma 2.5.)

If we restrict to $E(G) \leq P(G,G)$ and $K \cap E(G)$, then the restriction of Ψ does not remain generally surjective; see Lockhart [14] for an example in this connection.

However, we may prove immediately that this restriction to $E(G)$, namely the map $\tilde{\psi}$ in the sequence

$$0 \to K \cap E(G) \to E(G) \xrightarrow{\tilde{\psi}} End(G/[G,G])$$

remains surjective if

1. $G/[G,G]$ is cyclic; or
2. the exact sequence of groups

$$0 \to [G,G] \to G \to G/[G,G] \to 0$$

is split.

More generally, we get

Proposition 1.5. *$\tilde{\psi}$ is surjective if and only if for each $\alpha \in P(G,G)$ there exists an element $\beta \in E(G)$ such that $\beta - \alpha \in C^1(G,[G,G])$.*

We would like to characterize the set of all pseudo-homomorphisms of a group G. For the right near-ring N we put

1. $\delta(a,b,c) := a \cdot c - a \cdot (b+c) + a \cdot b$ for a, b, $c \in N$ and
2. $\delta(N) := \{a \in N \mid \delta(a,b,c) \in [N,N]_+$ for all $b,c \in N\}$.

If N is an abelian – additively written – group then $[N,N]_+ = \{0\}$ and the right near-ring with $\delta(N) = N$ is a ring.

Proposition 1.6. $P(G,G) = \delta(C^1(G,G))$.

Proof: Let $\alpha \in P(G,G)$ and $\beta, \gamma \in C^1(G,G)$; then for an arbitrary $x \in G$ we have

$$\delta(\alpha,\beta,\gamma)(x) = \alpha(\gamma(x)) - d_\alpha(\beta(x),\gamma(x)) - \alpha(\gamma(x)) - \alpha(\beta(x)) + \alpha(\beta(x) \in [G,G] .$$

Thus

$$\delta(\alpha,\beta,\gamma) \in C^1(G,[G,G]) = [C^1(G,G), C^1(G,G)] \text{ and } \alpha \in \delta(C^1(G,G)) .$$

Conversely, if $\alpha \in \delta(C^1(G,G))$ and $x, y \in G$ with $x \neq 0$, then there exist $\beta, \gamma \in C^1(G,G)$ such that $\beta(x) = x$ and $\gamma(x) = y$. We then have

$$\alpha(y) - \alpha(x+y) + \alpha(x) = \alpha(\gamma(x)) - \alpha(\beta(x) - \gamma(x)) + \alpha(\beta(x)) =$$
$$(\alpha \circ \gamma - \alpha \circ (\beta + \gamma) + \alpha \circ \beta)(x))) = \delta(\alpha,\beta,\gamma)(x) \in [G,G] .$$

Hence $\alpha \in P(G,G)$.

Remark 1.7. 1. If G is a non-cyclic simple group; i. e. $< G, + >$ is not abelian, then $P(G,G) = C^1(G,G) = E(G)$.
2. If $< G, + >$ is abelian, then $P(G,G) = End(G)$ is a ring.
3. If $[G,G] = G$ then $C^1(G,G) = P(G,G)$.
4. If $\alpha \in C^1(G,[G,G])$, then $\alpha \in P(G,G)$.

Let us give an example of a group G for which $P(G,G) \neq E(G)$.

Example 1.8. Let $< G, + >$ be the p-group generated by elements a, b, x subject to the relations

$$p^r \cdot a = p^s \cdot b = x = [a,b] \text{ and } p^t \cdot x = 0 \text{ with integers } r \geq s > t > 0 .$$

Then $\alpha \in End(G)$ is given by its action on the generators as follows

$$\alpha(a) = u_1 \cdot a + u_2 \cdot b + u_3 \cdot x \,,\ \alpha(b) = v_1 \cdot a + v_2 \cdot b + v_3 \cdot x \text{ and } \alpha(x) = w \cdot x \text{ where}$$
$$w \,,\ u_i \,,\ v_i \in \mathbb{Z} \text{ for } 1 \leq i \leq 3$$

such that

$$u_1 + u_2 \cdot p^{r-s} = w = u_1 \cdot v_2 - u_2 \cdot v_1 \, mod\, (p^t) \text{ and } v_1 \cdot p^s \cdot a = (w - v_2)x .$$

For an element $\alpha \in E(G)$ the action on the generators is satisfying only the relations

$$u_1 + u_2 \cdot p^{r-s} = w \, mod \, (p^t) \text{ and } v_1 \cdot p^s \cdot a0(w - v_2) \cdot x \, .$$

The element $\theta \in P(G, G)$ given by

$$\theta(a) = u \cdot x \, , \, \theta(b) = p^{r-s} + v \cdot x \, , \, \theta(x) = w \cdot x$$

is not in $E(G)$, since then $w = 0 \, mod \, (p^t)$ and therefore $p^r \cdot a = x = 0$ which is not the case. (We obtain such a θ by considering $\theta \in \Psi^{-1}(\overline{\theta})$, where $\overline{\theta} \in End(G/[G, F])$ is given by $\overline{\theta}(\overline{a}) = \overline{0}$ and $\overline{\theta}(\overline{b}) = p^{r-s} \cdot \overline{a}$.)

It arises naturally the following question:

Which are the groups for which $P(G, G) = E(G)$?

In the above considerations we have shown that there are p-groups for which the equality is not true and that for the simple non-abelian groups the equality holds.

2. Non-abelian cohomology.

Let G and M be two additive groups. We may define an **n-cochain** from G to M, by $\varphi : G^n \to M$, such that, if $0 \in \{x_1, \dots, x_n\}$, for $x_i \in G, i = 1, 2, \dots, n$, then $\varphi(x_1, \dots, x_n) = 0$.

It is obvious that an **1-cochain** is a mapping preserving 0, therefore, if we denote by $C^n(G, M)$ the set of n-cochains from G to M, then $C^1(G, M) = Ens_0(G, M)$.

For $\varphi \in C^n(G, M)$, we define $\partial\varphi \in C^{n+1}(G, M)$, by the formula:

$$\begin{aligned} \partial\varphi(x_0, x_1, \dots, x_n) &= \varphi(x_1, \dots, x_n) - \varphi(x_0 + x_1, x_2, \dots, x_n) + \\ &+ \varphi(x_0, x_1 + x_2, \dots, x_n) - \dots + (-1)^{n-1}\varphi(x_0, \dots, x_{n-2}, x_{n-1} + x_n) + \\ &+ (-1)^n \varphi(x_0, \dots, x_{n-1}). \end{aligned} \quad (1)$$

Denote

$$Z^n(G, M) := \{\varphi \in C^n(G, M) | \partial\varphi \in C^{n+1}(G, [M, M])\}, \text{ for } n \geq 1, \quad (2)$$

and we consider $Z^0(G, M) = C^0(G, M) = M$. This is the set of **$n$-cocycles** from G to M.

The following properties of n-cochains have been proved in [23] or could be proved by straightforward calculations:

Proposition 2.1. *Let G and M be two additive groups, generally non-abelian. Then:*

(i) $[C^n(G, M), C^n(G, M)] = C^n(G, [M, M])$;

(ii) The mapping $\Psi : C^n(G,M) \to C^n(G,M/[M,M])$ *given by*

$$\Psi(\varphi)(x_0,\dots,x_{n-1}) = \varphi(x_0,\dots,x_{n-1}) + [M,M],$$

is an epimorphism of groups and $Ker\Psi = C^n(G,[M,M])$;

(iii) $\Psi \circ \partial = \partial \circ \Psi$, *with* ∂ *given in (1);*

(iv) $\partial C^n(G,M) \subset Z^{n+1}(G,M)$;

(v) $Ker\Psi = C^n(G,[M,M])$;

(vi) $Z^n(G,M) = \Psi^{-1}(Z^n(G,M/[M,M]))$;

(vii) $\partial \in P(C^n(G,M), C^{n+1}(G,M))$.

Remark 2.2. $Z^1(G,M) = P(G,M)$.

Let us point out that the image of a subgroup by a pseudo-homomorphism (like δ) is not a subgroup in general. Thus the subset of $Z^n(G,M)$ given by applying δ to $C^n(G,M)$ is not a subgroup. Moreover, the sets

$\delta(Z^n(G,M)) = \{\delta\phi : \phi \in Z^n(G,M)\}$ and $C^{n+1}(G,[M,M])$ do not coincide ;

There are only the inclusions $\delta(Z^n(G,M)) \subseteq C^{n+1}(G,[M,M])$, which can be proper.

This is an obstructing difficulty in considering a cohomology theory in this context. Therefore for $n \geq 2$ we could consider the normal subgroups generated in $Z^n(G,M)$ by the subset $\delta(C^{n-1}(G,M))$, and we denote it by $\tilde{B}^n(G,M)$. For convenience we take $\tilde{B}^0(G,M) = \tilde{B}^1(G,M) = 0$. If we put $B^n(G,M) = \Psi^{-1}(\tilde{B}^n(G,M/[M,M]))$, then we get the following inclusions:

$$\delta(C^{n-1}(G,M)) \subseteq \tilde{B}^n(G,M) \subseteq B^N(G,M) \subset Z^n(G,M) \text{ for } n \geq 1\,.$$

In any case, we have a group epimorphism from

$$H^n(G,M) = Z^n(G,M)/\tilde{B}^n(G,M) \text{ to } H^n(G,M/[M,M])$$

which sends the coset of ϕ to the coset of $\Psi(\phi)$

We may change the commutator of M by a normal subgroup of M such that M/T is abelian. Then $[M,M] \subset T$.

Denote

$$P_T(G,M) := \{\varphi \in C^1(G,M) | \partial\varphi \in C^2(G,T))\}. \tag{3}$$

Proposition 2.3. *Let* G *and* M *be groups,* $T \triangleleft M$ *such that* M/T *is an abelian group. Then:*

(i) $< P_T(G,M), + >$ *is a group;*

(ii) When $M = G$ *and* $\alpha(T) \subseteq T$ *for all* $\alpha \in P_T(G,G)$, $< P_T(G,G), +, \circ >$ *is a right near-ring, and* $P(G,G) \subseteq P_T(G,G)$;

(iii) There exists an epimorphism of near-rings:

$$\Psi_T : P_T(G,G) \to End(G/T),$$

given by $\Psi_T(\varphi)(g+T) = \varphi(g) + T$, *and* $Ker\Psi_T = C^1(G,T)$ *is an ideal in* $P_T(G,G)$.

(See [23], for the proof.)

There exist such groups which contain a normal subgroup satisfying the hypotheses in the above proposition; for example, *a metacyclic group* G, *with* $E(G) = I(G)$, *the near-ring generated by* $Inn(G)$ *in* $Ens_0(G)$, *contains a normal subgroup* T *with the properties:*

(i) T *and* G/T *are cyclic groups.*

(ii) $[T,G] = T$.

(See, for more informations, Saad & al [22], Theorem 13.)

Proposition 2.4. *Let* G *and* M *be groups, and* $\rho \in Hom(G,M)$. *For each* $\sigma \in P(G,M)$ *(resp.* $P_T(G,M)$*), there exists* $\mu \in P(G,M)$ *(resp.* $P_T(G,M)$*), such that* $\sigma = \mu + \rho$. *Moreover* $\mu \in Z^1(G,M)$.

Proof. By calculations, we get for $\mu = \sigma - \rho$ that it belongs to $P(G,M)$ (resp. $P_T(G,M)$); indeed, for all $x,y \in G$,

$$\mu(x+y) = \mu(x) + \mu(y) + \rho(x) + \rho(y) + [\rho(y),\rho(x)]+$$

$$+[\rho(x),\sigma(y)] + d_\sigma(x,y) - \rho(y) - \rho(x).$$

The rest of the statement may be proved by verifying that $\delta\mu \in C^2(G,[M,M])$. Indeed

$$\delta\mu(x,y) = \sigma(y) + [\rho(y),-\rho(x)] + \{\rho(x) - d_\sigma(x,y) - \rho(x)\} + [-\rho(x),\sigma(y)] - \sigma(y)$$

$$\in [M,M](\text{ or } \in T \text{ if } \sigma \in P_T(G,M)).$$

We may see that each $\sigma \in P(G,M)$ is of the form $\rho + \mu$ where $\mu \in Z^1(G,M)$ and $\rho \in Hom(G,M)$ is fixed.

Remark 2.5. 1. Given a group homomorphism $\tau : G \to M$, a mapping $\mu \in Ens_0(G,M)$ which satisfies the condition:

$$\mu(x+y) = \mu(x) + {}^{\tau(x)}\mu(y), \text{ for all } x,y \in G, \tag{4}$$

is a pseudo-homomorphism from G to M.

2. If $\mu \in Ens_0(G,M)$ satisfies the condition:

$$\mu(x) = m + {}^{\tau(x)}(-m), \text{ for a fixed } m \in M,$$

then $\mu \in C^1(G,[M,M])$.

This remark shows that the usual 1-cocycles and 1-coboundaries are in fact pseudo-homomorphisms.

3. Some remarks.

In order to study $P(G, G)$ and $P_T(G, G)$ for a normal subgroup T of G and to determine their relationships with $E(G)$ and $Ens_0(G)$, some classes of groups seem to be of interest, namely those for which $E(G)$ or/and $I(G)$ have some additional properties. Here we quote some results in this connection, only for suggesting to be used in future researches.

A. Fröhlich [6] has studied first the case of a finite simple group; in this case $E(G) = Ens_0(G)$. Moreover, $I(G) = Ens_0(G)$ if and only if G is a finite simple group (Higman).

We point out another interesting (from the cohomology point of view) classes of groups: E-groups and I-groups.

Definition 3.1. *A group G is called an* E-**group** *(*I-**group***) if $E(G)$ (resp. $I(G)$) is a ring.*

In this case, taking $N = E(G)$, each N-group is a module.

The first example of E-groups is given by R. Faudree; this is a p-group of order p^8 and exponent p^2, with four generators. In 1971, Faudree [5] has also shown that, *for such a group G, each endomorphism which is not an automorphism is central.* Moreover, this group is nilpotent of class two.

In 1977, Malone [16] has proved:

Proposition 3.2. *In a group $< G, + >$, each element $g \in G$ commutes with its endomorphic images if and only if any two endomorphic images of each g commutes.*

This is also a condition of E-group, and it has been used to find other examples of E-groups (see [1], [2], [17]); among them , there are non-abelian 2-groups, infinite groups and other p-groups of exponent p^2.

Chandy [3] gave conditions for I-groups:

Proposition 3.3. *If G is nilpotent of class two, then $I(G)$ is a commutative ring.*

Proposition 3.4. *G is I-group, if and only if the centralizer of each element is a normal subgroup of G.*

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ABOUT THE DECOMPOSITION MATRIX OF $Sp(4,q)$

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ABSTRACT. This is a little note for the decomposition numbers of $Sp(4,q)$. Moreover this is one of the standard examples to use computer for the group representation theory.

1. INTRODUCTION

Let G the symplectic group $Sp(4,q)$ where q is the power of an odd prime p. The decomposition numbers of G in characteristics other than p are almost determined by White[6][7]. But in case the characteristic divides $q+1$, there is one variable in the decomposition matrix of the principal block. We will determine this variable α under some conditions. After this meeting, the same results are obtained in case p is even (see Waki[4]). All calculations of the scalar products in section 5 are done by *Mathematica*[3] with the character table of $Sp(4,q)$ of White[5].

2. NOTATION

An odd prime r which divides $q+1$ is fixed. Thus there are numbers d and s such that $q+1 = r^d s$ and s is not divisible by r. The subgroup H of G which is denoted by K in Srinivasan[2] is isomorphic to $SL(2,q) \times SL(2,q)$. The order of H is $q^2(q^2-1)^2$. We can see the fusion map between G and H in appendix B. We use the same notation of White[6] for the ordinary characters of G. The ordinary characters and conjugacy classes of G are given in Srinivasan[2]. The group C_n denotes the cyclic group of order n.

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3. A RESULT OF WHITE

Theorem 3.1 (White [6] [7]). *The decomposition matrix for the principal r-block of G is as follows. The unknown entry α is an integer satisfying $1 \leq \alpha \leq \frac{q-1}{2}$.*

Degrees	*Chars.*						*No. of Chars.*
1	1_G	1					1
$q(q-1)^2/2$	θ_{10}		1				1
$q(q^2+1)/2$	θ_{11}	1		1			1
$q(q^2+1)/2$	θ_{12}	1			1		1
q^4	θ_{13}	1	α	1	1	1	1
$(q^2+1)(q-1)^2$	χ_4		$\alpha-2$			1	$(r^d-1)(r^d-3)/8$
$(q^2+1)(q-1)$	χ_6		1		1		$(r^d-1)/2$
$q(q^2+1)(q-1)$	χ_7		$\alpha-1$		1	1	$(r^d-1)/2$
$(q^2+1)(q-1)$	ξ_1		1	1			$(r^d-1)/2$
$q(q^2+1)(q-1)$	ξ_1'		$\alpha-1$	1		1	$(r^d-1)/2$

4. A MAIN RESULT

Theorem 4.1. *In theorem 3.1, we have the following.*

(i) *If 3-Sylow subgroup of $Sp(4,q)$ is isomorphic to the elementary abelian group of order 3^2, then $\alpha = 1$ for $r = 3$.*

(ii) *If 5-Sylow subgroup of $Sp(4,q)$ is isomorphic to the elementary abelian group of order 5^2, then $\alpha = 2$ for $r = 5$.*

5. A PROOF OF THEOREM 4.1

Let $H = H_a \times H_b \cong SL(2,q) \times SL(2,q)$. We can find the character table of $SL(2,q)$ in appendix A. There is an r-block $\widetilde{b_1}$ in $SL(2,q)$ such that this block has the following decomposition matrix.

Degrees	*Chars.*	φ_1	φ_2	*No. of Chars.*
$(q-1)/2$	$\widetilde{\chi}_5$	1		1
$(q-1)/2$	$\widetilde{\chi}_6$		1	1
$q-1$	$\widetilde{\chi}_8(k)$	1	1	$\frac{r^d-1}{2}$

where $k \in I := \{s/2,\ s/2+s,\ \cdots,\ s/2+\frac{r^d-3}{2}s\}$.

There is an r-block b_1 which is constructed by the irreducible characters of H

$$\{\chi_{(5,5)}, \chi_{(5,6)}, \chi_{(5,8)}(k), \chi_{(6,5)}, \chi_{(6,6)}, \chi_{(6,8)}(k), \chi_{(8,5)}(k), \chi_{(8,6)}(k), \chi_{(8,8)}(k,l)\}$$

where $\chi_{(i,j)} = \widetilde{\chi}_i(h_a)\widetilde{\chi}_j(h_b)$ for each $h_a \in H_a$ and $h_b \in H_b$ and $k,l \in I$. It is not so easy to calculate the scalar products of the generic characters. But the values of characters $\widetilde{\chi}_5$, $\widetilde{\chi}_6$ and $\widetilde{\chi}_8(k)$ in the conjugacy class C_i are all 0. Moreover the restricted characters $\chi_{10\downarrow H}$, $\chi_{11\downarrow H}$, $\chi_{12\downarrow H}$ and $\chi_{13\downarrow H}$ take same values in the same kind of the conjugacy classes.

Remark 5.1. It is easy to check the following equations for each $k \in I$.

$$\sum_{i=1}^{\frac{q-1}{2}} -(\xi^{ik} + \xi^{-ik}) = (-1)^k + 1$$

$$\sum_{i=1}^{\frac{q-1}{2}} (-1)^i(\xi^{ik} + \xi^{-ik}) = -2$$

$$\sum_{i=1}^{\frac{q-1}{2}} (\xi^{ik} + \xi^{-ik})^2 = q - 3$$

where $\xi := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Then we can get the following lemma from appendices A and B.

Lemma 5.2.

$$\begin{aligned}
(\chi_{(5,6)}, \chi_{10\downarrow H}) &= 1 \\
(\chi_{(5,8)}(k), \chi_{10\downarrow H}) &= 0 \\
(\chi_{(8,6)}(k), \chi_{10\downarrow H}) &= 0 \\
(\chi_{(8,8)}(k,l), \chi_{10\downarrow H}) &= \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \\
(\chi_{(5,6)}, \chi_{11\downarrow H}) &= 0 \\
(\chi_{(5,8)}(k), \chi_{11\downarrow H}) &= 0 \\
(\chi_{(8,6)}(k), \chi_{11\downarrow H}) &= 0 \\
(\chi_{(8,8)}(k,l), \chi_{11\downarrow H}) &= \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \\
(\chi_{(5,6)}, \chi_{12\downarrow H}) &= 0 \\
(\chi_{(5,8)}(k), \chi_{12\downarrow H}) &= 0 \\
(\chi_{(8,6)}(k), \chi_{12\downarrow H}) &= 0 \\
(\chi_{(8,8)}(k,l), \chi_{12\downarrow H}) &= 0 \\
(\chi_{(5,6)}, \chi_{13\downarrow H}) &= 0 \\
(\chi_{(5,8)}(k), \chi_{13\downarrow H}) &= 1 \\
(\chi_{(8,6)}(k), \chi_{13\downarrow H}) &= 1 \\
(\chi_{(8,8)}(k,l), \chi_{13\downarrow H}) &= \begin{cases} 1 & k = l \\ 2 & k \neq l \end{cases}
\end{aligned}$$

where $k, l \in I$.

It is easy to define the decomposition matrix of b_1 by $\widetilde{b_1}$. From the above lemma, the multiplicities of the Brauer character $\varphi_{(1,2)} := \chi_{(5,6)}$ of H in $\theta_{10\downarrow H}$, $\theta_{11\downarrow H}$, $\theta_{12\downarrow H}$, and $\theta_{13\downarrow H}$ are $\frac{r^d+1}{2}$, $\frac{r^d-1}{2}$, 0 and $\frac{r^d(r^d-1)}{2}$ respectively. Thus we can get $\frac{r^d(r^d-1)}{2} \geq \alpha(\frac{r^d+1}{2}) + \frac{r^d-1}{2}$.

Proposition 5.3.

$$\alpha \leq \frac{(r^d-1)^2}{r^d+1}.$$

Remark 5.4. It is easy to check that an r–Sylow subgroup of G is isomorphic to $C_{r^d} \times C_{r^d}$. From this proposition, we can see that the variable α has just finitely many possibilities when we fix an r–Sylow subgroup of G. (See Donovan's Conjecture in Alperin[1])

The assumption of theorem 4.1 means r^d is 3 or 5. From proposition 5.3, $\alpha \leq 1$ if $r^d = 3$. Since $\alpha \geq 1$, α must be 1. If $r^d = 5$, proposition 1 shows $\alpha \leq \frac{8}{3}$. In case r^d is bigger than 3, $\alpha \geq 2$. Thus theorem 4.1 is proved.

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Appendix A. The character table of $SL(2,q)$

$SL_2(q)$	I	Z	P_1	P_2	$P_1 Z$	$P_2 Z$	C_i	D_i
# of conj.	1	1	1	1	1	1	$(q-3)/2$	$(q-1)/2$
# of elem.	1	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$
$\widetilde{\chi}_1$	1	1	1	1	1	1	1	1
$\widetilde{\chi}_2$	q	q	0	0	0	0	1	-1
$\widetilde{\chi}_3$	$\frac{q+1}{2}$	$\frac{(q+1)e}{2}$	b_q^+	b_q^-	$b_q^+ e$	$b_q^- e$	$(-1)^i$	0
$\widetilde{\chi}_4$	$\frac{q+1}{2}$	$\frac{(q+1)e}{2}$	b_q^-	b_q^+	$b_q^- e$	$b_q^+ e$	$(-1)^i$	0
$\widetilde{\chi}_5$	$\frac{q-1}{2}$	$-\frac{(q-1)e}{2}$	$-b_q^-$	$-b_q^+$	$b_q^- e$	$b_q^+ e$	0	$-(-1)^i$
$\widetilde{\chi}_6$	$\frac{q-1}{2}$	$-\frac{(q-1)e}{2}$	$-b_q^+$	$-b_q^-$	$b_q^+ e$	$b_q^- e$	0	$-(-1)^i$
$\widetilde{\chi}_7(k)$	$q+1$	$(-1)^k(q+1)$	1	1	$(-1)^k$	$(-1)^k$	$\zeta^{ik}+\zeta^{-ik}$	0
$\widetilde{\chi}_8(k)$	$q-1$	$(-1)^k(q-1)$	-1	-1	$-(-1)^k$	$-(-1)^k$	0	$-\xi^{ik}-\xi^{-ik}$

$e := (-1)^{\frac{q-1}{2}}$ $\quad b_q^+ := \frac{1+\sqrt{eq}}{2}$ $\quad b_q^- := \frac{1-\sqrt{eq}}{2}$ $\quad \zeta := \exp(\frac{2\pi\sqrt{-1}}{q-1})$ $\quad \xi := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Appendix B. The fusion map between G and H

	I	Z	P_1	P_2	$P_1 Z$	$P_2 Z$	C_k	D_l
I	A_1	D_1	A_{21}	A_{22}	D_{21}	D_{22}	$C_3(k)$	$C_1(l)$
Z	D_1	A'_1	D_{23}	D_{24}	A'_{21}	A'_{22}	$C'_3(k)$	$C'_1(l)$
P_1	A_{21}	D_{23}	A_{31}	A_{32}	D_{31}	D_{32}	$C_{41}(k)$	$C_{21}(l)$
P_2	A_{22}	D_{24}	A_{32}	A_{31}	D_{33}	D_{34}	$C_{42}(k)$	$C_{22}(l)$
$P_1 Z$	D_{21}	A'_{21}	D_{31}	D_{33}	A'_{31}	A'_{32}	$C'_{41}(k)$	$C'_{21}(l)$
$P_2 Z$	D_{22}	A'_{22}	D_{32}	D_{34}	A'_{32}	A'_{31}	$C'_{42}(k)$	$C'_{22}(l)$
C_i	$C_3(i)$	$C'_3(i)$	$C_{41}(i)$	$C_{42}(i)$	$C'_{41}(i)$	$C'_{42}(i)$	$B_3(i,k)$	$B_5(i,l)$
D_j	$C_1(j)$	$C'_1(j)$	$C_{21}(j)$	$C_{22}(j)$	$C'_{21}(j)$	$C'_{22}(j)$	$B_5(k,j)$	$B_4(j,l)$

These entries of the table are names of the conjugacy classes of G in case $q \equiv 1 \bmod 4$. Note that $B_3(i,i)$ means $B_8(i)$ and $B_4(j,j)$ means $B_6(j)$. We should exchange A_{31} to A_{32} and A'_{31} to A'_{32} in case $q \equiv 3 \bmod 4$.

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HOMOLOGICAL DEFINABILITY OF p-ADIC REPRESENTATIONS OF GROUPS WITH CYCLIC SYLOW p-SUBGROUP

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1. Introduction and preliminaries

It was shown in the paper [Ya1] that the p-adic representations of a cyclic p-group are almost completely determined by their homological parameters. This description appeared to be rather convenient; some applications of this result can be found in [Ya1, Ya2]. The purpose of this work is to extend the main theorem of the paper [Ya1] from cyclic p-groups to finite groups with cyclic Sylow p-subgroups.

1.1. **Notations.** We use throughout the paper the following notations:

p – a prime integer;
$\mathbb{Z}_p$ – the ring of p-adic integers;
G – a finite group with cyclic Sylow p-subgroup F;
p^{n-1} – the order of F;
$\sigma = \sigma_1$ – a fixed generator of F;
F_i – the subgroup of F generated by the element $\sigma_i = \sigma^{p^{i-1}}$;
N_i – the normalizer of the subgroup F_i in the group G;
$\Lambda = \mathbb{Z}_p[G]$ – the group ring of the group G over the ring $\mathbb{Z}_p$;
$\Sigma_i = \mathbb{Z}_p[N_i]$ – the group ring of the group N_i over the ring $\mathbb{Z}_p$ ($i > 0$).

Some reasonings of this paper are different for $i \geq 1$ and for $i = 0$. To cover the case $i = 0$ it will be convenient to denote by F_0, σ_0, N_0 and Σ_0 the group F, its generator σ, its normalizer N_1 and the group ring of the normalizer $\mathbb{Z}_p[N_0]$

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(so $F_0 = F_1$, $N_0 = N_1$, $\sigma_0 = \sigma$, $\Sigma_0 = \Sigma_1$). It is clear that

$$G = N_n \supseteq N_{n-1} \supseteq \cdots \supseteq N_2 \supseteq N_1 = N_0 \supseteq F_0 = F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n = 0\,,$$
$$\Sigma_0 = \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_n = \Lambda\,.$$

The following elements of Λ will play an important role in our investigation:

$$r_0 = 1 - \sigma\,, \quad \nu_n = \mu_0 = 1\,, \ \nu_0 = \mu_n = 0\,,$$
$$r_i = 1 + \sigma_i + \sigma_i^2 + \cdots + \sigma_i^{p-1} = 1 + \sigma^{p^{i-1}} + \sigma^{2p^{i-1}} + \cdots + \sigma^{(p-1)p^{i-1}}\,,$$
$$\nu_i = r_i r_{i+1} \dots r_{n-1} = \sum_{f \in F_i} f\,, \quad \mu_i = r_0 \dots r_{i-1} = 1 - \sigma_i \quad (1 \le i < n)\,.$$

Lemma 1.1. *If x is an element of one of the rings $\mathbb{Z}_p[F]$, Λ, Σ_j, such that $\mu_i x = 0$ (or $\nu_i x = 0$), then there exists an element y of this ring such that $\nu_i y = x$ (respectively $\mu_i y = x$). For any element $g \in N_i$ and any $j \ge i$*

$$g\nu_j = \nu_j g\,, \quad g\mu_j \in \mu_j \Sigma_i\,, \quad \mu_j g \in \Sigma_i \mu_j\,.$$

For any element $g \in N_i$ there exist elements $q, q' \in \Sigma_i = \mathbb{Z}_p[N_i]$ such that $r_i g = (g + q\mu_i) r_i$, $g r_i = r_i (g + \mu_i q')$.

Proof. The first statement is trivial since all the rings are free $\mathbb{Z}_p[F]$-modules. If $g \in N_i$ and $j \ge i$ then g belongs to the normalizer $N_j \supseteq N_i$ of the group F_j as well; hence, $g\sigma_j g^{-1} = \sigma_j^l$, $g^{-1}\sigma_j g^{-1} = \sigma_j^m$ for positive integers l, m, and

$$g\nu_j = \Big(\sum_{s=1}^{p^{n-j}} g\sigma_j^s g^{-1}\Big) g = \Big(\sum_{s=1}^{p^{n-j}} \sigma_j^{sl}\Big) g = \nu_j g\,,$$
$$g\mu_j = g\mu_j g^{-1} g = (g\sigma_j g^{-1} - 1) g = (\sigma_j^l - 1) g \in \mu_j \Sigma_i\,,$$
$$\mu_j g = g g^{-1} \mu_j g = g(g^{-1}\sigma_j g - 1) = g(\sigma_j^m - 1) \in \Sigma_i \mu_j\,.$$

In particular, $\nu_i g = g\nu_i$, $\nu_{i+1} g = g\nu_{i+1}$ and

$$r_i g \nu_{i+1} = r_i \nu_{i+1} g = \nu_i g = g\nu_i = g r_i \nu_{i+1}\,.$$

Therefore, there is an element $q \in \Sigma_i = \mathbb{Z}_p[N_i]$ such that $r_i g = g r_i + q\mu_{i+1} = (g + q\mu_i) r_i$. The existence of q' can be proved similarly.

1.2. **The category $\mathfrak{M}$.** Now let us define a new category $\mathfrak{M}$. Objects of this category are the diagrams

$$T_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} T_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \cdots \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} T_n$$

satisfying the following conditions:

(1) T_i is a finitely generated Σ_i-module ($1 \le i \le n$);
(2) $T_0 = T_n = 0$;
(3) the mappings α_i, β_i are homomorphisms of Σ_i-modules ($1 \le i \le n$);

(4) $\alpha_i\beta_i$ coincides with the action of the operator $r_i \in \Sigma_{i+1}$ $(0 \le i < n)$;
(5) $\beta_i\alpha_i$ coincides with the action of the operator $r_i \in \Sigma_i$ $(1 \le i < n)$.

Morphisms of the object

$$T_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} T_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \cdots \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} T_n$$

into the object

$$T'_0 \underset{\beta'_0}{\overset{\alpha'_0}{\rightleftarrows}} T'_1 \underset{\beta'_1}{\overset{\alpha'_1}{\rightleftarrows}} \cdots \underset{\beta'_{n-1}}{\overset{\alpha'_{n-1}}{\rightleftarrows}} T'_n$$

are the collections of Σ_i-homomorphisms $\gamma_i : T_i \to T'_i$ $(0 \le i \le n)$ such that $\alpha'_i\gamma_i = \gamma_{i+1}\alpha_i$, $\gamma_i\beta_i = \beta'_i\gamma_{i+1}$ for all $i < n$.

Proposition 1.2. *If* $T_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} T_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \cdots \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} T_n$ *is an object of the category* $\mathfrak{M}$ *then* $\mu_iT_i = \nu_iT_i = 0$ *for any* i $(0 \le i \le n)$.

Proof. Since $T_1 = T_n = 0$ and all homomorphisms are $\mathbb{Z}_p[F]$-homomorphisms, we have:

$$\mu_iT_i = r_0r_1\dots r_{i-1}T_i = \alpha_{i-1}\dots\alpha_1\alpha_0\beta_0\beta_1\dots\beta_{i-1} \in \alpha_{i-1}\dots\alpha_1\alpha_0T_0 = 0\,,$$
$$\nu_iT_i = r_{n-1}\dots r_iT_i = \beta_{i+1}\dots\beta_{n-1}\alpha_{n-1}\dots\alpha_{i+1}T_i \subseteq \beta_{i+1}\dots\beta_{n-1}T_n = 0\,.$$

1.3. **The functor** U. Let A be a Λ-module which is free and finitely generated as $\mathbb{Z}_p$-module; denote by A^i the $\mathbb{Z}_p$-submodule of A consisting of all elements $a \in A$ such that $\nu_ia = 0$. It is clear that $\Lambda^i = \mu_i\Lambda$, and more generally, $A^i \supseteq \mu_iA$ for any Λ-module A; denote by $U_i(A)$ the factor group A^i/μ_iA.

According to Lemma 1.1

$$\nu_iga = g\nu_ia = 0\,, \quad g\mu_ib \in \mu_i\Lambda A = \mu_iA\,.$$

for any elements $a \in A^i$, $b \in A$. Therefore, the groups A^i and μ_iA are Σ_i-submodules of A and consequently $U_i(A) = A^i/\mu_iA$ is a Σ_i-module.

It is obvious that $A^{i+1} \subseteq A^i$, $\mu_{i+1}A \subseteq \mu_iA$; therefore, the inclusion $A^{i+1} \to A^i$ induces a homomorphism $\tau_i : U_{i+1}(A) \to U_i(A)$, which is obviously a homomorphism of Σ_i-modules. On the other hand, $r_iA^i \subseteq A^{i+1}$, $r_i\mu_iA \subseteq \mu_{i+1}A$ and the operator $r_i \in \Lambda$ induces a homomorphism $\rho_i : U_i(A) \to U_{i+1}(A)$. It follows from Lemma 1.1 that for any elements $a \in A^i$, $g \in N_i$ the element gr_ia differs from the element r_iga only by a summand $r_i\mu_iq'a = \mu_{i+1}q'a \in \mu_{i+1}A$; therefore, ρ_i is a Σ_i-homomorphism.

Denote by $U(A)$ the following diagram:

$$U_0(A) \underset{\tau_0}{\overset{\rho_0}{\rightleftarrows}} U_1(A) \underset{\tau_1}{\overset{\rho_1}{\rightleftarrows}} \cdots \underset{\tau_{n-1}}{\overset{\rho_{n-1}}{\rightleftarrows}} U_n(A)\,.$$

The groups $U_i(A)$ and the homomorphisms ρ_i, τ_i have simple homological interpretation: $U_i(A) = H^1(F_i, A)$ for $i \geq 1$, and the homomorphisms ρ_i, τ_i are respectively restrictions and corestrictions.

Proposition 1.3. *$U(A)$ is an object of the category $\mathfrak{M}$.*

Proof. The requirements (1) – (5) of the definition of objects of the category $\mathfrak{M}$ are immediate corollaries of the definitions of the modules $U_i(A)$ and the homomorphisms ρ_i, τ_i.

2. Main theorems

We say that a Λ-module A is a p-adic integral representation of the group G if the module A regarded as a $\mathbb{Z}_p$-module is a free module of finite rank; we say that this representation is cohomologically trivial in the dimension 1 if $U(A) = 0$.

The main result of this paper is the following

Theorem 2.1. *For any object T of the category $\mathfrak{M}$ there exists a p-adic integral representation A of the group G such that $U(A) = T$. This representation is unique up to cohomologically trivial in the dimension 1 direct summands.*

The proof will be given later. Denote by $\mathfrak{R}(G)$ the set of isomorphism classes of indecomposable p-adic integral representation of the group G which are not cohomologically trivial in the dimension 1.

Theorem 2.2. *There exists a one-to-one correspondence between the elements of $\mathfrak{R}(G)$ and indecomposable objects of the category $\mathfrak{M}$.*

Proof. This theorem is an immediate corollary of Theorem 2.1.

Corollary 2.3. *Let G and G' be finite groups with cyclic Sylow p-subgroups. Let $P \subseteq G$ and $P' \subseteq G'$ be any subgroups of order p and let N and N' be the normalizers of P (in G) and P' (in G'). If the groups N and N' are isomorphic, then there exists a canonical one-to-one correspondence between the elements of the sets $\mathfrak{R}(G)$ and $\mathfrak{R}(G')$.*

Proof. The category $\mathfrak{M}$ depends in fact not on the group G but on the normalizers $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_{n-1}$ of the subgroups of the Sylow subgroup F. But the group N_{n-1} determines the group F and all the normalizers N_j up to isomorphism.

Theorem 2.4. *Any indecomposable cohomologically trivial in dimension 1 p-adic integral representation of the group G is isomorphic to a direct summand of one of the modules $\mathbb{Z}_p[G/F_i] = \Lambda/\Lambda\mu_i$.*

Proof. Let A be a left Λ-module. Denote by Γ the group ring $\mathbb{Z}_p[F]$. The ring $\Lambda = \mathbb{Z}_p[G]$ is a left and a right Γ-module; so we can define a left Λ-module $B = \Lambda \otimes_\Gamma A$; recall that operators from Λ acts on B by the rule

$$x(y \otimes_\Gamma a) = xy \otimes_\Gamma a \qquad (x, y \in \Lambda\,,\ a \in A)\,.$$

The mapping $x \otimes_\Gamma a \to xa$ defines a Λ-homomorphism $\varphi : B \to A$. It is clear that φ is an epimorphism; denote by X its kernel.

Let $G = F \cup Fg_2F \cup \cdots \cup Fg_kF$ be a decomposition of the group G into a union of double cosets. The module B regarded as a Γ-module decomposes into a direct sum $\Gamma \otimes_\Gamma A \oplus \Gamma g_2 \Gamma \otimes_\Gamma A \oplus \cdots \oplus \Gamma g_k \Gamma \otimes_\Gamma A$. The first summand $\Gamma \otimes_\Gamma A$ is canonically isomorphic to A, and the restriction of φ on it is an isomorphism of Γ-modules $\Gamma \otimes_\Gamma A \to A$. Thus, the extension

$$0 \longrightarrow X \longrightarrow B \xrightarrow{\ \varphi\ } A \longrightarrow 0 \qquad (*)$$

splits as an extension of Γ-modules.

Assume now that the module A regarded as a $\mathbb{Z}_p$-module is a free module of finite rank. Then the extension groups $\mathrm{Ext}^1_\Gamma(A, X)$ and $\mathrm{Ext}^1_\Lambda(A, X)$ are canonically isomorphic to the groups $H^1(F, \mathrm{Hom}_{\mathbb{Z}_p}(A, X))$ and $H^1(G, \mathrm{Hom}_{\mathbb{Z}_p}(A, X))$ (see [CE], XVI.7.(6) and the definition in XII.2). Let $w \in H^1(G, \mathrm{Hom}_{\mathbb{Z}_p}(A, X))$ be the element corresponding to the extension of Λ-modules $(*)$; then its restriction $\rho w \in H^1(F, \mathrm{Hom}_{\mathbb{Z}_p}(A, X))$ corresponds to the same extension regarded as an extension of Γ-modules. We have seen that the last extension splits, i.e., $\rho w = 0$. But the index $(G : F)$ is prime to p and consequently it is invertible in $\mathbb{Z}_p$; therefore, the restriction $\rho : H^1(G, \mathrm{Hom}_{\mathbb{Z}_p}(A, X)) \to H^1(F, \mathrm{Hom}_{\mathbb{Z}_p}(A, X))$ is a monomorphism ([CE], XII.8.(6)) and $w = 0$. Hence, the extension $(*)$ splits as an extension of Λ-modules. Therefore, we have proved that any p-adic integral representation A of the group G is a direct summand of the module $\Lambda \otimes_\Gamma A$.

Assume now that the representation A is cohomologically trivial in dimension 1. Then the module A regarded as a Γ-module decomposes into a direct sum of several modules each of which is isomorphic to one of the modules $\mathbb{Z}_p[F/F_i]$ (this follows, for example, from the results of [Ya1]). Therefore, A is a direct summand of a direct sum of several Λ-modules, each of which is isomorphic to one of the modules $\Lambda \otimes_\Gamma \mathbb{Z}_p[F/F_i] = \Lambda/\Lambda\mu_i = \mathbb{Z}_p[G/F_i]$. It remains to apply the Krull-Schmidt theorem, which holds for p-adic integral representations (see, for example, [R]).

3. The module $\Lambda/\Lambda r_i$

Lemma 3.1. *The Λ-module $X = \Lambda/\Lambda r_i$ regarded as a Σ_i-module is a direct sum of a submodule $\bar{X}_1$ generated by the element $z = 1 \pmod{\Lambda r_i}$ and a free Σ_i-submodule $\bar{Y}$. The generator z satisfies only the relations following from the* *relation* $r_i z = 0$. *The module* $\bar{X}_1$ *regarded as a* $\mathbb{Z}_p$-*module is free and finitely generated; besides,* $r_i\bar{X}_1 = 0$ *and* $\bar{X}_1^{i+1} = 0$.

Proof. Let $G = N_i \cdot 1 \cdot F_i \cup N_i g_2 F_i \cup \cdots \cup N_i g_k F_i$ be a representation of the group G as a union of double cosets. The group ring $\Lambda = \mathbb{Z}_p[G]$ can be regarded as a left $\mathbb{Z}_p[N_i]$- and a right $\mathbb{Z}_p[F_i]$-module. This bimodule decomposes into a direct sum $\mathbb{Z}_p[G] = X_1 \oplus X_2 \oplus \cdots \oplus X_k$ of its $\mathbb{Z}_p[N_i] - \mathbb{Z}_p[F_i]$-submodules $X_1 = \mathbb{Z}_p[N_i] \cdot \mathbb{Z}_p[F_i] = \mathbb{Z}_p[N_i]$, $X_j = \mathbb{Z}_p[N_i] g_j \mathbb{Z}_p[F_i]$ $(j > 1)$. Therefore, the module $\Lambda/\Lambda r_i$ regarded as a left $\mathbb{Z}_p[N_i]$-module is the direct sum of the left $\mathbb{Z}_p[N_i]$-modules $\bar{X}_j = X_j/X_j r_i$, $1 \leq j \leq k$.

Assume first that $j > 1$; in this case g_j does not belong to the normalizer of the group F_i. Moreover, the group $g_j F_i g_j^{-1}$ is not contained in the group N_i. Indeed, F is a Sylow p-subgroup of the group N_i; F_i is the only subgroup of F, the order of which is equal to p^{n-i}, if $i \geq 1$, and equal to p^{n-1}, if $i = 0$. If $g_j F_i g_j^{-1} \subseteq N_i$ then $g_j F_i g_j^{-1}$ is contained in a Sylow subgroup F' of the group N_i. All Sylow subgroups are conjugate; therefore, there exists an element $h \in N_i$ such that $F' = hFh^{-1}$. Thus, $g_j F_i g_j^{-1} \subseteq hFh^{-1}$, and $g_j F_i g_j^{-1} = hF_i h^{-1}$ since the order of the group $g_j F_i g_j^{-1}$ is equal to the order of the group F_i. But F_i is a normal subgroup of the group N_i; hence, $hF_ih^{-1} = F_i$ and $g_jF_ig_j^{-1} = F_i$, i.e., $g_j \in N_i$ in contradiction to the assumption.

Thus the intersection $N_i \cap g_j F_i g_j^{-1}$ is a proper subgroup of the cyclic group F_i, and $g_j^{-1} F g_j \cap F_i = F_l$ for an integer $l > 1$, $i < l \leq n$. Any element of the bimodule X_j can be uniquely represented in the form

$$\sum_{s=0}^{p^{l-i}-1} \sum_{f \in N_i} a_{s,f} f g_j \sigma_i^s, \quad a_{s,f} \in \mathbb{Z}_p$$

for $i \geq 1$ and in the form

$$\sum_{s=0}^{p^{l-1}-1} \sum_{f \in N_0} a_{s,f} f g_j \sigma^s, \quad a_{s,f} \in \mathbb{Z}_p$$

for $i = 0$; therefore, the bimodule X_j regarded as a left Σ_i-module is a free Σ_i-module with the basis $\{g_j\sigma_i^s,\ 0 \leq s < p^{l-i}\}$ for $i \geq 1$ and the basis $\{g_j\sigma^s,\ 0 \leq s < p^{l-1}\}$ for $i = 0$. Hence, $X_j r_i$ is the Σ_i-submodule of X_j generated by the elements

$$g_j r_i \sigma_i^s = g_j(1 + \sigma_i + \cdots + \sigma_i^{p-1})\sigma_i^s \quad (0 \leq s \leq p^{l-i} - p)$$

for $i > 1$, and by the elements

$$g_j(1 - \sigma^s) \quad (1 \leq s < p^{l-1})$$

for $i = 0$. It is clear that X_j is the direct sum of $X_j r_i$ and a free Σ_i-submodule Y of the module X_j generated by the elements $g_j\sigma_i^t$, $0 \leq t < p - 1$, for $i \geq 1$, and generated by the element g_j for $i = 0$. The factor module $\bar{X}_j = X_j/X_j r_i$ is isomorphic to Y and consequently $\bar{X}_j$ is a free Σ_i-module for $j > 1$.

The Σ_i-module $\bar{X}_1$ is equal to $\Sigma_i/\Sigma_i r_i$; therefore, this module is generated by the element $z = 1 \pmod{\Sigma_i r_i} = 1 \pmod{\Lambda r_i}$ satisfying the only relation $r_i z = 0$. It is clear that $\bar{X}_1$ is a free $\mathbb{Z}_p$-module of finite rank. Let g be any element of the group N_i; if $i \geq 1$ then by Lemma 1.1 there is an element $q \in \Sigma_i$ such that $r_i g = (g + q\mu_i)r_i$, and $r_i g z = (g + q\mu_i)r_i z = 0$. The same is true for $i = 0$; indeed, $g^{-1}\sigma g = \sigma^u$ for a positive integer u and $(1-\sigma)gz = g(1-\sigma^u)z = g(1 + \sigma + \cdots + \sigma^{u-1})(1-\sigma)z = 0$. Thus, $r_i\bar{X}_1 = r_i\mathbb{Z}_p[N_i]z = 0$.

It remains to prove that $\bar{X}_1^{i+1} = 0$. Let $x \in \bar{X}_1^{i+1}$; then $\nu_{i+1}x = 0$. Besides, $x \in \bar{X}_1$ and consequently $r_i x = 0$. Show that $x = 0$. Assume at first that $i \geq 1$; the polynomials $v(t) = 1 + t + \cdots + t^{p-1}$ and $w(t) = 1 + t^p + \cdots + t^{p^{n-i}-p}$ are relatively prime; therefore, there exist polynomials $v_1(t)$, $w_1(t)$, the coefficients of which are integers, and an integer $d \neq 0$ such that $d = v_1(t)v(t) + w_1(t)w(t)$. We have now:

$$dx = v_1(\sigma_i)v(\sigma_i)x + w_1(\sigma_i)w(\sigma_i)x = v_1(\sigma_i)r_i x + w_1(\sigma_i)\nu_{i+1}x = 0\,.$$

If $i = 0$ then the polynomials $v(t) = 1 - t$, $w(t) = 1 + t^p + \cdots + t^{p^{n-1}-1}$ are relatively prime, and again $d = v_1(t)v(t) + w_1(t)w(t)$, $dx = 0$ for some polynomials $v_1(t)$, $w_1(t)$ and an integer $d \neq 0$. But the module $\bar{X}_1$ is a torsion free $\mathbb{Z}_p$-module; hence, $x = 0$.

4. The proof of Theorem 2.1. Existence

4.1. **Auxiliary theorem.** To prove the existence of the module A we shall prove at first a stronger theorem.

Theorem 4.1. *Let* $T_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} T_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \cdots \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} T_n$ *be an object of the category* $\mathfrak{M}$. *For every* i, $0 \leq i \leq n$, *there exist a* Λ-*module* B_i *and* Σ_j-*epimorphisms* $\pi_i^j : B_i^j \to T_j$ $(i \leq j \leq n)$ *such that:*

a$_i$) *the following diagram is commutative:*

$$\begin{array}{ccccccc} B_i^i & \xrightarrow{r_i} & B_i^{i+1} & \xrightarrow{r_{i+1}} & \cdots & \xrightarrow{r_{n-1}} & B_i^n \\ \downarrow{\scriptstyle \pi_i^i} & & \downarrow{\scriptstyle \pi_i^{i+1}} & & & & \downarrow{\scriptstyle \pi_i^n} \\ T_i & \xrightarrow[\alpha_i]{} & T_{i+1} & \xrightarrow[\alpha_{i+1}]{} & \cdots & \xrightarrow[\alpha_{n-1}]{} & T_n \end{array}$$

b$_i$) *for any* j, $i \leq j < n$, *the restriction of the homomorphism* π_i^j *on* B_i^{j+1} *coincides with the homomorphism* $\beta_j\pi_i^{j+1}$;

c$_i$) $r_j \operatorname{Ker}\pi_i^j = \operatorname{Ker}\pi_i^{j+1}$ *for any* j, $i \leq j < n$;

d$_i$) *the module* B_i *regarded as a* $\mathbb{Z}_p$-*module is a free module of finite rank;*

e$_i$) *the Λ-module B_i is generated by B_i^i;*

f$_i$) *the factor group B_i/B_i^i regarded as a $\mathbb{Z}_p[F]$-module is isomorphic to the direct sum of several copies of the group ring $\mathbb{Z}_p[F/F_i]$ regarded as a $\mathbb{Z}_p[F]$-module;*

g$_i$) $\mu_i B_i \subseteq \operatorname{Ker} \pi_i^i$.

4.2. **Preliminaries.** We shall prove Theorem 4.1 by inverse induction on i; the module $B_n = 0$ and the homomorphism $\pi_n^n = 0$ satisfy the requirements of the theorem for $i = n$. Let $0 < i < n$ and assume that a Λ-module B_{i+1} and homomorphisms π_{i+1}^j ($j \geq i+1$), satisfying the requirements a$_{i+1}$)–g$_{i+1}$), are already constructed.

Lemma 4.2. *For $j \leq i$ the groups B_{i+1}^j, $B_{i+1}^{i+1} + \mu_j B_{i+1}$ coincide.*

Proof. It is clear that $B_{i+1}^{i+1} + \mu_j B_{i+1} \subseteq B_{i+1}^j$. Let b be any element of the group B_{i+1}^j. Then the class b (mod B_{i+1}^{i+1}) belongs to the group $(B_{i+1}/B_{i+1}^{i+1})^j$. By the statement f$_{i+1}$) of the inductive hypothesis the group B_{i+1}/B_{i+1}^{i+1} regarded as $\mathbb{Z}_p[F]$-module is isomorphic to the direct sum of several copies of the module $\mathbb{Z}_p[F/F_{i+1}]$; therefore $(B_{i+1}/B_{i+1}^{i+1})^j = \mu_j(B_{i+1}/B_{i+1}^{i+1})$. It follows that there exists an element $b_1 \in B_{i+1}$ such that $b - \mu_j b_1 \in B_{i+1}^{i+1}$. Thus $B_{i+1}^j \subseteq B_{i+1}^{i+1} + \mu_j B_{i+1}$.

4.3. **The modules** W, V. Denote by W the submodule of the direct sum $T_i \oplus B_{i+1}^{i+1}$ consisting of all elements $t \oplus b$ such that $\alpha_i(t) = \pi_{i+1}^{i+1}(b)$ and by V the submodule of the same direct sum consisting of all elements $\beta_i \pi_{i+1}^{i+1}(b) \oplus r_i b$, $b \in B_{i+1}^{i+1}$. Further, denote by ϵ_1, ϵ_2 the canonical projections of the direct sum $T_i \oplus B_{i+1}$ onto T_i, B_{i+1}^i, and by $\bar{\epsilon}_1$, $\bar{\epsilon}_2$ the restrictions of ϵ_1, ϵ_2 on W. It is obvious that $r_i W \subseteq V \subseteq W$, $\bar{\epsilon}_1 W = T_i$, $\bar{\epsilon}_2 W = (\pi_{i+1}^{i+1})^{-1} \operatorname{Im} \alpha_i$, $\alpha_i \bar{\epsilon}_1 = \pi_{i+1}^{i+1} \bar{\epsilon}_2$. The normalizer N_i of the group F_i is contained in the normalizer N_{i+1} of the group F_{i+1}; therefore, $\Sigma_i = \mathbb{Z}_p[N_i] \subseteq \mathbb{Z}_p[N_{i+1}] = \Sigma_{i+1}$ and the groups T_i, B_{i+1} and their direct sum $T_i \oplus B_{i+1}$ can be regarded as Σ_i-modules. We shall show that V, W are Σ_i-submodules of $T_i \oplus B_{i+1}$. Indeed, for any elements $g \in N_i$, $t \oplus b \in W$ we have $\alpha_i(gt) = g\alpha_i(t) = g\pi_{i+1}^{i+1}(b) = \pi_{i+1}^{i+1}(gb)$; so $gW \subseteq W$. By Lemma 1.1 there is an element $q' \in \Sigma_i$ such that $gr_i = r_i(g + \mu_i q')$. For any element $b \in B_{i+1}^{i+1}$ we have $gb \in B_{i+1}^{i+1}$ and

$$g\beta_i \pi_{i+1}^{i+1}(b) \oplus gr_i b = \beta_i \pi_{i+1}^{i+1}(gb) \oplus r_i(gb + \mu_i q' b) = \beta_i \pi_{i+1}^{i+1}(gb + \mu_i q' b) \oplus r_i(gb + \mu_i q' b),$$

because $\beta_i \pi_{i+1}^{i+1}(\mu_i q' b) = \mu_i \beta_i \pi_{i+1}^{i+1}(q' b) \subseteq \mu_i T_i = 0$. Hence $gV \subseteq V$.

4.4. **The construction of the module B_i.** Choose elements $e_1, \dots, e_m \in W$ so that these elements together with the module V generate W as a Σ_i-module. Denote by B_i the Λ-module containing B_{i+1} and generated by B_{i+1} and elements $x_1, \dots, x_m$, satisfying only the relations $r_i x_j = \bar{\epsilon}_2(e_j)$, $1 \le j \le m$.

Lemma 4.3. *The modules B_i/B_{i+1}, B_i regarded as $\mathbb{Z}_p$-modules, are free modules of finite rank.*

Proof. It follows from the definition of the module B_i that the Λ-module B_i/B_{i+1} is generated by the elements $\bar{x}_1 = x_1 \pmod{B_{i+1}}, \dots, \bar{x}_m = x_m \pmod{B_{i+1}}$, and these generators satisfy only the relations following from the relations $r_i \bar{x}_j = 0$, $1 \le j \le m$. Therefore, the module B_i/B_{i+1} is isomorphic to the direct sum of m copies of the module $\Lambda/\Lambda r_i$. But Lemma 3.1 shows that $\Lambda/\Lambda r_i$ is a free $\mathbb{Z}_p$-module of finite rank. Hence, B_i/B_{i+1} is a free $\mathbb{Z}_p$-module of finite rank as well. It follows that the module B_i regarded as $\mathbb{Z}_p$-module decomposes into a direct sum of the module B_{i+1} and a module isomorphic to B_i/B_{i+1}. Therefore, B_i is a free $\mathbb{Z}_p$-module of finite rank.

Denote by C_0 the Σ_i-submodule of B_i generated by B_{i+1}^{i+1} and the elements $x_1, \dots, x_m$, and by C the module $C_0 + B_{i+1}$. It follows from the definition of B_i that $r_i x_j = \bar{\epsilon}_2(e_j) \in B_{i+1}^{i+1}$ $(1 \le j \le m)$ and that all relations between the generators follow from these equalities.

Lemma 4.4. *The module B_i regarded as a Σ_i-module decomposes into a direct sum of the module C and a free Σ_i-module Y. Besides, $r_i C_0 \subseteq B_{i+1}^{i+1}$, $C^i = C_0 + \mu_i B_{i+1}$, $C^j = B_{i+1}^j$ for $j > i$.*

Proof. The factor module $B_i/C = (B_i/B_{i+1})/(C/B_{i+1})$ is isomorphic to the direct sum of m copies of the Σ_i-module $(\Lambda/\Lambda r_i)/\bar{X}_1$ where $\bar{X}_1$ is the Σ_i-submodule of $(\Lambda/\Lambda r_i)$ generated by the element $1 \pmod{\Lambda r_i}$. According to Lemma 3.1 the module $(\Lambda/\Lambda r_i)/\bar{X}_1$ is a free Σ_i-module; therefore, B_i/C is a free Σ_i-module, and B_i decomposes into a direct sum $B_i = C \oplus Y$, where Y is a free Σ_i-module. The other statements of Lemma 4.4 are trivial.

Lemma 4.5. *For all indices j, $0 \le j \le n$, the group Y^j coincides with $\mu_j Y$. If $j \le i$ then $C^j = C_0 + \mu_j B_{i+1}$ and $B_i^j = B_i^i + \mu_j B_i$. If $j > i$ then $C^j = B_{i+1}^j$, $B_i^j = B_{i+1}^j + \mu_j Y$.*

Proof. The first statement of Lemma 4.5 is trivial because Y is a free Σ_i-module and consequently a free $\mathbb{Z}_p[F]$-module.

Let g be an element of the group N_i; using Lemma 1.1 we can find an element $q \in \Sigma_i$ such that $r_i g = (g + q\mu_i) r_i$. Therefore,

$$r_i g x_s = (g + q\mu_i) r_i x_s = (g + q\mu_i)\bar{\epsilon}_2(e_s) \in B_{i+1}^{i+1}$$

$(g + q\mu_i \in \Sigma_i \subseteq \Sigma_{i+1}$, and B_{i+1}^{i+1} is a Σ_{i+1}-module). Thus, we have proved that $r_i C_0 \subseteq B_{i+1}^{i+1}$.

Let $j \leq i$; since $\nu_j C_0 = r_j \dots r_{i-1}\nu_{i+1} r_i C_0 \subseteq r_j \dots r_{i-1}\nu_{i+1} B_{i+1}^{i+1} = 0$ and $\nu_j \mu_j B_{i+1} = 0$, the module $C_0 + \mu_j B_{i+1}$ is contained in C^j. Conversely, let $c \in C$ be an element of C^j. The module C is the sum of its submodules C_0 and B_{i+1}; therefore, the element c can be represented in the form $c = c_0 + b$, where $c_0 \in C_0$, $b \in B_{i+1}$. But we have seen that all elements of C_0 belong to C^j; hence $b = c - c_0 \in C^j$. It follows that $b \in C^j \cap B_{i+1} \subseteq B_{i+1}^j$; according to Lemma 4.2, $B_{i+1}^j = B_{i+1}^{i+1} + \mu_j B_{i+1}$, and the element b decomposes into the sum $b_0 + \mu_j b_1$ with $b_0 \in B_{i+1}^{i+1}$, $b_1 \in B_{i+1}$. Thus, $c = c_0 + b_0 + \mu_j b_1 \in C_0 + B_{i+1}^{i+1} + \mu_j B_{i+1} = C_0 + \mu_j B_{i+1}$.

Since $B_i = C \oplus Y$, we have:

$$B_i^j = C^j \oplus Y^j = (C_0 + \mu_j B_{i+1}) \oplus \mu_j Y =$$
$$((C_0 + \mu_i B_{i+1}) \oplus \mu_i Y) + \mu_j (B_{i+1} \oplus Y) \subseteq B_i^i + \mu_j B_i .$$

The converse inclusion is trivial.

If $c \in C^{i+1}$ then the class $c \pmod{B_{i+1}} \in C/B_{i+1}$ belongs to the group $(C/B_{i+1})^{i+1}$; but C/B_{i+1} is isomorphic to the direct sum of m copies of the Σ_i-module $\bar{X}_1$, and Lemma 3.1 shows that $\bar{X}_1^{i+1} = 0$. Therefore, $(C/B_{i+1})^{i+1} = 0$ and $c \in B_{i+1} \cap C^{i+1} = B_{i+1}^{i+1}$. Thus, $C^{i+1} = B_{i+1}^{i+1}$ and it follows immediately that $C^j = B_{i+1}^j$ for any $j > i+1$. Since $B_i = C \oplus Y$, we obtain for $j \geq i+1$ that $B_i^j = C^j \oplus Y^j = B_{i+1}^j \oplus \mu_j Y$.

4.5. **The homomorphisms ξ and π_i^i.** Define a Σ_i-homomorphism $\xi : C_0 \to W$ by setting $\xi(x_s) = e_s$, $\xi(b) = \beta_i \pi_{i+1}^{i+1}(b) \oplus r_i b$ for $1 \leq s \leq m$, $b \in B_{i+1}^{i+1}$. This definition is not contradictory. To show this we have to verify that it preserves the relations $r_i x_s = \bar{\epsilon}_2(e_s)$. There are two ways to calculate the action of ξ on the element $r_i x_s = \bar{\epsilon}_2(e_s) \in B_{i+1}^{i+1}$. We have $\xi(r_i x_s) = r_i \xi(x_s) = r_i e_s$. On the other hand, $\pi_{i+1}^{i+1} \bar{\epsilon}_2(e_s) = \alpha_i \bar{\epsilon}_1(e_s)$ by the definition of W; therefore,

$$\xi(\bar{\epsilon}_2(e_s)) = \beta_i \pi_{i+1}^{i+1} \bar{\epsilon}_2(e_s) \oplus r_i \bar{\epsilon}_2(e_s) = \beta_i \alpha_i \bar{\epsilon}_1(e_s) \oplus r_i \bar{\epsilon}_2(e_s) =$$
$$r_i \bar{\epsilon}_1(e_s) \oplus r_i \bar{\epsilon}_2(e_s) = r_i e_s ,$$

and both ways give the same result.

The image of ξ contains the elements $e_1, \dots, e_m$ and the group $V = \xi(B_{i+1})$; therefore, this image coincides with W, i.e., ξ is an epimorphism of C_0 onto W.

Define a Σ_i-homomorphism $\pi_i^i : B_i^i = C_0 + \mu_i B_i \to T_i$ by setting $\pi_i^i(c) = \bar{\epsilon}_1 \xi(c)$ for $c \in C_0$, $\pi_i^i(\mu_i b) = 0$ for $b \in B_i$. This definition is consistent as well. Indeed, let $c \in C_0 \cap \mu_i B_i$. But $\mu_i B_i = \mu_i C \oplus \mu_i Y$; therefore, $c \in C_0 \cap \mu_i C = C_0 \cap (\mu_i C_0 + \mu_i B_{i+1})$, and $c = \mu_i c_0 + \mu_i b$ fore some elements $c_0 \in C_0$, $b \in B_{i+1}$. The element $\mu_i b = c - \mu_i c_0$ belongs to the group $C_0 \cap B_{i+1} \subseteq B_{i+1}^{i+1}$. According to the requirement f_{i+1}) of the inductive hypothesis the group B_{i+1}/B_{i+1}^{i+1} regarded as $\mathbb{Z}_p[F]$-module is isomorphic to the direct sum of several copies of the module

$\mathbb{Z}_p[F/F_{i+1}]$. Therefore, for any element $\bar{b} \in B_{i+1}/B_{i+1}^{i+1}$ such that $\mu_i \bar{b} = 0$ there is an element $\bar{d} \in B_{i+1}/B_{i+1}^{i+1}$ such that $\bar{b} = r_i \bar{d}$. In particular, this is true for the class $b \pmod{B_{i+1}^{i+1}}$; hence, there exists an element $b_1 \in B_{i+1}$, such that the element $b_2 = b - r_i b_1$ belongs to $B_{i+1}^{i+1} \subseteq C_0$, and we have

$$\pi_i^i(c) = \bar{\epsilon}_1 \xi(c) = \bar{\epsilon}_1 \xi(\mu_i(c_0 + b_2) + \mu_i r_i b_1) = \bar{\epsilon}_1 \xi(\mu_i(c_0 + b_2) + \mu_{i+1} b_1) =$$
$$\beta_i \pi_{i+1}^{i+1}(\mu_i(c_0 + b_2)) + \beta_i \pi_{i+1}^{i+1}(\mu_{i+1} b_1) = \mu_i \beta_i \pi_{i+1}^{i+1}(c_0 + b_2) \in \mu_i T_i = 0\,,$$

since the homomorphisms β_i, π_{i+1}^{i+1} are Σ_i-homomorphisms, the element μ_i belongs to Σ_i and $\mu_{i+1} B_{i+1} \subseteq \operatorname{Ker} \pi_{i+1}^{i+1}$ (the last inclusion is the statement g_{i+1}) of the inductive hypothesis).

Thus, we have proved that the homomorphism π_i^i was defined correctly. Its image contains the group $\pi_i^i(C_0) = \bar{\epsilon}_1 \xi(C_0) = \bar{\epsilon}_1(W) = T_i$; hence, π_i^i is an epimorphism onto T_i.

4.6. The homomorphisms π_i^j, $j > i$.

Lemma 4.6. *If $j > i$ then $\pi_{i+1}^j \mu_j C = 0$.*

Proof. The Σ_i-module C is generated by B_{i+1} and the elements x_s, $1 \le s \le m$. By Lemma 1.1, $\mu_j g x_s \in \Sigma_i \mu_j x_s$; hence, the group $\mu_j C$ is generated as a Σ_i-module by $\mu_j B_{i+1}$ and the elements $\mu_j x_s$. According to the statement g_{i+1}) of the inductive hypothesis, $\mu_j B_{i+1} \subseteq \operatorname{Ker} \pi_{i+1}^j$; so it is sufficient to prove that $\pi_{i+1}^j \mu_j x_s = 0$. But the last statement is true, since

$$\pi_{i+1}^j \mu_j x_s = \pi_{i+1}^j \mu_i r_{i+1} \dots r_j r_i x_s = \mu_i \pi_{i+1}^j r_{i+1} \dots r_j \bar{\epsilon}_2(e_s) =$$
$$\mu_i \alpha_j \dots \alpha_{i+1} \pi_{i+1}^{i+1} \bar{\epsilon}_2(e_s) = \mu_i \alpha_j \dots \alpha_{i+1} \alpha_i \bar{\epsilon}_1(e_s) \in \mu_i \alpha_j \dots \alpha_{i+1} \alpha_i T_i =$$
$$\alpha_j \dots \alpha_{i+1} \alpha_i \mu_i T_i = 0\,.$$

By Lemma 4.5 $B_i^j = C^j \oplus \mu_j Y = B_{i+1}^j \oplus \mu_j Y$. We take for $\pi_i^j : B_i^j \to T_j$ the homomorphism, the restriction of which on B_{i+1}^j coincides with π_{i+1}^j and the restriction on $\mu_j B_i = \mu_j C \oplus \mu_j Y$ is the zero homomorphism. Lemma 4.6 shows that this definition is consistent: if $b \in B_{i+1}^j \cap \mu_j B_i = \mu_j C$, then both ways to calculate $\pi_i^j(b)$ give the same element 0. Both groups B_{i+1}^j and $\mu_j B_i$ are Σ_j-modules, and the restrictions of π_i^j to these groups are Σ_j - homomorphisms; therefore, π_i^j is a Σ_j - homomorphism as well. The homomorphism π_i^j is an epimorphism because even its restriction on B_{i+1}^j is an epimorphism onto T_j.

4.7. End of the proof of Theorem 4.1. To complete the proof of Theorem 4.1 we have to verify that the Σ_i-module B_i and the epimorphisms π_i^j satisfy the requirements a_i) – g_i) of this theorem.

g$_i$). Recall that the module B_i is the sum of three submodules C_0, B_{i+1}, Y; therefore $\mu_i B_i = \mu_i C_0 + \mu_i B_{i+1} + \mu_i Y$. Let $c \in C_0$; since $C_0 \subseteq B_i^i$, the element $\pi_i^i(c)$ make sense and $\pi_i^i(\mu_i c) = \mu_i \pi_i^i(c) \subseteq \mu_i T_i = 0$. Therefore, the group $\mu_i C_0$ is contained in the kernel of π_i^i. By the definition of π_i^i the groups $\mu_i B_{i+1}$, $\mu_i Y$ are contained in the kernel of π_i^i as well and $\mu_i B_i \subseteq \operatorname{Ker} \pi_i^i$.

a$_i$). Let $1 \leq s \leq m$, $b \in B_{i+1}^{i+1}$, $b_1 \in B_i$. It follows from the definitions and the property g$_i$), which has been already proved that

$$\alpha_i \pi_i^i(x_s) = \alpha_i \bar{\epsilon}_1(e_s) = \pi_{i+1}^{i+1} \bar{\epsilon}_2(e_s) = \pi_{i+1}^{i+1}(r_i x_s),$$
$$\alpha_i \pi_i^i(b) = \alpha_i \beta_i \pi_{i+1}^{i+1}(b) = r_i \pi_{i+1}^{i+1}(b) = \pi_{i+1}^{i+1}(r_i b),$$
$$\alpha_i \pi_i^i(\mu_i b_1) = 0 = \pi_{i+1}^{i+1}(\mu_{i+1} b_1) = \pi_{i+1}^{i+1}(r_i \mu_i b_1).$$

The group B_i^i is generated as Σ_i-module by the elements x_s and the groups B_{i+1}^{i+1}, $\mu_i B_i$, and the homomorphisms $\alpha_i \pi_i^i$, π_{i+1}^{i+1} are Σ_i-homomorphisms; therefore, $\alpha_i \pi_i^i$ and $\pi_{i+1}^{i+1} r_i$ coincide on B_i^i.

Assume now that $i < j < n$; by Lemma 4.5 any element of the group B_i^j can be represented in the form $b + \mu_j b_1$ with $b \in B_{i+1}^j$, $b_1 \in B_i$. We must prove that $\alpha_j \pi_i^j(b + \mu_j b_1) = \pi_{i+1}^{j+1}(r_j(b + \mu_j b_1))$. Since $\pi_i^j \mu_j B_i = 0$ and $\pi_{i+1}^{j+1}(r_j \mu_j B_i) = \pi_{i+1}^{j+1} \mu_{j+1} B_i = 0$, we need only to prove that $\alpha_j \pi_i^j(b) = \alpha_j \pi_{i+1}^j(b) = \pi_{i+1}^{j+1}(r_j b)$; but this is exactly the statement a$_{i+1}$) of the inductive hypothesis.

b$_i$). Any element of the group B_i^{j+1} can be represented in the form $b + \mu_{j+1} b_1$ with $b \in B_{i+1}^{j+1}$, $b_1 \in B_i$. We obtain, using b$_{i+1}$) for $j > i$ and the definition of π_i^i for $j = i$:

$$\pi_i^j(b + \mu_{j+1} b_1) = \pi_i^j(b) + \pi_i^j(\mu_{j+1} b_1) = \pi_{i+1}^j(b) = \beta_j \pi_{i+1}^{j+1}(b) = \beta_j \pi_i^{j+1}(b) = \beta_j \pi_i^{j+1}(b + \mu_{j+1} b_1) \quad (j > i),$$
$$\pi_i^i(b + \mu_{i+1} b_1) = \pi_i^i(b) + \pi_i^i(\mu_{i+1} b_1) = \beta_i \pi_{i+1}^{i+1}(b) = \beta_i \pi_i^{i+1}(b) = \beta_i \pi_i^{i+1}(b + \mu_{i+1} b_1).$$

c$_i$). It is clear that $r_i \operatorname{Ker} \pi_i^j \subseteq \operatorname{Ker} \pi_i^{j+1}$. Let b be an element of B_i^{j+1} such that $\pi_i^{j+1}(b) = 0$. Since $B_i^{j+1} = B_{i+1}^{j+1} + \mu_{j+1} B_i$, the element b can be represented in the form $b = b_1 + \mu_{j+1} b_2$, where $b_1 \in B_{i+1}^{j+1}$, $b_2 \in B_i$. But $\pi_i^{j+1}(\mu_{j+1} b_2) = 0$ by the definition of π_i^{j+1}; therefore, $\pi_{i+1}^{j+1}(b_1) = \pi_i^{j+1}(b_1) = 0$. Let us show that there exists an element $a \in \operatorname{Ker} \pi_i^j$ such that $r_i a = b_1$. For $j > i$ this follows from the statement c$_{i+1}$) of the inductive hypothesis; the case $j = i$ is a little more complicated. The element $0 \oplus b_1 \in T_i \oplus B_{i+1}^{i+1}$ belongs to the module W since $\alpha_i(0) = \pi_{i+1}^{i+1}(b_1) = 0$. Since the homomorphism $\xi : C_0 \to W$ is an epimorphism, there exists an element $a \in C_0$ such that $\xi(a) = 0 \oplus b_1$. According to our definitions, $\pi_i^i(a) = \bar{\epsilon}_1 \xi(a) = 0$ and $r_i a = \bar{\epsilon}_2 \xi(a) = b_1$. We have now in both cases: $b = r_i(a + \mu_j b_2)$ and $a + \mu_j b_2 \in \operatorname{Ker} \pi_i^j$. Hence $\operatorname{Ker} \pi_i^{j+1} \subseteq r_i \operatorname{Ker} \pi_i^j$.

d$_i$). The statement d$_i$) was proved earlier (Lemma 4.3).

e$_i$). The Λ-module B_i is generated by the elements $x_1, \dots, x_m$ and the module B_{i+1}; by the inductive hypothesis f$_{i+1}$) the Λ-module B_{i+1} is generated by B_{i+1}^{i+1}. Since the elements $x_1, \dots, x_m$ and the group B_{i+1}^{i+1} are contained in $C_0 \subseteq B_i^i$, the module B_i is generated by B_i^i.

f$_i$). The module B_i decomposes into a direct sum $C \oplus Y = (C_0 + B_{i+1}) \oplus Y$ and $B_i^i = (C_0 + \mu_i B_{i+1}) \oplus \mu_i Y$; therefore, $B_i/B_i^i = (C_0 + B_{i+1})/(C_0 + \mu_i B_{i+1}) \oplus Y/\mu_i Y$. But Y is a free Σ_i-module and consequently a free $\mathbb{Z}_p[F]$-module; hence, $Y/\mu_i Y$ is a direct sum of several copies of the module $\mathbb{Z}_p[F]/\mu_i\mathbb{Z}_p[F] = \mathbb{Z}_p[F/F_i]$. Since $B_{i+1} \cap C_0 = B_{i+1}^{i+1}$, the module $D = (C_0 + B_{i+1})/(C_0 + \mu_i B_{i+1})$ is isomorphic to the module $B_{i+1}/(B_{i+1}^{i+1} + \mu_i B_{i+1}) = (B_{i+1}/B_{i+1}^{i+1})/\mu_i(B_{i+1}/B_{i+1}^{i+1})$. According to the statement f$_{i+1}$) of the inductive hypothesis B_{i+1}/B_{i+1}^{i+1} is a direct sum of several copies of the module $\mathbb{Z}_p[F/F_{i+1}]$; it follows that the module D, regarded as $\mathbb{Z}_p[F]$-module, is isomorphic to a direct sum of several copies of the module $\mathbb{Z}_p[F/F_{i+1}]/\mu_i\mathbb{Z}_p[F/F_{i+1}] = \mathbb{Z}_p[F/F_i]$.

4.8. **The proof of Theorem 2.1. Existence.** Let $A = B_0$ be a Λ-module satisfying the requirements of Theorem 4.1 for $i = 0$. Since $T_0 = 0$, $\operatorname{Ker} \pi_0^0 = B_0$; it follows from the condition c$_0$) that $\operatorname{Ker} \pi_0^i = r_{i-1} \dots r_1 r_0 \operatorname{Ker} \pi_0^0 = \mu_i B_0$. Hence, π_0^i induces an isomorphism $U_i(B_0) = B_0^i/\mu_i B_0 \xrightarrow{\xi_i} T_i$. It is obvious that the collection of isomorphisms $\xi_0, \xi_1, \dots, \xi_n$ is an isomorphism in the category $\mathfrak{M}$.

5. The proof of Theorem 2.1. Uniqueness

Let $\mathfrak{M}'$ and U' be the category and the functor constructed for the group F as the category $\mathfrak{M}$ and the functor U were defined above for the group G.

For the case $G = F$ Theorem 2.1 was proved in [Ya1]. Moreover, the following statement can be easily obtained from its proof.

Let X be an object of the category $\mathfrak{M}'$ and A, B be $\mathbb{Z}_p[F]$-modules such that $U'(A) = U'(B) = X$. Then there exists a $\mathbb{Z}_p[F]$-homomorphism $\xi : A \to B$ such that the homomorphism $U'(A) \to U'(B)$ induced by ξ is the identity isomorphism.

Now let X be an object of the category $\mathfrak{M}$ and A, B be Λ-modules such that $U(A) = U(B) = X$. Since X can be regarded as an object of the category $\mathfrak{M}'$ and A, B can be regarded as $\mathbb{Z}_p[F]$-modules, we can apply the above statement. It follows that there is a $\mathbb{Z}_p[F]$-homomorphism $\xi : A \to B$ inducing the identity isomorphism $X = U'(A) \to U'(B) = X$. For any i the homomorphism $U_i(A) = U_i'(A) \to U_i'(B) = U_i(B)$, induced by ξ, is the identity isomorphism as well, and therefore it is a Σ_i-isomorphism.

Recall that the group $\mathrm{Hom}(A,B)$ can be made into a G-module in a canonical way:

$$(g\eta)(a) = g\eta(g^{-1}a) \quad \text{for any } g \in G\,,\ \eta \in \mathrm{Hom}(A,B)\,,\ a \in A\,.$$

For the $\mathbb{Z}_p[F]$-homomorphism $\xi : A \to B$ define its averaging $\bar{\xi}$ as follows. Let $G = \bigcup_{s=1}^{m} h_s F$ be a decomposition of G into a union of left cosets; then $\bar{\xi} = \sum_{s=1}^{m} h_s\xi$. It is obvious that the homomorphism $\bar{\xi}$ is a Λ-homomorphism and that it does not depend on the choice of representatives h_s in the cosets.

Lemma 5.1. *Denote by ξ^i, $\bar{\xi}^i : H^1(F_i, A) \to H^1(F_i, B)$ the homomorphisms, induced by ξ, $\bar{\xi}$. Then $\bar{\xi}^i = (G : F)\xi^i$ for each i.*

Proof. Let $G = \bigcup_{s=1}^{v} F_i g_s F$ be a decomposition of G into a union of double cosets. For any s the intersection $F_i \cap g_s F g_s^{-1}$ is a subgroup of F_i; therefore this intersection coincides with the group F_{l_s} for an integer l_s, $i \le l_s < n$. Denote by $O(s)$ the index $(F_i : F_{l_s}) = p^{l_s - i}$. The elements σ_i^t $(1 \le t \le O(s))$ are representatives of all left cosets of the subgroup $F \cap g_s F g_s^{-1} = F_{l_s}$ in the group F_i; therefore, $F_i = \bigcup_{t=1}^{O(s)} \sigma_i^t (F \cap g_s F g_s^{-1})$ is a decomposition of F_i into a union of left cosets. Then $G = \bigcup_{s=1}^{v} (\bigcup_{t=1}^{O(s)} \sigma_i^t) g_s F$ is a decomposition of G into a union of left cosets and $\sum_{s=1}^{v} O(s) = (G : F)$ (as in [CE],XII.9).

Let $a \in A^i$; then $\sum_{t=1}^{O(s)} \sigma_i^t a \in A^{l_s}$. The element g_s belongs to the normalizer of the group $F_{l_s} = F_i \cap g_s F g_s^{-1}$ and therefore $g_s^{-1} \sum_{t=1}^{O(s)} \sigma_i^t a \in A^{l_s}$; since ξ^{l_s} is a Σ_{l_s}-homomorphism, $g_s \in \Sigma_{l_s}$ and ξ is a F-homomorphism, there exists an element $b_s \in B$ such that

$$g_s\xi(g_s^{-1} \sum_{t=1}^{O(s)} \sigma_i^t a) = \xi(g_s(g_s^{-1} \sum_{t=1}^{O(s)} \sigma_i^t a)) + \mu_{l_s} b_s =$$

$$\sum_{t=1}^{O(s)} \sigma_i^t \xi(a) + \mu_{l_s} b_s = O(s)\xi(a) + \sum_{t=1}^{O(s)} (\sigma_i^t - 1)\xi(a) + \mu_{l_s} b_s\,.$$

Obviously, all the summands but the first one belong to the group $(\sigma_i - 1)B = \mu_i B$. Therefore there is an element $c_s \in B$ such that

$$g_s\xi(g_s^{-1} \sum_{t=1}^{O(s)} \sigma_i^t a) = O(s)\xi(a) + \mu_i c_s\,.$$

We have now

$$\bar{\xi}(a) = ((\sum_{s=1}^{v}(\sum_{t=1}^{O(s)} \sigma_i^t) g_s)\xi)(a) = \sum_{s=1}^{v}(\sum_{t=1}^{O(s)} \sigma_i^t g_s \xi(g_s^{-1}\sigma_i^{-t} a)) =$$

$$\sum_{s=1}^{v}(\sum_{t=1}^{O(s)} (\sigma_i^t - 1) g_s \xi(g_s^{-1}\sigma_i^{-t} a)) + \sum_{s=1}^{v}(\sum_{t=1}^{O(s)} g_s \xi(g_s^{-1}\sigma_i^{-t} a))\,.$$

Obviously, the first sum belongs to the group $\mu_i B$; denote this sum by $\mu_i b$. Thus,

$$\bar{\xi}(a) = \sum_{s=1}^{v}(\sum_{t=1}^{O(s)} g_s\xi(g_s^{-1}\sigma_i^{-t}a)) + \mu_i b = \sum_{s=1}^{v} O(s)\xi(a) + \sum_{s=1}^{v}\mu_i c_s + \mu_i b =$$
$$(\sum_{s=1}^{v} O(s))\xi(a) + \mu_i(b + \sum_{s=1}^{v} c_s) = (g : F)\xi(a) + \mu_i(b + \sum_{s=1}^{v} c_s);$$

hence the elements $\bar{\xi}(a)$ and $(G : F)\xi(a)$ define the same element of the group $B^i/\mu_i B = U_i(B)$. Lemma 5.1 is proved.

It follows from the previous lemma that the Λ-homomorphism $\zeta = 1/(G : F)\xi : A \to B$ induces the identity isomorphism $X = U(A) \to U(B) = X$ (the division is possible because p does not divide the index $(G : F)$ and, therefore, the index is invertible in the ring of p-adic integers). Similarly there exists a Λ-homomorphism $\theta : A \to B$ which induces the identity isomorphism $X = U(B) \to U(A) = X$. The uniqueness in Theorem 2.1 is an immediate corollary of the following lemma.

Lemma 5.2. *Let A, B be Λ-modules free and finitely generated as $\mathbb{Z}_p$-modules, and let $\zeta : A \to B$, $\theta : B \to A$ be Λ-homomorphisms such that the homomorphisms $U(A) \to U(B)$, $U(B) \to U(A)$, induced by ζ, θ are the identity isomorphisms. Then there exist direct decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ and homomorphisms $\zeta' : A \to B$, $\theta' : B \to A$ such that $A_2 = \operatorname{Ker}\zeta'$, $B_2 = \operatorname{Ker}\theta'$, $U(A_2) = U(B_2) = 0$ and the restrictions of the homomorphisms ζ', θ' on A_1 and B_1 are isomorphisms $A_1 \to B_1$, $B_1 \to A_1$.*

Proof. For any positive integer m denote by $\mathfrak{A}_m$ the set of pairs of homomorphisms $\zeta_m : A/p^m A \to B/p^m B$, $\theta_m : B/p^m B \to A/p^m A$ such that

(1) $\zeta_m\theta_m\zeta_m = \zeta_m$, $\theta_m\zeta_m\theta_m = \theta_m$;
(2) there exist homomorphisms $\zeta'' : A \to B$, $\theta'' : B \to A$ such that the homomorphisms $U(A) \to U(B)$, $U(B) \to U(A)$, induced by ζ'', θ'', are the identity isomorphisms, and homomorphisms $A/p^m A \to B/p^m B$, $B/p^m B \to A/p^m A$, induced by ζ'', θ'', coincide with ζ_m, θ_m.

The set $\mathfrak{A}_m$ is finite; it is not empty. Indeed, the modules $A/p^m A$, $B/p^m B$ are finite; therefore, there are only a finite number of homomorphisms from $A/p^m A$ into $B/p^m B$. Hence, there exists a homomorphism $\zeta_m : A/p^m A \to B/p^m B$ such that the homomorphism $A/p^m A \to B/p^m B$ induced by $\zeta(\theta\zeta)^w$ coincides with ζ_m for infinitely many positive integers w. In particular, there are positive integers $u < v$ such that ζ_m is induced by $\zeta(\theta\zeta)^u$ and by $\zeta(\theta\zeta)^{u+v}$. Take for θ_m the homomorphism $B/p^m B \to A/p^m A$ induced by $\theta(\zeta\theta)^{2v-u-1}$. Easy calculations show that $\zeta_m\theta_m\zeta_m = \zeta_m$, $\theta_m\zeta_m\theta_m = \theta_m$.

If $m_1 \geq m$ there is a canonical mapping $\mathfrak{A}_{m_1} \to \mathfrak{A}_m$. The projective limit of these nonempty finite sets is nonempty. Therefore we can choose the pairs $(\zeta_m, \theta_m) \in \mathfrak{A}_m$, so that the homomorphisms $\zeta_{m_1}, \theta_{m_1}$ induce the homomorphisms ζ_m, θ_m for $m_1 \geq m$. There exist homomorphisms $\zeta' : A \to B$, $\theta' : B \to A$ such that the homomorphisms $A/p^m A \to B/p^m B$, $B/p^m B \to A/p^m A$, induced by ζ', θ', coincide with ζ_m, θ_m for all m. It follows from the construction that $\zeta'\theta'\zeta' = \zeta'$, $\theta'\zeta'\theta' = \theta'$. The homomorphisms ζ', θ' and the modules $A_1 = \operatorname{Im}\theta'$, $A_2 = \operatorname{Ker}\zeta'$, $B_1 = \operatorname{Im}\zeta'$, $B_2 = \operatorname{Ker}\theta'$ satisfy all the requirements of Lemma 5.2.

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