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of GROUPS, ALGEBRAS, and ORDERS

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Proceedings of the Workshop and Meeting on Representation Theory of Groups, Algebras, and Orders

held at the

Ovidius University, Constanţa,

September 25 - October 6, 1995

Edited by Klaus W. Roggenkamp and Mirela Ştefănescu

PREFACE

From September 25 to October 6, 1995 a Workshop and a Meeting on Representation Theory of Groups and Algebras had been held at the Ovidius University in Constanţa, Romania. The WORKSHOP consisted of five series of four lectures each on the topics

1. "Computer algebra and representation theory"
2. "Auslander–Reiten sequences and derived categories"
3. "Structure of blocks with cyclic defect and Green correspondence"
4. "Stable and derived equivalences of blocks"
5. "Clifford theory and the Zassenhaus Conjectures"

and a demonstration of the Computer-Algebra-System MAPLE and the Group-System GAP.

The intention of this workshop was to report on some of the recent developments in representation theory of groups and algebras, and to introduce the audience to the use of computers in group and representation theory.

The emphasis of the INTERNATIONAL MEETING in the second week was on representations of groups and algebras.

We would like to express our gratitude to Jon Carlson, Vlastimil Dlab, Karl Gruenberg, Idun Reiten and Claus Ringel for each giving a series of three lectures during the meeting.

Special thanks go to the University of Constanţa which supplied us with a pleasant environment for the meeting.

The financial support of the Volkswagen–Stiftung has made this meeting possible and we are very grateful. Also the Deutsche Forschungsgemeinschaft has supported many of the participants from Germany, we also thank it.

Viviana Ene, Christina Flaut and Michael Kauer have been the genies behind the scene without whose help and engagement the meeting would not have run as smoothly as it did.

There were many more people who helped us with the organization of the meeting. We do not name all of them, but we thank them.

Maybe it is worthwhile to present some data about the University with name of the Latin poet Ovid. Publius Ovidius Naso has been banished by the Roman emperor Augustus in the first decades of the Christian era (approx. 9 a. d. - 16/17 a. d.) to the ancient Greek (Greece) then Roman colony of Tomis on the coast of the Black Sea (then Pontus Euxinus), where he also died[1]. Constanţa has been built on the ruins of Tomis. It is now the biggest Romanian harbour. The University was founded in 1990, growing out of a Pedagogical Institute and a Technical University. It has now seven faculties, Letters and Theology, History and Law Sciences, Economical Sciences, Natural Sciences, Medical Faculty, Engineering and Mechanics and Mathematics, with altogether more than 5.000 students. A new campus is now in construction and the University is developing each day.

An exposition of the lectures from the workshop will constitute PART I of these proceedings.

The contributions from the talks during the meeting will make up PART II of the proceedings and should give the reader a flavour of the lectures during the meeting.

We highly appreciate, that the participants did send us their contributions to be published in the "ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII OVIDIUS, CONSTANŢA", thus promoting the young University of Constanţa.

We should like to express our gratitude to all the referees for all their help and to the "ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII OVIDIUS, CONSTANŢA" for publishing this volume.

The TEXnical preparation of these volumes was done with great care and many time consuming trial and error checks by Michael Kauer and Martin Wursthorn. Our special thanks go to them for their courageous engagement in preparing these notes.

Constanţa, March 1996

Mirela Ştefănescu
Klaus Roggenkamp

[1] Contrary to the participants of this meeting, Ovid did not at all enjoy his stay in Tomis. We have included for leisurely reading some of Ovid's 'lamento' about this place from his 'Tristia', p. ix.

CONTENTS

Ovid's description of his life, being banished to Tomis – nowadays called Constanţa

Ovid, 'Tristia'[2]

IX. The Origin of Tomis. Here too then there are Greek cities (who would believe it?) among the names of the wild barbarian world. But the ancient name, more ancient than the founding of the city, was given to this place 'tis certain, from the murder of Absyrtus. For in the ship[3] wicked Medea fleeing her forsaken sire brought to a haven her oars, they say, in these waters. Him in the distance the lookout espied and said, "A stranger approaches from Colchis; I recognize the sails!"

The Colchian maid conscious of her guilt smote her breast with a hand that had dared and was to dare many things unspeakable, and thought her heart still retained its great boldness, there was a pallor of dismay upon the girl's face.

And so at the sight of the approaching sails, she said, "I am caught!" and "I must delay my father by some trick!" As she was seeking what to do, turning her countenance on all things she chanced to bend her gaze upon her brother. When aware of his presence she exclaimed, "The victory is mine! His death shall save me!" Forthwith while he in his ignorance feard no such attack she pierced his innocent side with the hard sword. Then she tore him limb from limb, scattering the fragments of his body throughout the fields so that they must be sought in many places. And to apprise her father she placed upon a lofty rock the pale hands and gory head.

Thus was the sire delayed by his fresh grief, lingering while he gathered those lifeless limbs, on a journey of sorrow.

So was this place called Tomis because here, they say, the sister cut to pieces her brother's body.[4].

X. The Rigours of Tomis. If there be still any there who remembers banished Naso, if my name without me still survives in the city, let him know that beneath the stars which never touch the sea I am living in the midst of the barbarian world. When grim winter has thrust forth his squalid face, and the earth is marblewhite with frost, and Boreas and the snow prohibit dwelling beneath the Bear, then 'tis clear that these tribes are hard pressed by the shivering pole. The snow lies continuously, and once fallen, neither sun nor rains may melt it, for Boreas hardens and renders it eternal. So when an earlier fall is not yet melted another has come, and in many places 'tis wont to remain for two years.

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[2] We would like to thank Karl Gruenberg to supply us with the english translation of this part of the 'Tristia', quoted from an english translation by A. L. Wheeler, second edition by G. P. Goold

[3] The Argonauts

[4] Ovid derives Tomis (Tomi) from $\tau\varepsilon\mu\nu\omega$, "to cut"

With skins and stitched breeches they keep out the evils of the cold; of the whole body only the face is exposed. Often their hair tinkles with hanging ice and their beards glisten white with the mantle of frost. Exposed wine stands upright, retaining the shapes of the jar, and they drink, not draughts of wine, but fragments.

Why tell of brooks frozen fast with the cold and how brittle water is dug out of the pool? The very Hister freezes as the winds stiffen his dark flood, and winds its way into the sea with covered waters. Where ships had gone before now men go on foot and the waters congealed with cold, feel the hoof-beat of the horse. I may scarce hope for credence, but since there is no reward for a falsehood, the witness ought to be believed.

I have seen the vast sea stiff with ice, a slippery shell holding the water motionless. And seeing is not enough; I have trodden the frozen sea, and the surface lay beneath an unwetted foot. At such times the curving dolphins cannot launch themselves into the air; if they try, stern winter checks them; and though Boreas may roar and toss his wings, there will be no wave on the beleaguered flood. Shut in by the cold the ships will stand fast in the marble surface nor will any oar be able to cleave the stiffened waters. I have seen fish clinging fast bound in the ice, yet some even then still lived.

So, when the Hister has been levelled by the freezing Aquilo the barbarian enemy with his swift horses rides to the attack — an enemy strong in steeds and in far flying arrows — and lays waste far and wide the neighbouring soil. Some flee, and with non to protect their lands their unguarded resources are plundered, the small resources of the country, flocks and creaking carts all the wealth the poor peasant has. Some are driven, with arms bound behind them, into captivity, gazing back in vain upon their farms and their homes; some fall in agony pierced with barbed shafts, for there is a stain of poison upon the winged steel. What they cannot carry or lead away they destroy, and the hostile flame burns the innocent hovels. Even when peace prevails, there is timorous dread of war, nor does any man furrow and soil with down-pressed share. A foe this region either sees or fears when it does not see; idle lies and soil abandoned in stark neglect. Not here the sweet grapes lying hidden in the leafy shade nor the frothing must brimming the deep vats! Fruits are denied in this region. This then, though the great world is so broad, is the land discovered for my punishment!

XI. To an Enemy. A barbarous land, the unfriendly shores of Pontus, and the Maenalian bear with her companion Boreas behold me. No interchange of speech have I with the wild people; all places are charged with anxiety and fear. As a timid stag caught by ravenous bears or a lamb surrounded by the mountain wolves is stricken with terror, so am I in dread, hedged about on all sides by warlike tribes, the enemy almost pressing against my side. Were it a slight punishment that I am deprived of my dear wife, my native land, and my loved ones; were I supporting no ills but the naked wrath of Caesar, is the naked wrath of Caesar too small an ill?

XII. Springtime in Tomis. The cold is now weakening beneath the zephyrs and at the year's end a winter more endless than those of old curbs its rigour. Now merry boys and girls are plucking the violets that spring up unsown in the fields, the meadows are abloom with many-coloured flowers, the chatty birds from unschooled throats utter a song of spring, and the swallow, to put off the name of evil mother[5], builds beneath the rafters the tiny house that cradles her young. The grain that lay in hiding beneath the furrows sends forth from the unfrozen soil its tender tips. Wherever grows the wine, the bud is just pushing from the shoot and wherever grows the tree, the branches are just budding. In yonder land there is now rest, and the noisy wars of the wordy forum are giving place to festivals one after another; now there is exercise on horse, now there is play with light arms, with the ball or the swift circling hoop; now the young men, reeking of slippery oil, are bathing wearied limbs in Virgin water[6]. The stage is full of life.

But mine is to feel the snow melted by the spring sun and water which is not dug all hard from the pool. The sea, too, is no longer solid with ice, nor as before does the Sauromatian herdsman drive his creaking wagon across the Ister. Some ships will begin to voyage even as far as here, and soon there will be a friendly bark on Pontus' shore. Eagerly I shall run to meet the mariner and when I've greeted him, shall ask why he comes, who and from what place he is. Rarely does a sailor cross the wide seas from Italy, rarely visit this harbourless shore. Yet if he knows how to speak with the voice of Greek or Roman (this last will surely be the sweeter), whoever he is, he may be one to tell faithfully some rumour, one to share and pass on some report. May he, I pray, have power to tell of Caesar's triumphs and vows paid to Jupiter of the Latins; that thou, rebellious Germany, at length hast lowered thy sorrowing head beneath the foot of our leader[7]. He who tells me such things as these — Things it will grieve me that I have not seen — shall be forthwith a guest within my home. Ah me! is Naso's (Ovid's) home now in the Scythian world, and dost thou, Pontus, assign me thy soil as an abode? Ye gods, give Caesar the will that not here may be my hearth and home but only the hostelry of my punishment!

[5] Procne

[6] From the aqueduct called Virgo

[7] Tiberius, who took the field against the Germans after the defeat of Varus, A. D. 9

XIII. A Birthday at Tomis. Lo! to no purpose — for what profit was there in my birth? my birthday god[8] attends his anniversary. Cruel one, why hast thou come to increase the wretched years of an exile? To them thou shouldst have put an end. Hadst thou any love for me or any sense of shame, thou wouldst not be following me beyond my native land, and where first I was known by thee as an ill-starred child, there shouldst thou have tried to be my last, and when I was forced to leave Rome, thou too, like my friends shouldst have said in sorrow "Farewell".

What hast thou to do with Pontus? Is it that Caesar's wrath sent thee too to the remotest land of the world of cold? Thou awaitest, I suppose, thine honour in its wonted guise: a white robe hanging from my shoulders, a smoking altar garlanded with chaplets, the grains of incense snapping in the holy fire, and myself offering the cakes that mark my birthday and framing kindly petitions with pious lips. Not such is my condition nor such my hours, that I can rejoice at thy coming. An altar of death girdled with funereal cypress is suited to me and a flame made ready for the up-reared pyre. Nor is it a pleasure to offer incense that wins nothing from gods, nor in such misfortunes do words of good omen come to my lips. Yet if I must ask thee something on this day, return thou no more to such a land, I pray, so long as all but the remotest part of the world, the Pontus, falsely called Euxine[9] possesses me.

XIV. Epilogue to an Unnamed Friend. Cherisher and revered protector of learned men what doest thou - thou that hast ever befriended my genius? As thou once wert wont to extol me when I was in safety, now too dost thou take heed that I seem not wholly absent? Dost thou assemble my verse except only that "Art" which ruined its artificer? Do so, I pray, thou patron of new bards; so far as may be, keep my body[10] in the city. Exile was decreed to me, exile was not decreed to my books; they did not deserve their master's punishment. Oft is a father exiled on a foreign shore, yet may the exile's childpen live in the city. Pallasfashion[11] were my verses born from me without a mother; these are my offspring, my family. These I commend to thee; the more bereft they are, the greater burden will they be to thee their guardian.

[8] The genius natalis to whom the Roman offered sacrifice on his birthday. The genius was believed to be a spiritual counterpart of the individual.

[9] Euxine means "hospitable".

[10] i. e. my poems

[11] Pallas was born from the head of Zeus

List of lecturers

Workshop.

1. W. Kimmerle: "Computer algebra and representation theory"
2. S. König: "Auslander–Reiten sequences and derived categories",
3. M. Linckelmann: "Stable and derived equivalences of blocks"
4. K. W. Roggenkamp: "Elementary properties of group rings and orders"
5. K. W. Roggenkamp: "Clifford theory and a counterexample to the isomorphism problem of infinite groups"
6. K. W. Roggenkamp: "Zassenhaus Conjecture and Čech cohomology"
7. M. Wursthorn: "Representation theory with MAPLE: an example"
8. A. Zimmermann: "Structure of blocks with cyclic defect and Green correspondence"

Meeting.

1. I. Armeanu: Ambivalent groups
2. V. Bavula: Filter dimension; an analogon to Bernstein's inequality for simple affine algebras
3. A. Bovdi: The group of units in a modular group ring
4. J. Carlson: Infinitely generated modules over group rings
5. V. Dlab: Quasi–hereditary algebras
6. K. W. Gruenberg: Some applications of integral representations of finite groups
7. D. Happel: Perpendicular categories of exceptional modules
8. L. Hille: Moduli spaces of quiver representations
9. A. Jones: Some integral representations
10. P. Koshlukov: Speciality of Jordan pairs
11. A. Marcus: A Clifford theory of tilting complexes
12. G. Militaru: Quantum Yetter–Drinfel'd modules and Doi–Hopf modules
13. S. Raianu: Crossed coproducts
14. I. Reiten: Quasi–tilted algebras
15. C. M. Ringel: Recent advances on representation of hereditary algebras
16. M. Roczen: Simple and simple elliptic singularities over a field of positive characteristic
17. R. Rouquier: Some examples of Rickard complexes
18. W. Rump: A deformation of the preprojective algebra
19. V. Sergeichuk: Multiplicative basis of a finitely presented vectroid
20. D. Stefan: The set of types of semisimple and cosemisimple Hopf–algebras is finite
21. M. Stefanescu: Cohomology and nearrings
22. K. Waki: About the decomposition matrix of $Sp(4,q)$
23. A. Yakovlev: Homological description of modules over group rings

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ELEMENTARY PROPERTIES OF GROUP-RINGS AND ORDERS

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ABSTRACT. Here we recall some general results from the theory of orders and representation theory of groups, as separable, maximal, hereditary and Green orders, vertices, sources and blocks, and we introduce some basic facts about group cohomology, which may be of use for the understanding of the following contributions, and in particular those in Volume II. No proofs are given.

1. DEFINITIONS AND NOTATIONS

We assume that the reader is familiar with BASIC ALGEBRA and ALGEBRAIC NUMBER THEORY. We also freely use constructions from HOMOLOGICAL ALGEBRA. Moreover, we also assume the basic facts from ordinary representation theory and character theory.

We shall first FIX THE NOTATION, which is used throughout these notes: For a commutative ring R we denote by $max(R)$ the set of maximal ideals of R; this set becomes a topological space under the ZARISKI TOPOLOGY.

If S is a ring and M, N are left S-modules, we shall often use the NOTATION

$$Hom_S(M,N) = (M,N)_S\,.$$

For the arithmetic we shall denote by ζ_n a primitive n-th root of unity over $\mathbb{Q}$, and for groups we shall denote by C_n the cyclic group with n elements.

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As general references for orders and their representations we list Curtis-Reiner [C-R1; 82], [C-R2; 87], Reiner [Re; 75], Gruenberg [Gr; 70], Roggenkamp-Huber-Dyson [RHD; 70 1], Roggenkamp [Ro; 70 2], [Ro; 92dmv], [Ro; 94], Reiner-Roggenkamp [ReRo; 79].

Definition 1.1. *A* DEDEKIND DOMAIN *is an integral domain, such that every finitely generated torsion free module is projective. It has a field of fractions K. By A we denote a finite dimensional separable K-algebra*[1]
*An R-*ORDER Λ IN A *is an R-algebra $\Lambda \subset A$ such that*

1. *Λ is finitely generated as R-module and*
2. $K\Lambda = A$ [2].

A (LEFT) Λ-LATTICE *is Λ-module, which is finitely generated and torsion free over R. We denote the category of left Λ-lattices and (homo-)morphisms by* ${}_{\Lambda}\mathcal{M}^0$.

Note 1.2. 1. For R one should think of as the ring of algebraic integers in an algebraic number field or of the polynomial ring in one variable over a field.
2. Since an R-order Λ is finitely generated over R, every element $\lambda \in \Lambda$ is INTEGRAL OVER R. But contrary to the commutative situations, the set of all elements in A, which are integral over R does not form a ring. The matrices
$$a := \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \text{ and } b := \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}$$
are surely integral over $\mathbb{Z}$, but $(a+b)^2$ is not.
3. A typical example of an R-order is the GROUP-RING RG over R of a finite group G, i. e. it is R-free with basis $\{x_g\}_{g\in G}$, and the multiplication is induced from that in G [3]. In order that the underlying algebra $A := KG$ is separable, we must however ASSUME that $(char(K), |G|) = 1$ [4].
4. Since R is a Dedekind domain, every Λ-lattice is projective as R-module.
5. In case R is a principal ideal domain and $\Lambda = RG$, the Λ-lattices correspond categorically to the REPRESENTATIONS OF G OVER R i. e. homomorphisms $G \longrightarrow Gl(n, R)$ [5]. For an arbitrary Dedekind domain, every R-representation of G gives rise to an RG-lattice, but not conversely, since

[1] This means that A is semi-simple and the center of A is a finite product of separable field extensions of K i. e. $A = \prod_{i=1}^{n}(D_i)_{n_i}$, where D_i are skew-fields, whose centers are separable over K.
[2] This means that Λ contains a K-basis of A.
[3] We will in general identify x_g with g.
[4] This means that the group order and the characteristic of K are relatively prime.
[5] This is the group of invertible $n \times n$-matrices over R.

not every RG-lattice comes from a representation – note: not every R-lattice is R-free.

6. One word of caution is advisable: In ${}_\Lambda\mathcal{M}^0$ an epimorphism $M \longrightarrow N$ is not necessarily surjective, but every surjective homomorphism is an epimorphism[6].
7. Thus, if we use the notion of projective with respect to epimorphisms, free $\mathbb{Z}$-modules would not be projective. We have to use the notion of SURJECTIVE.
8. The same remark applies for a short exact sequence of Λ-lattices:

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0 .$$

   Here we require α to be monic[7], and β must be surjective.
   PROJECTIVE Λ-LATTICES are then those lattices M'', where every sequence as above is split. A Λ-lattice is projective if and only if it is a direct summand of a finitely generated free left Λ-module.
9. On ${}_\Lambda\mathcal{M}^0$ we have a duality from left Λ-lattices to right Λ-lattices $\mathcal{M}^0_\Lambda$:

$$* : {}_\Lambda\mathcal{M}^0 \longrightarrow \mathcal{M}^0_\Lambda : M \longrightarrow Hom_R(M, R) := M^* .$$

   The reason is that Λ-lattices are projective as R-modules, and so the dual $Hom_R(-, R)$ is an exact functor.
10. An INJECTIVE (with respect to short exact sequences of lattices) LEFT Λ-LATTICE is of the form P^* for a projective right Λ-lattice P. It has with respect to lattices the splitting property of short exact sequences: i. e. if a left Λ-lattice I is an injective Λ-lattice, then $Ext^1_\Lambda(M, I) = 0$ for every $M \in {}_\Lambda\mathcal{M}^0$. These injective Λ-lattices are definitely not injective Λ-modules.
11. Λ is said to be a GORENSTEIN ORDER, provided, each projective left Λ-lattice is also an injective Λ-lattice. The notion of Gorenstein is symmetric with respect to left and right lattices. Note that a Gorenstein order can never have finite global dimension[8].
12. Since Λ-lattices are torsion-free over R, we have for the global dimensions

$$gl.dim(\Lambda) = gl.dim.({}_\Lambda\mathcal{M}^0) + 1 .$$

GROUP-RINGS ARE SPECIAL types of orders, since they have additional structures, which we shall discuss briefly:

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[6] The map $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ is an epimorphism, but not surjective.

[7] This is the same as injective.

[8] A ring $\mathfrak{A}$ is said to have global dimension bounded by n, provided every finitely generated left $\mathfrak{A}$-module has homological dimension bounded by n i. e. a projective resolution of length less or equal to n.

Note 1.3. 1. For a group-ring RG we have another duality: With G we can associate the opposite group G^{op} [9] Thus each left RG-module is in a natural way a right RG^{op}-module. More precisely, we have the ANTIPODAL MAP $g \to g^{-1}$. Using this one can change from right to left modules and vice versa.

2. If we combine this with the above duality, we can make for $M \in {}_{RG}\mathcal{M}^0$ then the module M^* again into a left RG-lattice, called the CONTRAGREDIENT OF M.
3. An important property of a group-ring RG of a finite group is that it is Gorenstein. It is even symmetric; i. e. $RG \simeq RG^*$. This means that the contragredient of a projective module is again a projective module. For an indecomposable RG-lattice we thus get – in case the Krull-Remak-Schmidt-Theorem is valid – a bijection on the indecomposable projective modules, this is called the NAKAYAMA TRANSFORMATION.
4. We also have SPECIAL CONSTRUCTIONS of RG-modules, which comes from the fact thatwe have the ANTIPODAL MAP $g \to g^{-1}$: Given two left RG-modules M and N, we can make $Hom_R(M, N)$ into a left RG-module, by defining:
$$(g \cdot \phi)(m) := g \cdot \phi(g^{-1} \cdot m) \text{ for } g \in G, \phi \in Hom_R(M, N).$$
5. We also can make the tensor product $M \otimes_R N$ into a left RG-module, by defining
$$g \cdot (m \otimes n) = g \cdot m \otimes g \cdot n \text{ and then extending } R\text{-linearly}^{10}.$$
6. The tensor product- and the hom- constructions are linked by the formula:
$$Hom_R(M, N^*) \simeq M \otimes_R N \text{ as } RG\text{-modules.}$$
7. A very important property of the tensor product of two RG-modules is the following observation: Assume that M is RG-projective and N is R-projective, then
$$M \otimes_R N \text{ is } RG\text{-PROJECTIVE}.$$

Definition 1.4. *For a maximal ideal $\mathfrak{p}$ of R we denote by $R_\mathfrak{p}$ the* LOCALIZATION *of R at $\mathfrak{p}$ and by $\widehat{R}_\mathfrak{p}$ its* COMPLETION. *Both the localization and the completion are principal ideal domains. For*
$$M \in {}_\Lambda\mathcal{M}^0 \text{ we put } M_\mathfrak{p} := R_\mathfrak{p} \otimes_R M \text{ and } \widehat{M}_\mathfrak{p} := \widehat{R}_\mathfrak{p} \otimes_R M.$$
Similar notation is used for Λ:
$$\Lambda_\mathfrak{p} := R_\mathfrak{p} \otimes_R \Lambda \text{ and } \widehat{\Lambda}_\mathfrak{p} = \widehat{R}_\mathfrak{p} \otimes_R \Lambda.$$

[9] $h \cdot_{op} g = g \cdot h$.

[10] Tough $g \in G$ acts diagonally, the scalars $r \in R$ act as usual.

Note 1.5. 1. We also point out that $\widehat{\Lambda}_{\mathfrak{p}}$-lattices are $\widehat{R}_{\mathfrak{p}}$-free.

2. Given a Λ-lattice M we have
$$M = \bigcap_{\mathfrak{p}\in max(R)} M_{\mathfrak{p}} .$$
This means that a lattice is determined by the local data. Note that here the local lattices are given PHYSICALLY. If one replaces the local lattices by isomorphic copies, the intersection will in general be NOT ISOMORPHIC to the original one. This will lead to the notion of GENUS.
3. Similarly, Λ is the intersection of its localizations.
4. Conversely, given $\Lambda_{\mathfrak{p}_i}$-lattices $M(\mathfrak{p}_i) \subset V$ lying in a FIXED A-module V for $1 \leq i \leq n$, then there exists a Λ-lattice M with localizations $M(\mathfrak{p}_i)$ for this finite number of maximal ideals $\mathfrak{p}_i$.
5. If $M \in {}_{\Lambda_{\mathfrak{p}}}\mathcal{M}^0$ then
$$M = (K \otimes_R M) \cap \widehat{M}_{\mathfrak{p}} .$$
This means that M is determined by the A-module $K \otimes_R M$ and the completion of M. Moreover, given $\widehat{M} \in {}_{\widehat{\Lambda}_{\mathfrak{p}}}\mathcal{M}^0$ and an A-module V with
$$\widehat{K}_{\mathfrak{p}} \otimes_K V = \widehat{K}_{\mathfrak{p}} \otimes_{\widehat{R}_{\mathfrak{p}}} \widehat{M} ,$$
then
$$M := V \cap \widehat{M}$$
is a $\Lambda_{\mathfrak{p}}$-lattice with completion $\widehat{M}$.
6. A Λ-lattice P is projective if and only if $P_{\mathfrak{p}}$ is projective for every $\mathfrak{p} \in max(R)$ if and only if $\widehat{P}_{\mathfrak{p}}$ is projective for every $\mathfrak{p} \in max(R)$. (This follows from Note 2.10 below.)

2. Lattices: Local versus global

Let us first consider an Artin K-algebra A – not necessarily semi-simple. Let ${}_A\mathcal{M}^f$ be the category of finitely generated left A-modules. Though the representation theory of A is not at all understood in general, there are some features, which make life a bit easier (cf. [Ba; 68], [ARS; 95]):

1. Cancelation holds in ${}_A\mathcal{M}^f$ i. e.
$$X \oplus Z \simeq X \oplus Y \text{ implies } X \simeq Y .$$
2. The Krull-Remak-Schmidt Theorem holds; i. e. every module decomposes UNIQUELY – up to isomorphism – into indecomposable modules.

3. Every $X \in {}_A\mathcal{M}^f$ has a PROJECTIVE COVER:

$$0 \longrightarrow \Omega_1(X) \xrightarrow{\alpha} P(X) \xrightarrow{\beta} X \longrightarrow 0$$

where $P(X)$ is projective and β is an ESSENTIAL EPIMORPHISM; i. e. if $Im(\alpha)+N = P(X)$ for a submodule N of $P(X)$ then $N = P(X)$. By this property both $P(X)$ and $\Omega_1(X)$ are uniquely determined by X. If A is self-injective (Gorenstein), then $\Omega_1(X)$ has no projective direct summand.

Proposition 2.1. *Let Λ be an order over the complete local Dedekind domain R. Then*

1. *cancelation holds in ${}_\Lambda\mathcal{M}^0$,*
2. *the Krull-Remak-Schmidt Theorem is valid in ${}_\Lambda\mathcal{M}^0$,*
3. *projective covers exist for $M \in {}_\Lambda\mathcal{M}^0$.*

Hence the representation theory of orders over complete Dedekind domains has similar nice properties as that of Artin algebras.

An important application of this result is

Lemma 2.2 (Noether-Deuring).

Let S be a finite R-free extension of the complete Dedekind domain R and let $M, N \in {}_\Lambda\mathcal{M}^0$. Then

$$S \otimes_R M \simeq_{S\otimes_R \Lambda} S \otimes_R N \text{ if and only if } M \simeq_\Lambda N.$$

We now assume that R is a SEMI-LOCAL DEDEKIND DOMAIN; i. e. R has only finitely many maximal ideals. Then R is a principal ideal domain.

Proposition 2.3. *Let Λ be an R-order, where R is a semi-local Dedekind domain. Then in ${}_\Lambda\mathcal{M}^0$*

1. *cancelation holds,*
2. *the Krull-Remak-Schmidt Theorem does not hold in general,*
3. *projective covers do not exist in general.*

The first statement follows from the close relation between Λ-modules and their completions:
Let $J = rad(R)$ – note that this is different from zero, since R is semi-local. The J-ADIC COMPLETION of R is

$$\widehat{R} := lim.proj.(R/J^n) = \prod_{\mathfrak{p}\in max(R)} \widehat{R}_{\mathfrak{p}}$$

is the product of the completions of R at its maximal ideals. If

$$\widehat{\Lambda} := \widehat{R} \otimes_R \Lambda = \bigoplus_{\mathfrak{p} \in max(R)} \widehat{\Lambda}_{\mathfrak{p}}$$

is the completion of Λ, then we define for $M \in {}_{\Lambda}\mathcal{M}^0$ the completion as $\widehat{M} := \widehat{R} \otimes_R M$. Note that $\widehat{M} \simeq \bigoplus_{\mathfrak{p} \in max(R)} \widehat{M}_{\mathfrak{p}}$. We then have the following result:

Proposition 2.4. *Let $M, N \in {}_{\Lambda}\mathcal{M}^0$ then*

$$M \simeq N \text{ if and only if } \widehat{M} \simeq \widehat{N} .$$

In particular, if R is local, then two Λ-lattices are isomorphic if and only if their completions are isomorphic.
The cancelation property follows easily from this result.

We now turn to the GLOBAL SITUATION; i. e. R is an ARBITRARY DEDEKIND DOMAIN with – in general – infinitely many maximal ideals; e. g. $R = alg.int(K)$ is the ring of algebraic integers in an algebraic number field K, finite dimensional over $\mathbb{Q}$.

Note 2.5. Let Λ be an R-order in a separable K-algebra, for a Dedekind domain R. Then in ${}_{\Lambda}\mathcal{M}^0$

1. cancelation does not hold in general – not even for projective Λ-lattices,
2. the Krull-Remak-Schmidt Theorem does not hold in general,
3. there do not exist projective covers in general.

The theory of GENERA is the ARITHMETIC OF ORDERS and makes integral representation theory DIFFICULT and also FASCINATING – it plays global representation theory versus local representation theory.

Definition 2.6. *We say that two Λ-lattices M, N lie in the* SAME GENUS *(in signs $M \vee N$), provided $M_{\mathfrak{p}} \simeq N_{\mathfrak{p}}$ for every $\mathfrak{p} \in max(R)$. The* GENUS *of a Λ-lattice M consists of the isomorphism types of Λ-lattices in the same genus as M.*

If R is semi-local, then $M \vee N$ if and only if $M \simeq N$.

The question of when two lattices lie in the same genus is a very delicate one and involves class groups of finite extensions of R [Ja; 68a], [Ja; 68b].

If R is the ring of algebraic integers in the algebraic number field K, then the genus of R consists precisely of the non-zero ideals in R and the non-isomorphic modules in the genus of R constitute the CLASS GROUP of R. Given two non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$, CLASS FIELD THEORY implies that $\mathfrak{a}^{(n)} \simeq \mathfrak{b}^{(n)}$ [11] for some natural number n.

Let now R be a Dedekind domain and let Λ be an R-order. The next result generalizes in some sense class field theory.

Proposition 2.7. *Let $M, N \in {}_\Lambda\mathcal{M}^0$, then the following are equivalent:*

1. $M \vee N$ *i. e. M and N lie in the same genus,*
2. $M^{(n)} \simeq N^{(n)}$ *for some $n \in \mathbb{N}$.*

Though lattices in the same genus are in general not isomorphic, they nevertheless share an important property:

Proposition 2.8. *Let $M \vee N$. Then M is indecomposable if and only if N is indecomposable. This means that decomposition is a* GENUS PROPERTY.

This result is a consequence of the following observation:

Lemma 2.9. *Given an exact sequence*

$$\mathcal{E}: \quad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of Λ-lattices. Then $\mathcal{E}$ is split if and only if for every $\mathfrak{p} \in max(R)$ the sequence

$$\mathcal{E}_\mathfrak{p}: \quad 0 \longrightarrow M'_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow M''_\mathfrak{p} \longrightarrow 0$$

is split.

Note 2.10. The reason for this result is twofold:

1. $Ext^1_\Lambda(M'', M')$ is a finitely generated R-torsion module,
2. $R_\mathfrak{p} \otimes_R Ext^1_\Lambda(M'', M') \simeq Ext^1_{\Lambda_\mathfrak{p}}(M''_\mathfrak{p}, M'_\mathfrak{p})$, since $R_\mathfrak{p}$ is R-flat.

We have pointed out that projective covers for $M \in {}_\Lambda\mathcal{M}^0$ do not exist in general; there is for group-rings though a special property of the kernels of projectives mapping onto M.

We first recall:

Lemma 2.11 (Schanuel's Lemma).

Given $M \in {}_\Lambda\mathcal{M}^0$ and two exact sequences

$$0 \longrightarrow \Omega_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \Omega_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0,$$

[11] $M^{(n)}$ denotes the direct sum of n copies of the module M.

with P_i projective, then

$$P_2 \oplus \Omega_1 \simeq P_1 \oplus \Omega_2 .$$

We now turn to the ARITHMETIC SITUATION for group rings: R is the ring of algebraic integers in an algebraic number field and G is a finite group. BY Λ WE DENOTE THE GROUP-RING RG.

An important property of projective RG-lattices is given by:

Lemma 2.12. *Let P be an indecomposable projective left Λ-lattice, then P lies in the same genus as the left module ${}_\Lambda\Lambda$.*

Definition 2.13. *Let $M \in {}_\Lambda\mathcal{M}^0$ and let us be given a* PROJECTIVE RESOLUTION

$$0 \longrightarrow \Omega \longrightarrow P \stackrel{\alpha}{\longrightarrow} M \longrightarrow 0$$

with P projective. We decompose $\Omega = C(\alpha) \oplus Q$, where Q is projective and $C(\alpha)$ has no projective summands. Then $C(\alpha)$ is called a CORE OF M.

Lemma 2.14. *Let C be a core of M. Then there exists a projective resolution of M of the form*

$$0 \longrightarrow C \longrightarrow P \longrightarrow M \longrightarrow 0 .$$

Proof: Since C is a core, there exists an exact sequence

$$\mathcal{E} \quad : \quad 0 \longrightarrow C \oplus Q \longrightarrow P' \longrightarrow M \longrightarrow 0 .$$

Let $\alpha : C \oplus Q \longrightarrow C$ be the projection. Then the push-out of $\mathcal{E}$ along α has the desired property. q.e.d.

An important property of cores is given in the following

Proposition 2.15. 1. *Two cores C and C' of M lie in the same genus.*
2. *Let C be a core of M and assume that $C' \vee C$, then C' is also a core of M.*
3. *This means that the* CORES OF M CONSTITUTE A FULL GENUS CLASS.

Proof: Assume that we are given two projective resolutions

$$\mathcal{E}_1 \quad : \quad 0 \longrightarrow C_1 \oplus Q_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0 \text{ and}$$

$$\mathcal{E}_2 \quad : \quad 0 \longrightarrow C_2 \oplus Q_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0 ,$$

where C_1, C_2 are cores of M. Then Schanuel's Lemma 2.11 implies

$$C_1 \oplus Q_1 \oplus P_2 \simeq C_2 \oplus Q_2 \oplus P_1 .$$

Using the fact that the projective modules are locally free that the cores do not have a locally free direct summand, and that decomposition is a property of the genus (cf. Proposition 2.8) we conclude $C_1 \vee C_2$.

Conversely, let $C \vee C_1$. The sequence $\mathcal{E}_1$ represents an element in $Ext^1_\Lambda(M, C_1 \oplus Q_1)$. Because of Note 2.10 we have

$$Ext^1_\Lambda(M, C_1) \simeq Ext^1_\Lambda(M, C).$$

Thus $\mathcal{E}_1$ corresponds – up to equivalence of short exact sequences – to a unique extension $\mathcal{E}$ in $Ext^1_\Lambda(M, C \oplus Q_1)$, say

$$\mathcal{E} \quad : \quad 0 \longrightarrow C \oplus Q_1 \longrightarrow X \longrightarrow M \longrightarrow 0.$$

Since 'being projective' is a local property (cf. Notes 1.5, 4.)), we conclude that X is projective. And hence C is a core of M. q.e.d.

3. Some special orders

For the results in this section we refer to the books [Re; 75], [RHD; 70 1].

Definition 3.1. *Let Λ be an R-order. Λ is called*

1. HEREDITARY, *if every Λ-lattice is projective;*
2. MAXIMAL, *if it is a maximal object among the R-orders in A under the inclusion relation;*
3. SEPARABLE, *if Λ is projective as $\Lambda^e := \Lambda \otimes_R \Lambda^{op}$-module[12]. Λ^e is called the* EVELOPPING ORDER *to Λ.*

Proposition 3.2. *The order Λ is maximal (hereditary, separable) if and only if $\Lambda_\mathfrak{p}$ is maximal (hereditary, separable) for every $\mathfrak{p} \in max(R)$ if and only if $\widehat{\Lambda}_\mathfrak{p}$ is maximal (hereditary, separable) for every $\mathfrak{p} \in max(R)$.*

Theorem 3.3. 1. *Let R be a Dedekind domain and let Λ be an R-order in the* SEPARABLE *K-algebra A. Then Λ is contained in a – not necessarily unique – maximal R-order. (We point out that it is not enough to require that A is semi-simple.)*

2. *Let $R := \widehat{R} := \widehat{R}_\mathfrak{p}$ be complete and let $A = D$ be a skew-field. Then there exists a* UNIQUE MAXIMAL *R-*ORDER *$\Omega := \Omega(D)$ in D. (It consists of all elements in D which are integral over R[13].) Moreover, $rad(\Omega) = \omega_0 \cdot \Omega = \Omega \cdot \omega_0$[14] is a principal ideal for some $\omega_0 \in \Omega$.*

---

[12]This means that the Hochschild cohomology groups $H^n_{\Lambda^e}(\Lambda, -)$ vanish for $n \geq 1$.

[13]These are elements in D satisfying a monic polynomial over R.

[14]This is the Jacobson radical.

3. *Assume that Λ is a connected*[15] *maximal order in the separable K-algebra A, then A is simple, say $A = (D)_n$*[16] *for a skew-field D and Λ is conjugate in A to $(\Omega)_n$, Ω as above.*
4. *Assume that Λ is a basic*[17] *connected hereditary order, then $A = (D)_n$ is simple and Λ is conjugate in A to*

$$\begin{pmatrix} \Omega & \Omega & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \Omega & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \Omega & \cdots & \Omega & \Omega \\ & & & \ddots & & \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \underline{\omega_0} & \Omega \end{pmatrix}_n , \qquad (1)$$

where $\underline{\omega_0} = \omega_0 \cdot \Omega = \Omega \cdot \omega_0 := rad(\Omega)$ and Ω is the unique maximal order in D. Moreover, we have

$$rad(\Lambda) = \begin{pmatrix} \underline{\omega_0} & \Omega & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \Omega & \Omega \\ & & & \ddots & & \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \underline{\omega_0} & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \underline{\omega_0} & \underline{\omega_0} \end{pmatrix}_n . \qquad (2)$$

5. *Let Λ be a separable connected $\widehat{R}$-order, then $A = (L)_n$ is simple and Λ is conjugate in A to $(S)_n$, where S is the ring of algebraic integers in the unramified extension L of $\widehat{K}$*[18].

We shall next list an interesting characterization of hereditary orders. But first we have to make a definition.

Definition 3.4. *Let R be a complete Dedekind domain and let Λ be an R-order. If $J := rad(\Lambda)$ we define the* LEFT-ORDER OF *$rad(\Lambda)$ – sometimes this is also called the* LEFT RING OF MULTIPLIERS *of $rad(\Lambda)$ – as*

$$\Lambda_l = \Lambda_l(\Lambda) := \{x \in A \mid x \cdot J \subset J\} .$$

This surely is an R-order in A.

Proposition 3.5. *An R order Λ over the complete Dedekind domain R is hereditary if and only if $\Lambda_l(\Lambda) = \Lambda$ if and only if $rad(\Lambda)$ is a projective left Λ-lattice.*

---

[15] This means that Λ is indecomposable as ring.

[16] This denotes the full matrix ring over D.

[17] This means that in the (unique) decomposition of Λ into indecomposable left modules, there are no multiplicities.

[18] This means that S is $\widehat{R}$-free of finite rank and $rad(S) = rad(\widehat{R}) \cdot S$.

This can sometimes be used to reduce questions about an arbitrary order to questions about a hereditary order, by going up the FINITE CHAIN of radical orders, until one reaches a hereditary order.

We can GENERALIZE THE STRUCTURE OF A HEREDITARY ORDER in the following way:

Definition 3.6. *Let $\widehat{R}$ be complete, and let Ω be a local* [19] *$\widehat{R}$-order, and let ω_0 be a regular* [20] *element and assume that $\underline{\omega_0} := \omega_0 \cdot \Omega = \Omega \cdot \omega_0$. Then the order*

$$H(\Omega, \omega_0) := \begin{pmatrix} \Omega & \Omega & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \Omega & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \Omega & \cdots & \Omega & \Omega \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \underline{\omega_0} & \Omega \end{pmatrix}_n \qquad (3)$$

is called a GENERALIZED HEREDITARY ORDER OF SIZE n .
Then there is a special ideal

$$\alpha(\Omega, \omega_0) := \begin{pmatrix} \underline{\omega_0} & \Omega & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \Omega & \cdots & \Omega & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \Omega & \Omega \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \underline{\omega_0} & \Omega \\ \underline{\omega_0} & \underline{\omega_0} & \underline{\omega_0} & \cdots & \underline{\omega_0} & \underline{\omega_0} \end{pmatrix}_n . \qquad (4)$$

We note that $H(\Omega, \omega_0)/\alpha(\Omega, \omega_0)$ is a product of n copies of the local rings $\Omega/\underline{\omega_0}$.

Such a generalized hereditary order can be characterized as follows:

Proposition 3.7 ([Ro; 92]). *Let Δ be a basic $\widehat{R}$-order, and assume that Δ has a full*[21] *two-sided ideal I such that*

1. *Δ/I is a product of local algebras,*
2. *$I = \Delta \cdot \alpha$,*
3. *$I \subset rad(\Delta)$,*

then Δ is a generalized hereditary order.

These generalized hereditary orders play an important rôle in the STRUCTURE OF BLOCKS WITH CYCLIC DEFECT of group-rings.

[19] This means that $\Omega/rad(\Omega)$ is a skew-field – equivalently Ω is indecomposable as left module.
[20] This means that ω_0 is a unit in A .
[21] This means $K \cdot I = A$.

For the description of these blocks the next construction is essential. We start with an example:

Example 3.8. Let $G = C_p \rtimes C_q$, where $q|(p-1)$ and p is an odd prime, be the semidirect product[22]. Here C_p should be identified with the p-th roots of unity over $\mathbb{Z}$, and C_q should be viewed as the appropriate subgroup of the Galois-group of $\mathbb{Q}[\zeta_p]$. The semidirect product structure is then given by the Galois-action. Let $S = \mathbb{Z}[\zeta_p]^{C_q}$ be the fixed ring of $\mathbb{Z}[\zeta_p]$ under C_q, and denote by Ω the completion of S at p - note that p is totally ramified in S. Let $\pi \cdot \Omega = rad(\Omega)$ and put $\omega_0 = \pi$. Then $\Omega/\pi \cdot \Omega \simeq \mathbb{Z}/p \cdot \mathbb{Z}$. Let $\Delta := H(\Omega, \omega_0)$ be the corresponding GENERALIZED HEREDITARY ORDER of size q. Then there is a natural epimorphism[23]

$$\Delta \xrightarrow{\alpha} \prod^{q} \mathbb{Z}/p \cdot \mathbb{Z}\,.$$

We let $R = \widehat{\mathbb{Z}}_p$ and $\Lambda = \prod^q R$. Then we also have an epimorphism

$$\Lambda \xrightarrow{\beta} \prod^{q} \mathbb{Z}/p \cdot \mathbb{Z}\,.$$

We thus can form the pull-back

$$\begin{array}{ccc} \Pi & \longrightarrow & \Delta \\ \downarrow & & \downarrow \alpha \\ \Lambda & \xrightarrow{\beta} & \prod^{q} \mathbb{Z}/p \cdot \mathbb{Z}\,. \end{array}$$

Lemma 3.9. *The order Π is isomorphic to the integral group-ring RG.*

We shall now generalize this construction:

Definition 3.10. *Let $\{H_i := H(\Omega_i, \omega_i)\}_{1 \le i \le n}$ and $\{D_j := H(\Delta_j, \delta_j)\}_{1 \le j \le m}$ be generalized hereditary orders of size n_i and m_i resp. Assume that*

$$\overline{\Omega} \simeq \Omega_i/(\omega_i \cdot \Omega_i) \simeq \Delta_j/(\delta_j \cdot \Delta_j)\,,\ 1 \le i \le n\,,\ 1 \le j \le m$$

is the same for all $\{i,j\}$, and that

$$\sum_{i=1}^{m} m_i = \sum_{i=0}^{n} n_i =: \nu\,.$$

[22] As set this is the product, and the multiplication is twisted by the action of C_q on C_p.
[23] At the beginning we have pointed out the epimorphisms are not necessarily surjective; however, we shall nevertheless use 'epimorphism' to mean surjective epimorphisms.

Let

$$H := \prod_{i=1}^{n} H_i \text{ and } D := \prod_{i=1}^{m} D_i \, .$$

We then have epimorphisms

$$\alpha : H \longrightarrow B := \prod^{\nu} \overline{\Omega} \text{ and } \beta : D \longrightarrow B \, .$$

The order Λ, *defined as pull-back*

$$\begin{array}{ccc} \Lambda & \longrightarrow & H \\ \downarrow & & \downarrow \alpha \\ D & \xrightarrow{\beta} & B \, , \end{array}$$

is called a GENERALIZED GREEN ORDER.

It will turn out that BLOCKS WITH CYCLIC DEFECT are a special type of generalized Green orders, they are so called GREEN ORDERS.

Note 3.11. The above definition needs – for the experts – an EXPLANATION. Let B be a block with cyclic defect and Brauer tree T, which we orient clockwise. To each vertex corresponds a generalized hereditary order. We now subdivide these hereditary orders into the orders $\{H_i\}$ and $\{D_j\}$ in such a way that there are only edges between $\{H_i\}$ and $\{D_j\}$. This is possible, since T is a tree. The identification in the pull-back is then according to GREEN'S WALK AROUND THE BRAUER TREE [Gr; 74]. So B is a generalized Green order.

We now RETURN TO MAXIMAL, HEREDITARY AND SEPARABLE ORDERS.

Note 3.12. The above results (in Theorem 3.3) show that separable R-orders are maximal and that maximal orders are hereditary.

We shall next define IDEALS in R, which measure, HOW FAR AWAY AN ORDER IS FROM BEING SEPARABLE AND HEREDITARY RESP..

Definition 3.13. 1. *The* HIGMAN IDEAL $H(\Lambda)$ *of* Λ *is defined as the annihilator*

$$ann_R(H_{\Lambda^e}(\Lambda, -)) := H(\Lambda) \, .$$

2. *We put*

$$h(\Lambda) := \bigcap_{M,N \in_\Lambda \mathcal{M}^0} ann_R(Ext^1_\Lambda(M, N)) \, .$$

Lemma 3.14. *We have* $0 \neq H(\Lambda) \subseteq h(\Lambda) \subseteq R$. *Moreover, for a maximal ideal* $\mathfrak{p}$ *the order* $\Lambda_{\mathfrak{p}}$ *is separable (resp. hereditary) if and only if* $(\mathfrak{p}, H(\Lambda)) = 1$ *(resp.* $(\mathfrak{p}, h(\Lambda)) = 1$*)* [24].

This implies in particular that $Ext^1_\Lambda(N, M)$ is R-torsion for M , $N \in \mathcal{M}^0$.

The lemma needs a word of explanation: We have the exact augmentation sequence of Λ^e-modules:

$$0 \longrightarrow I(\Lambda) \longrightarrow \Lambda^e \xrightarrow{\epsilon} \Lambda \longrightarrow 0$$

where ϵ is the AUGMENTATION; i. e. multiplication, and $I(\Lambda)$ is the AUGMENTATION IDEAL. Thus

$$H(\Lambda) = ann_R(H_{\Lambda^e}(\Lambda, I(\Lambda)),$$

which is a non zero ideal, since A is separable.
Moreover, we have a natural identification:

$$H_{\Lambda^e}(\Lambda, Hom_R(M, N)) = Ext^1_\Lambda(M, N).$$

Whence the statements follows. q.e.d.

For details we refer to [Ro; 94].

Remark 3.15. 1. For the group-ring RG of a finite group G , the Higman ideal is given by $H(RG) = |G| \cdot R = h(\Lambda)$. This is shown by modifying the argument in the proof of Maschke's Theorem.
2. Consequently, a group-ring is hereditary if and only if it is maximal if and only if it is separable.

We conclude this section with an explicit description of a group-ring:

Example 3.16. 1. With the group-ring RG we have associated the AUGMENTATION SEQUENCE

$$0 \longrightarrow I_R(G) \longrightarrow RG \xrightarrow{\epsilon} R \longrightarrow 0,$$

where $\epsilon : g \longrightarrow 1$ is the AUGMENTATION MAP and $I_R(G)$ is the AUGMENTATION IDEAL. It is generated freely over R by the elements

$$\{g - 1\}_{g \in G \setminus \{1\}} .$$

2. Let G be a finite group such that $|G| \cdot R = R$. Then RG is a product of full matrix-rings over unramified extensions of R.

[24] This notation means that the ideals are relatively prime.

3. Let $C_p := < c\,,\, c^p = 1 >$ be the cyclic group of order p for a rational prime p and denote by ζ_p a primitive p-th root of unity. Then there is a natural homomorphism
$$\mathbb{Z}C_p \xrightarrow{\phi_1} \mathbb{Z}[\zeta_p]\,,\; c \longrightarrow \zeta_p\,.$$
We also have the TRIVIAL representation: $c \xrightarrow{\phi_2} 1$. We thus have an embedding:
$$\mathbb{Z}C_p \xrightarrow{(\phi_1,\phi_2)} \mathbb{Z}[\zeta_p] \times \mathbb{Z}\,.$$
So the group-ring will be described as a sub-ring of $\mathbb{Z}[\zeta_p] \times \mathbb{Z}$:
$$\mathbb{Z}C_p = \{(x,y) \mid x \in \mathbb{Z}[\zeta_p], y \in \mathbb{Z} \,:\, x\alpha = y\beta\}$$
where α and β resp. is reduction modulo the ideal generated by $\zeta_p - 1$ and p resp. This means that $\mathbb{Z}C_p$ is the pull-back
$$\begin{array}{ccc} \mathbb{Z}C_p & \longrightarrow & \mathbb{Z}[\zeta_p] \\ \downarrow & & \downarrow \alpha \\ \mathbb{Z} & \xrightarrow{\beta} & \mathbb{F}_p \end{array},$$
where $\mathbb{F}_p$ is the field with p elements.
4. Let $G = M \rtimes H$ be the semidirect product of H acting on the abelian group M – written multiplicatively – by conjugation; i. e.
$$(m_1, h_1) \cdot (m_2, h_2) = (m_1 \cdot {}^{h_1}m_2, h_1 \cdot h_2)\,,\; h_i \in H\,,\; h_i \in H\,,$$
then the group-rings RG is a TWISTED TENSOR PRODUCT
$$RG = RM \otimes_R RH\,,$$
where the addition is as in the usual tensor product, but the multiplication on the group generators is twisted according to the above action and then extended linearly.

4. Induction and restriction

Definition 4.1. *Let H be a subgroup of G, and denote by $\{g_i\}_{1\leq i\leq n}$ left coset representatives of H in G.*

1. *Let M be an RH-module. The* INDUCED MODULE FROM M TO G *is defined as the RG-module*
$$M\uparrow_H^G := RG \otimes_{RH} M\,.$$

Since RG is RH-free on $\{g_i\}_{1\leq i\leq n}$, *we have*

$$M\uparrow_H^G = \bigoplus_{i=1}^{n} g_i \otimes M,$$

where $g_i \otimes M$ *is an* $R({}^{g_i}H)$*-module*[25].

2. *If* $g_i \in N_G(H)$ [26], *then* $g_i \otimes M =: {}^{g_i}M$ *is again an RH-module, called the* CONJUGATE MODULE.
3. *Let M be an RG-module, then the* RESTRICTION $M\downarrow_H^G$ *is the RH-module obtained by restricting to* $RH \subseteq RG$.

INDUCTION AND RESTRICTION ARE EXACT FUNCTORS.

The following FORMULA OF MACKEY is important in applications:

Proposition 4.2 (Mackey's formula). *Let U and V be subgroups of G, and let M be an RU-module. Then*

$$M\uparrow_U^G\downarrow_V^G = \bigoplus_{i=1}^{m} ((g_i \otimes M)\downarrow^{{}^{g_i}U}_{({}^{g_i}U)\cap V})\uparrow^V_{({}^{g_i}U)\cap V},$$

where the sum is taken over the $U\backslash G/V$ *double cosets and* $\{g_i\}_{1\leq i\leq m}$ *are double coset representatives.*

The next formula is FROBENIUS' RECIPROCITY and it describes homomorphisms to an induced module:

Proposition 4.3. *Let* $H \leq G$ *and assume that M is an RG-module and N is an RH-module. Then there is a natural isomorphism*

$$Hom_{RG}(M, N\uparrow_H^G) = Hom_{RH}(M\downarrow_H^G, N),$$

which extends also to the derived functors[27].

Let now $\widehat{R}$ be COMPLETE of characteristic zero with residue field of characteristic $p > 0$ and put $\Lambda := \widehat{R}G$. Again using a modification of Maschke's argument, one can show:

Proposition 4.4. *Let* $M \in {}_\Lambda\mathcal{M}^0$, *and let P be a Sylow p-subgroup of G. Then M is a direct summand of* $M\downarrow_P^G\uparrow_P^G$. *This means that for M indecomposable, M is a direct summand of a lattice, induced from an indecomposable lattice of the Sylow p-subgroup.*

An important consequence of this result is – we use the notation from above –

---

[25] ${}^{g_i}H := g_i \cdot H \cdot g_i^{-1}$.

[26] This is the normalizer of H in G.

[27] In cohomology of groups (derived functors) this is called Shapiro's Lemma.

Proposition 4.5. *The number of indecomposable non-isomorphic $\hat{R}G$-lattices is finite if and only if the number of indecomposable non-isomorphic $\hat{R}P$-lattices is finite for a Sylow p-subgroup P of G.*

This was used by A. Jones [Jo; 63] to show

Theorem 4.6. *The number of non-isomorphic indecomposable $\mathbb{Z}G$-lattices is finite if and only if the Sylow p-subgroups of G are cyclic of order $\leq p^2$ for every prime p.*

In Proposition 4.4 we have seen that every indecomposable Λ-lattice is a direct summand of a module induced from a Sylow p-subgroup. However, it can also be induced from a smaller group. This leads to the

Definition 4.7. *Let $M \in {}_\Lambda\mathcal{M}^0$ be indecomposable. A subgroup V of G is called a* VERTEX OF M, *provided it is minimal among the subgroups H, such that M is a direct summand of a module N induced from H, say $N = \Sigma\uparrow_H^G$ for an indecomposable H-module Σ. The module Σ is called a* SOURCE OF M *with respect to the vertex $V = H$.*

Proposition 4.8. *Let $M \in {}_\Lambda\mathcal{M}^0$ be indecomposable. Then all vertices of M are conjugate, and they are p-groups. Each representative of this conjugacy class is called – by abuse of language –* THE *vertex of M. Let V be a vertex of M and let Σ_1 and Σ_2 be two sources – with respect to the* **SAME VERTEX**. ***then*** *they are conjugate in the normalizer of V; i. e. $\Sigma_1 \simeq {}^x\Sigma_2$ for some $x \in N_G(V)$.*

We shall now apply this to bimodules. We denote by $G^e := G \times G^{op}$. Then RG is a left module for $R(G^e) \simeq (RG)^e$ (cf. Definition 3.1). This is tantamount to saying that RG is an RG-bimodule.

Definition 4.9. *Let $\hat{R}$ be a complete Dedekind domain of characteristic zero with residue field of characteristic $p > 0$ and let G be a finite group. A* BLOCK B *of $\hat{R}G$ is an indecomposable ring direct summand of $\hat{R}G$. This is the same as saying that B is an indecomposable direct summand of $\hat{R}G$ as left $\hat{R}G^e$-module.*

The DEFECT GROUP OF A BLOCK B of $\hat{R}G$ is the vertex of B as left module over $\hat{R}G^e$, more precisely,

Proposition 4.10. *Let $G_0 = \{(g, g^{-1}) \mid \subset G^e\}$ be the diagonal subgroup. Then any block B of $\hat{R}G$ has a vertex contained in G_0. So the vertex of B, called* THE DEFECT GROUP D OF B [28], *can be viewed as a p-subgroup of G, if one identifies G_0 with G. The vertex of B is called the* DEFECT GROUP *of B.*

[28] Recall that the vertex is a conjugacy class of subgroup, and hence also the defect group of a block is a conjugacy class of p-subgroups of G.

As a matter of fact not every p-subgroup of G can occur as a vertex of a block. Defect groups are INTERSECTIONS of two Sylow p-subgroups.

Note 4.11. 1. The last statement implies that for a group with a normal Sylow p-subgroup P, all blocks have defect group P.
2. A block of defect zero – i. e. it has the trivial group as defect group – is a full matrix ring over an unramified extension of $\widehat{R}$; i. e. it is separable.
3. Since the group-ring $\widehat{R}G$ contains the trivial representation $\widehat{R}$, there is a unique block B_0, PRINCIPAL BLOCK which contains the trivial representation.
4. Let G be a solvable group and let $O_{p'}(G)$ be the largest normal subgroup of order prime to p, then the principal block is $B_0 = \widehat{R}(G/O_{p'}(G))$.
5. In general, the remaining blocks can often be described for solvable groups by using CLIFFORD THEORY (cf. [Ro; 96,II]).

5. Cohomology

In this section we shall give a brief account of GROUP COHOMOLOGY. For a detailed presentation of group cohomology we refer to the books [Ben; 91], [Ev; 91]. Here R is an arbitrary integral domain.

Definition 5.1. 1. *For $n \geq 0$, the n-th* COHOMOLOGY GROUP OF G WITH COEFFICIENTS IN THE FINITELY GENERATED RG-MODULE M *is defined as*

$$H^n_R(G, M) := Ext^n_{RG}(R, M).$$

Here R is the trivial RG-module. These cohomology groups are R-modules; they are even right modules for $End_{RG}(M)$ in addition. $H^n(G, -)$ is a covariant functor. For $n = 0$ the group

$$H^0_R(G, M) := Hom_{RG}(R, M)$$

consists of the FIXED POINTS OF G IN M. *We sometimes write M^G for the fixed points.*
2. *More generally, if A is a ring and L, M, N are left R-modules, then for $n \geq 1$, the elements in $Ext^n_A(M, N)$ can be interpreted as long exact sequences of length $n+1$, starting with N and ending with M. Thus the juxtaposition of long exact sequences (*YONEDA-PRODUCT*) gives a pairing*

$$Ext^n_A(N, L) \times Ext^m_A(M, N) \longrightarrow Ext^{n+m}_A(M, L),$$

which turns out to be associative [Ben; 84]. *Moreover,*

$$Ext^n_A(M, N) \text{ is a bimodule over } (End_A(N), End_A(M)).$$

3. *This way,*
$$Ext^*_A(M) := \bigoplus_{i\geq 0} Ext^i_A(M, M)$$
is a graded algebra over $End_A(M)$.
4. *Moreover, the direct sum*
$$\bigoplus_{i\geq 0} Ext^i_A(M, N) \text{ is a graded right module for } Ext^*_A(M).$$
5. *For the group-ring* RG, *the* COHOMOLOGY RING OF G OVER R *is defined as*
$$H^*_R(G) := \bigoplus_{i\geq 0} H^i_R(G, R).$$
It turns out, that $H^*_R(G)$ *is a commutative ring in the graded sense; i. e.*
$$\alpha \cdot \beta = (-1)^{m\cdot n} \cdot \beta \cdot \alpha,$$
where α *and* β *are homogeneous elements of degree* m *and* n *resp.; i. e.* $\alpha \in H^m_R(G, R)$ *and* $\beta \in H^n_R(G, R)$.
It should be noted that for an – even simple – RG*-module* M *the graded ring* $Ext^*_{RG}(M, M)$ *is in general not commutative in the graded sense.*
6. *This shows that for* $char(R) \neq 2$ *the sum* $\bigoplus_{i\ odd} H^i_R(G, R)$ *is a nilpotent ideal in* $H^*_R(G)$ *– already the squares are zero.*
7. *The quotient*
$$H^{ev}_R(G) := \bigoplus_{i\ even} H^i_R(G, R)$$
is then a genuine commutative ring. By a deep result of Quillen [Qui; 71] *it is a finitely generated* $R = H^0_R(G, R)$*-algebra, generated by homogeneous elements.*
8. *The prime ideal spectrum of* $H^{ev}_R(G)$ *with the Zariski topology is called the* COHOMOLOGY VARIETY OF G OVER R. *If* R *is noetherian, it is a noetherian space* [29].
9. *Let now* M *be an* R*-projective* RG*-module, then the* R*-module*
$$Ext^*_{RG}(M) := \bigoplus Ext^i_{RG}(M, M)$$
can be made into a module for $H^*(G, R)$ *in the following way: Given a cohomology class* $\zeta \in H^n(G, R)$. *Then it is represented by a long exact sequence:*
$$\mathcal{E} : 0 \longrightarrow R \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow R \longrightarrow 0.$$

[29] This means that the lattice of open sets is noetherian; i. e. ascending chains terminate (cf. [Ba; 68]).

Since M is R-projective, the functor $M \otimes_R -$ is exact. Hence $M \otimes_R \mathcal{E}$ represents an element in $Ext^n_{RG}(M, M)$. If we now have an element $\mathcal{E}' \in Ext^m_{RG}(M, M)$, we can use the Yoneda composition with $M \otimes_R \mathcal{E}$ to obtain an element in $Ext^{n+m}_{RG}(M, M)$. This way $Ext^*_{RG}(M, M)$ becomes a graded module for $H^*(G, R)$.

10. *Thus*

$$ann_{H^{ev}_R(G)}(Ext^*_{RG}(M, M))$$

is an ideal, and its variety is called the COHOMOLOGY VARIETY OF M [Ben; 91].

Note 5.2. 1. The cohomology variety of M carries very important informations on M. For example, it gives a measure of how far M is from being projective [Ca; 83].

2. The low dimensional cohomology groups $H^i(G, M)$ have important group theoretical interpretations – we have $\mathbb{Z}$ as ring of coefficients.
   (a) $H^0(G, M)$ are the fixed points of G in M.
   (b) $H^1(G, M)$ describes the derivations modulo inner derivations (see for example [Ro; 94], [Gr; 70], [H-S; 70]).
   (c) $H^2(G, M)$ describes group extensions E with a normal abelian subgroup M and $E/M \simeq G$.

We shall elaborate a bit on H^2. We have described $H^2(G, M)$ as $Ext^2_{\mathbb{Z}G}(\mathbb{Z}, M)$, which is by dimension shift isomorphic to $Ext^1_{\mathbb{Z}G}(I(G), M)$, where $I(G)$ is the integral augmentation ideal of $\mathbb{Z}G$ (cf. Example 3.16, 2.).

There is however a PURELY GROUP THEORETICAL DESCRIPTION OF $H^2(G, M)$ in terms of 2-cocycles and 2-coboundaries.

Given a short exact sequence of groups with M an abelian group, written multiplicatively:

$$\mathcal{E}: \quad 0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} G \longrightarrow 0.$$

Then M becomes a G-module by conjugating with an inverse image in E of $g \in G$. This operation is well defined, since M is abelian.

Most naturally a 2-cocycle arises as obstruction of a set theoretical splitting of f to β in the short exact sequence $\mathcal{E}$.

Definition 5.3. *If f is a set theoretical splitting of β, then*

$$f(g) \cdot f(h) = f(gh) \cdot \sigma(g, h) \text{ [30]}.$$

[30] This is the usual definition contrary to our use of a 2-cocycle above which arises, since we have written maps on the left.

for a unique $\sigma(g,h) \in M$. *The map* $\sigma : G \times G \longrightarrow M$ *is then called a* 2-COCYCLE OF G WITH VALUES IN M.

Using the associativity in G, one obtains – viewing M as G-module – the relation

$$\sigma(g, hk) \cdot \sigma(h,k) = \sigma(gh,k) \cdot {}^{f(k)^{-1}}\sigma(g,h) .$$

If one takes another splitting g , then these splittings differ by a 2-COBOUNDARY OF G WITH VALUES IN M, which is of the form

$$\sigma(g,h) = \mu(gh)^{-1} \cdot {}^{h^{-1}}\mu(g) \cdot \mu(h)$$

for all $g, h \in G$ for SOME FUNCTION $\mu : G \longrightarrow M$.

Note that $g \in G$ acts on $m \in M$ by conjugation with $f(g)$. We have written this action as ${}^{f(g)}m = {}^{g}m$.

Direct computation shows that every 2-coboundary is a 2-cocycle and both sets form abelian groups (this uses heavily that M is abelian).

The SECOND COHOMOLOGY GROUP $H^2_{gr}(G,M)$ of a group G with values in a G-module M is defined to be the group of 2-cocycles modulo the group of 2-coboundaries. This group CLASSIFIES THE GROUP EXTENSIONS in the described way.

There is a natural isomorphism between $H^2(G,M)$ and $H^2_{gr}(G,M)$, which is explicitly described in [ReRo; 79]; it gives a natural construction of how to pass from module extensions

$$0 \longrightarrow M \longrightarrow X \longrightarrow I(G) \longrightarrow 0$$

to group extensions

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 0 .$$

The crucial map here is the set-theoretic map $g \in G \longrightarrow g - 1 \in I(G)$, which is used to form pull-backs, in order to pass from module extensions to group extensions.

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STRUCTURE OF BLOCKS WITH CYCLIC DEFECT GROUPS AND GREEN CORRESPONDENCE

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1. Introduction

This paper is based on a course I gave at the Ovidius university in Constanta on "Structure of blocks with cyclic defect groups and Green correspondence". However, I added many details to the script of the course I have given there. We shall divide the material into three parts.

- In the first part we will present a new way due to Auslander and Kleiner [1] to derive a form of Green correspondence. The classical Green correspondence follows easily from that and this more general point of view might have some impact on other fields of interest. The classical Green correspondence will be used successfully in the following sections.
- In the second part we shall present the classical paper of Green [5]. Here, the Green correspondence in its classical form is used intensively. We just cite from the classical paper of Dade [2] the fundamental properties of blocks with cyclic defect groups. We also give results of Michler [13, 14] on the structure of blocks with cyclic defect groups.
- The third part deals with K. W. Roggenkamp's paper [16] on Green orders in which he firstly defined Green orders, secondly used the results of Green to prove that a block with cyclic defect group is a Green order, and thirdly determines the structure of a Green order in great detail. As far as is known to the author, this is the most far reaching result on the structure of blocks with cyclic defect groups. For the proof we follow [16].

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The reader is assumed to know the basic facts on categories, such as the definition of a category, functors and natural transformations. Recommended references are [15, 12]. Also the basic notions in noetherian ring theory are assumed to be known such as the notion of a radical of a ring and a module, a socle and a top. As a reference we refer to [7]. Furthermore, some basic algebraic number theory is assumed to be known such as the basic definitions of a Dedekind domain and the ramification index for local fields. [6] is recommended as an abundant reference. Besides the theory of Dade, all proofs are included. In this sense the paper is self-contained.

The course I gave in Constanta is contained, for the part dealing with the Green correspondence in the Sections 2.1, 2.2, 2.3, 2.4.1, 2.4.2 and the beginning parts of 2.4.4. For the reader who is merely interested in the classical Green correspondence we suggest studying only Sections 2.1, 2.3 and 2.4.1. Of course, the proof is given in abstract terms, as in [1]. The most interesting part of the abstract Green correspondence is collected in the Sections 2.4.2 and 2.4.3. For the classical theory on blocks with cyclic defect groups, the material presented in Constanta is located in the Sections 3.1, 3.2 and 3.3. Sections 3.4 and 3.5 are devoted to the proof of Green's walk around the Brauer tree, as in [5]. For Roggenkamp's description of blocks with cyclic defect groups the parts which were presented in Constanta are located in Sections 4.1 and 4.2. The rest of the sections contain mainly proofs, which especially in Section 2 are technical, and which are not needed for the understanding of the other parts.

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2. Green correspondence

2.1. **Motivation.** Let G be a finite group and let R be a complete discrete valuation ring of characteristic 0 with residue field k of characteristic p. A large part of modular representation theory of finite groups deals with trying to relate the representation theory of G to the representation theory of the 'p-local structure' of G. The first and perhaps most elementary attempt to do so is the Green correspondence.

By $RF-\overline{mod}^{\circ}$ we denote for any finite group F the *stable module category*. The objects of $RF-\overline{mod}^{\circ}$ are the same as those of $RF-mod^{\circ}$, namely finitely generated R-projective RF-modules, called in the sequel RF-lattices. To define the morphism set we have to put an equivalence relation onto the morphism sets of $RF - \overline{mod}^o$. Let M and N be two RF–lattices. Two morphisms $f, g \in Hom_{RF}(M, N)$ are called equivalent, $f \cong g$, if and only if $f - g$ factors through a projective RF–module. Then,

$$Hom_{RF-\overline{mod}^o}(M,N) =: \overline{Hom}_{RF}(M,N) := Hom_{RF}(M,N)/_{\cong}.$$

Theorem 2.1. *(Green) Let D be a Sylow p–subgroup of G and let $H := N_G(D)$ be the normalizer of D in G. Assume that for all $g \in G \backslash H$ we have $gDg^{-1} \cap D = \{1\}$. Then, induction*

$$ind_H^G := RG \otimes_{RH} - : RH - \overline{mod}^o \longrightarrow RG - \overline{mod}^o$$

and restriction

$$res_H^G : RG - \overline{mod}^o \longrightarrow RH - \overline{mod}^o$$

are mutually inverse equivalences of categories. These equivalences of categories preserve the indecomposability of modules. More precisely, for every indecomposable non projective RG–module M there is an indecomposable non projective RH–module $f(M)$ such that $f(M)|res_H^G(M)$ [1] and $res_H^G(M)/f(M)$ is a projective RH–module. For every indecomposable non projective RH–module N there is an indecomposable non projective RG–module $g(N)$ such that $g(N)|ind_H^G(N)$ and $(ind_H^G N)/g(N)$ is a projective RG–module.

Remark 2.2.
- The situation described by the hypotheses of Theorem 2.1 is commonly known and will be referred to as the TI–situation. Here TI stands for 'trivial intersection'.
- One should note that even the classical Green correspondence is more general than expressed here. The theorem above is just a special case where the correspondence appears in a quite illustrative way.

This theorem is a very special case of a much more general statement which was proven by M. Auslander and M. Kleiner in [1]. They give a categorical and more general approach to the theory of J. A. Green [4] which establishes an equivalence between certain quotient categories of finitely generated RG–modules and finitely generated RH–modules where H is a subgroup of G containing the normalizer in G of a certain p–subgroup D of G. The theorem above is the case for which the categorical equivalence of Green is most easily formulated.

2.2. Adjoint functors.

The method of Auslander and Kleiner intensively uses adjoint functors. We shall give a brief account on this subject.

We assume the reader to be familiar with the notion of a category, functors and of natural transformations between functors. As basic reference one might see [15] or [12].

[1] For any ring R and any two R-modules M and N we say that $M|N$, if there is an R-module K such that $M \oplus K = N$.

Definition 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be two functors. The category of sets is called $\mathcal{E}ns$.

If there is an equivalence

$$\mathcal{B}(F-,-) \simeq \mathcal{A}(-,G-)$$

of bifunctors

$$\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{E}ns$$

then the functor F is said to be *left adjoint* to the functor G and the functor G is said to be *right adjoint* to F. The pair (F,G) is said to be an adjoint pair.

Let (F,G) be an adjoint pair. By the defining relation we get an isomorphism of bifunctors

$$\eta' : \mathcal{B}(F-,F-) \simeq \mathcal{A}(-,GF-)$$

and hence we get a natural transformation

$$\eta : 1_{\mathcal{A}} \longrightarrow GF$$

by just putting $\eta(A) := \eta'(id_{FA})$. The natural transformation η is called the *unit of the adjointness*. Of course, it depends not only on the two functors F and G but also on the choice of the isomorphisms in the defining relation.

We give an example. Let R be a commutative ring and let G be a finite group with subgroup H. We denote by $\iota : RH \longrightarrow RG$ the canonical embedding. Since an RG–module M is an R–module M together with a ring homomorphism $RG \longrightarrow End_R(M)$, one defines the restriction $res^G{}_H(M)$ just as $RH \longrightarrow RG \longrightarrow End_R(M)$. The corresponding mapping is denoted by ι_*. One should observe that this means that the RH–module structure of $res^G_H(M)$ is just M as R–module and H operates on M as a subset of G.

One defines functors

$$ind_H^G := RG \otimes_{RH} - : RH - mod \longrightarrow RG - mod$$

and

$$res_H^G := \iota_* : RG - mod \longrightarrow RH - mod\,.$$

We claim that (ind_H^G, res_H^G) is an adjoint pair. This fact is commonly known as *Frobenius reciprocity*.

We have to give for all RG–modules M and for all RH–modules N natural isomorphisms

$$Hom_{RG}(ind_H^G(N), M) \simeq Hom_{RH}(N, res_H^G(M))\,.$$

We define

$$\begin{array}{ccccc} Hom_{RG}(ind_H^G(N), M) & \simeq & Hom_{RH}(N, res_H^G(M)) & & \\ \phi & \stackrel{\Phi}{\longrightarrow} & (n \longrightarrow \phi(1 \otimes n)) & & \forall n \in N \\ (g \otimes n \longrightarrow g \cdot \psi(n)) & \stackrel{\Psi}{\longleftarrow} & \psi & & \forall n \in N, g \in G \end{array}$$

and observe that the second mapping is well defined since ψ is RH-linear. Now, one immediately checks that $\Phi\Psi(\psi) = \psi$ and $\Psi\Phi(\phi) = \phi$ for all $\phi \in Hom_{RG}(ind_H^G(N), M)$ and $\psi \in Hom_{RH}(N, res_H^G(M))$. For the functoriality of Φ and Ψ we observe that for a homomorphism $\alpha : N \longrightarrow N'$ and for a homomorphism $\beta : M \longrightarrow M'$ we get

$$\begin{aligned} \beta([\Phi(\phi)](\alpha(n))) &= [\beta \circ \phi](1 \otimes \alpha(n)) \\ &= [\Phi(\beta \circ \phi \circ (id_{RG} \otimes \alpha))](n) \end{aligned}$$

and

$$\begin{aligned} \beta[\Psi(\psi)](g \otimes \alpha(n)) &= \beta(g \cdot (\psi \circ \alpha)(n)) \\ &= (g \cdot [\beta \circ \psi \circ \alpha](n)) \\ &= [\Psi(\beta \circ \psi \circ \alpha)](n) . \end{aligned}$$

This proves the functoriality.

2.3. **Some more background from modular representation theory.**

Though we do not need to use modular representation theory to formulate and prove the Green correspondence for adjoint functors we shall give some background to see what the Green correspondence is about and to be able to give examples.

In this subsection we shall use the following notation.

- R is a commutative Noetherian ring.
- G is a finite group.
- For any subgroup S of G we set $mod(G, S) := mod(RG, S)$ the full[2], additive subcategory of $RG - mod$ whose objects are finitely generated RG-modules M for which there is an RS-module L such that M is a direct summand of $RG \otimes_{RS} L$.

With these notations we state the following results which also provide a brief introduction into some of the elementary techniques in modular representation theory.

1. Choosing R a local complete discrete valuation ring of characteristic 0 with $pR \neq R$ or a field of characteristic p, for a prime number p, and choosing D a Sylow p-subgroup of G, we claim that $mod(G, D) = RG - mod$.

   Proof. Given $M \in mod(G, D)$. Then, the mapping

$$\begin{array}{ccc} RG \otimes_{RD} M & \longrightarrow & M \\ g \otimes m & \longrightarrow & gm \end{array}$$

[2] A functor $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is full, if it is surjective on the morphism sets.

is split by[3]

$$\begin{array}{ccc} M & \longrightarrow & RG \otimes_{RD} M \\ m & \longrightarrow & \frac{1}{|G:D|} \sum_{Dh \in D\backslash G} h^{-1} \otimes hm \,. \end{array}$$

The last map is clearly well defined and is a G–linear map since

$$g \cdot \sum_{Dh \in D\backslash G} h^{-1} \otimes hm = \sum_{Dh \in D\backslash G} (hg^{-1})^{-1} \otimes (hg^{-1})gm \,.$$

Additionally, running over a coset $\{h\}$ is the same as to running over the coset $\{hg^{-1}\} = \{h\}g^{-1}$.

Given a finitely generated RG-module M we call a group D which is of minimal order amongst all the subgroups D' with $M \in mod(G, D')$ the *vertex* of M.

2. (D. Higman) Let M be a finitely generated RG–module and let S be a subgroup of G. We claim that M is a direct summand of $ind_S^G(L)$ for some finitely generated RS–module L if and only if M is a direct summand of $ind_S^G res_S^G(M)$.

   Proof. Clearly, if $M | ind_H^G res_S^G(M)$, then there is the RS–module $L = res_H^G M$ such that $M | ind_S^G(L)$.

   Conversely, let L be a finitely generated RS–module such that $M | ind_S^G L$. Then, by Mackey's formula,

$$\begin{aligned} ind_S^G res_S^G M &= ind_S^G res_S^G ind_S^G L \\ &= ind_S^G \Big(\bigoplus_{HgH \in H\backslash G/H} ind_{H \cap {}^gH}^H res_{H \cap {}^gH}^H \, {}^gL \Big) \\ &= ind_H^G L \oplus \text{ others} \\ &= M \oplus \text{ others} \,. \end{aligned}$$

   How unique the vertices are is the subject of the following item.

3. We assume now that R is a complete discrete valuation ring of characteristic 0 with $pR \neq R$ for a prime number p or a field of characteristic p. Given an indecomposable RG–module M we claim that vertices of M are conjugate to each other.

   Proof. $M | ind_D^G V$ and $M | ind_{D'}^G W$ with $V \in Ob(RD - mod)$ and $W \in Ob(RD' - mod)$, D and D' being both vertices of M. But, using Mackey's formula,

$$res_{D'}^G ind_D^G V = \bigoplus_{DgD'} ind_{{}^gD \cap D'}^{D'} res_{{}^gD \cap D'}^{{}^gD} \, {}^gV$$

[3] We denote by $D\backslash G$ the left cosets of G by D on which G acts on the right and by G/D the right cosets of G by D on which G acts on the left.

and

$$res^G_{D'} ind^G_{D'} W = \bigoplus_{D'hD'} ind^{D'}_{{}^hD\cap D'} res^{{}^hD}_{{}^hD'\cap D'} {}^hW = W \oplus \text{ others }.$$

$res^G_{D'} M$ is a direct summand of both modules. Direct summands X in the above equation have vertices smaller than D' or are isomorphic to W. If $X|res^G_{D'} M$ for $X \not\simeq W$ then $M|ind^G_{D''}$ for some smaller subgroup D'' of D' and we reach a contradiction. Hence there must be some g_0 with

$$W|ind^{D'}_{{}^{g_0}D\cap D'} res^{{}^{g_0}D}_{{}^{g_0}D\cap D'} {}^{g_0}V .$$

Hence, $M|ind^G_{{}^{g_0}D\cap D'} res^{{}^{g_0}D}_{{}^{g_0}D\cap D'} {}^{g_0}V$ and by the minimality of D, we get ${}^{g_0}D \cap D' = D'$.

4. An indecomposable ring direct factor B of RG is called a *block* of RG. Of course, then B is an $R(G\times G)$-module by putting $(g,h)\cdot m = gmh^{-1}$ where $(g,h) \in G\times G$ and $m \in B$.

   We claim that there is always a vertex of B in $\{(g,g) \in G\times G | g \in G\} =: \Delta(G)$.

   Proof. We view $R(G\times G)$ as RG-right-module by letting G act as $\Delta(G)$.

$$\begin{array}{rcl} R(G\times G)\otimes_{R\Delta(G)} R & \longrightarrow & RG \\ ((g,h)\otimes r) & \longrightarrow & grh^{-1} = rgh^{-1} \end{array}$$

for $r \in R$, $(g,h) \in G\times G$, is split by

$$\begin{array}{rcl} R(G\times G)\otimes_{R\Delta(G)} R & \longleftarrow & RG \\ (\sum_{g\in G} r_g(g,1))\otimes 1 & \longleftarrow & \sum_{g\in G} r_g g . \end{array}$$

The splitting is a module homomorphism as one immediately verifies using that we tensor over $R\Delta(G)$.

   The vertex of a block as $R(G\times G)$-module is called a *defect group* of the block.

   We assume now that R is a complete discrete valuation ring of characteristic 0 with $pR \neq R$ for a prime number p or a field of characteristic p. Then the defect groups D are p-groups and the integer $log_p(|D|)$ is called the *defect of the block.*

5. Now we assume again that R is a complete discrete valuation ring of characteristic 0 with $pR \neq R$ for a prime number p or a field of characteristic p. If B is a block of G with defect group D and if M is an indecomposable B-module, then there is a vertex of M contained in D.

Proof[4]. Since B has defect group D, we see by 3. that

$$B|[R(G \times G) \otimes_{R\Delta(D)} B].$$

Hence,

$$\begin{aligned} M &= B \otimes_{RG} M | R(G \times G) \otimes_{R\Delta(D)} B \otimes_{RG} M \\ &= RG \otimes_{RD} M . \end{aligned}$$

With these preparations we shall illustrate the Green correspondence in the situation of Theorem 2.1.

Example. We fix a prime number p and set $G := SL_2(p)$ the group of 2 by 2 matrices over the prime field of characteristic p with determinant 1.

We look at the modular representations of G over k being the prime field of characteristic p.[5]

We set $GL_2(p)$ the group of invertible 2 by 2 matrices over the prime field of characteristic p. Then we get an exact sequence

$$1 \longrightarrow SL_2(p) \longrightarrow GL_2(p) \stackrel{det}{\longrightarrow} k^* \longrightarrow 1 .$$

Now, k^* has order $p-1$ and $GL_2(p)$ has order $(p^2-1)\cdot(p^2-p)$ since an invertible matrix is determined by its action on the 2-dimensional natural module and the first basis vector can be mapped to all of k^2 besides the zero element, the second basis vector can be mapped to k^2 besides the one dimensional space which is already spanned by the image of the first basis vector.

Hence, $|SL_2(p)| = (p-1)\cdot p \cdot (p+1)$.

The Sylow p-subgroup of G is hence cyclic of order p. In fact, it is easy to find one explicitly:

$$D := \{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} | y \in k \} .$$

The normalizer H of D in G is

$$H := \{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} | x \in k^* ; y \in k \} .$$

We shall illustrate the Green correspondence on the (natural) module

$$M = \begin{pmatrix} k \\ k \end{pmatrix}$$

on which G acts by matrix multiplication.

Clearly, M is indecomposable. By Theorem 2.1 we know that $res^G_H(M) \simeq f(M) \oplus P_H$ where P_H is a projective RH-module.

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[4] This proof was pointed out to me by M. Linckelmann.

[5] This characteristic is commonly called the describing characteristic and in the theory of algebraic groups the describing characteristic always provides a huge framework of techniques coming from algebraic geometry.

Claim 2.4. *Let P be a p-group and let R be a complete discrete valuation ring of characteristic 0 with residue field of characteristic p. Then, RP is a local ring. Every projective RP-module is free.*

Proof. We have to show that $R/radR$ is the only simple RP-module. We proceed by induction on $|P|$.

The statement is true for the group with 1 element.

Let P be arbitrary. Let $1 \neq c$ be a central element of P of order p, which exists by the conjugacy class number formula, just counting the size of the conjugacy classes and observing that their order equals the index of the stabilizer of an element which is a subgroup, and let V be a simple RP-module. Then, since c is central, $V_1 := (c-1) \cdot V$ is also an RP-module. If $V_1 = 0$, then V is an indecomposable $R(P/<c>)$ module and V is isomorphic to $R/radR$ by the induction hypothesis. Else, $V_1 = V$ by the simplicity of V. Hence,

$$V = (c-1) \cdot V = (c-1)^2 \cdot V = \cdots = (c-1)^p \cdot V = (c^p - 1) \cdot V = 0\,,$$

which is a contradiction.

The above claim shows that a projective kG-module has as k-rank at least the order of a Sylow p-subgroup. In fact, the restriction of a projective RG-module to a Sylow p-subgroup is again projective, hence free.

This argument (or elementary computations) shows, that the 2-dimensional kH-module $res_H^G(M)$ is indecomposable.

Conversely, let N be the natural two dimensional kH-module. Then, $N = res_H^G(M)$. We look for its Green correspondent in kG. As above,

$$\begin{array}{rcl} ind_H^G N = ind_H^G res_H^G M & \longrightarrow & M \\ g \otimes m & \longrightarrow & gm \end{array}$$

is split and $M | ind_H^G N$. Hence, the Green correspondent of N in kG is $M = g(N)$.

But, since the index of H in G is $p+1$, $dim_k(ind_H^G N) = 2 \cdot (p+1)$ and $dim_k(ind_H^G(N)/M) = 2p$ and $ind_H^G(N)/M$ is a projective module of dimension $2p$. Observe that this matches our observation in Claim 2.4.

2.4. **The Green correspondence for adjoint pairs of functors.** We shall follow the paper [1].

All categories we deal with are assumed to be additive[6]. We start with three (additive) categories $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{G}$ and functors

$$\mathcal{D} \xrightarrow{S'} \mathcal{H} \xrightarrow{S} \mathcal{G}$$

as well as

$$\mathcal{D} \xleftarrow{T'} \mathcal{H} \xleftarrow{T} \mathcal{G}$$

where (S, T) and (S', T') will form adjoint pairs.

In our later application to group theory these categories and functors will be specialized as follows.

Let k be a field of characteristic $p \geq 0$, let G be a finite group, and assume for simplicity that kG is indecomposable as ring, let D be the Sylow p–subgroup of G, let H be a group with $D \leq H \leq G$. The results become non trivial only if we assume that $H \geq N_G(D) := \{g \in G | gD = Dg\}$. One should think of

$$\mathcal{D} = kD - mod, \mathcal{H} = kH - mod, \mathcal{G} = kG - mod$$

and

$$S' = kH \otimes_{kD} -; \; S = kG \otimes_{kH} -;$$

$$T = res^G_H(-); \; T' := res^H_D(-),$$

where res^G_H and res^H_D are the restriction functors of kG–modules to kH–modules or of kH–modules to kD–modules respectively, and the adjointness is just Frobenius reciprocity as explained earlier in Subsection 2.2.

For technical reasons in later applications we fix isomorphisms, natural in both variables,

$$\alpha(N, M) : \mathcal{G}(SN, M) \longrightarrow \mathcal{H}(N, TM); \;\; \forall N \in Ob(\mathcal{H}), M \in Ob(\mathcal{G})$$

and

$$\gamma(L, N) : \mathcal{H}(S'L, N) \longrightarrow \mathcal{D}(L, T'N); \;\; \forall N \in Ob(\mathcal{H}), L \in Ob(\mathcal{D}) .$$

[6] A category is called additive if it has a zero object, there are finite products and coproducts, finite products over a set of objects and finite coproducts over this set are isomorphic by the natural map, and for every object A there is an endomorphism s_A of A such that, denoting by Δ_A the diagonal mapping and by ∇_A the codiagonal mapping, $\Delta_A(1_A \oplus s_A)\nabla_A = 0$. In additive categories the set of morphisms carries a structure of an abelian group by setting $f + g = \Delta_A(f \oplus g)\nabla_B$ for $f, g \in Mor(A, B)$

Throughout Section 2.4 we assume that

$$TS = 1_{\mathcal{H}} \oplus U$$

for an endofunctor U of $\mathcal{H}$ and that the induced natural transformation

$$\eta_I : 1_{\mathcal{H}} \xrightarrow{\eta} TS = 1_{\mathcal{H}} \oplus U \xrightarrow{proj} 1_{\mathcal{H}}$$

is an isomorphism.

Notation 2.5.
- All subcategories in Section 2.4 are meant to be full and additive. If $\mathcal{A}$ and $\mathcal{B}$ are full subcategories of the category $\mathcal{C}$, then we say that $\mathcal{A}$ *divides* $\mathcal{B}$ if for all $M \in Ob(\mathcal{A})$ there is a $X \in Ob(\mathcal{B})$ such that $M|X$, i.e. M is a direct summand of X. If $Ob(\mathcal{A})$ has only one element M, then we also say that M divides $\mathcal{B}$. We use the notation $\mathcal{A}|\mathcal{B}$.
- Let $\mathcal{C}'$ be a subcategory of the category $\mathcal{C}$. We denote by $\mathcal{C}/\mathcal{C}'$ the category whose objects are the same as those of $\mathcal{C}$ and the morphisms are equivalence classes of morphisms of $\mathcal{C}$. Two morphisms are said to be equivalent if their difference factors through an object of $\mathcal{C}'$.
- Let $\mathcal{E}$ and $\mathcal{F}$ be categories and let $U : \mathcal{E} \longrightarrow \mathcal{F}$ be a functor. For any subcategory $\mathcal{Y}$ of $\mathcal{F}$ let $U^{-1}(\mathcal{Y})$ be the full additive subcategory of $\mathcal{E}$ generated by objects $M \in Ob(\mathcal{E})$ with $U(M)|\mathcal{Y}$.

2.4.1. *The theorem in group theoretical terms.* We shall give the Green correspondence in the classical situation, before we turn to the more abstract setting.

Theorem 2.6. *(Green) Let G be a finite group and let R be a complete discrete valuation ring of characteristic 0 with residue field of characteristic $p > 0$ or let R be a field of characteristic p. Let D be a p-subgroup of G and let $H \geq N_G(D)$. Set*

$$\mathcal{X} := \{X \leq D \cap gDg^{-1} | g \in G \setminus H\}$$
$$\mathcal{Y} := \{Y \leq H \cap gDg^{-1} | g \in G \setminus H\}$$
$$\mathcal{Z} := \{Z \leq D\}\,.$$

Set $mod(G,\mathcal{F})$ the category of finitely generated RG−modules with vertex in $\mathcal{F}$ for $\mathcal{F} \in \{\mathcal{X},\mathcal{Y},\mathcal{Z}\}$.

Then,

$$ind_H^G : mod(H,\mathcal{Z})/mod(H,\mathcal{X}) \longrightarrow mod(G,\mathcal{Z})/mod(G,\mathcal{X})$$

is an equivalence of categories and

$$res_H^G : mod(G,\mathcal{Z})/mod(G,\mathcal{X}) \longrightarrow mod(H,\mathcal{Z})/mod(H,\mathcal{Y})$$

is an equivalence of categories.

For every indecomposable object M in $mod(H,\mathcal{Z})\setminus mod(H,\mathcal{X})$ there is an indecomposable object $g(M)$ in $mod(G,\mathcal{Z})\setminus mod(G,\mathcal{X})$ which is a direct summand of $ind_H^G(M)$.

For every indecomposable object N in $mod(G,\mathcal{Z})\setminus mod(G,\mathcal{X})$ there is an indecomposable object $f(N)$ in $mod(H,\mathcal{Z})\setminus mod(H,\mathcal{Y})$ which is a direct summand of $res_H^G(N)$.

We shall prove the theorem in the sequel.

2.4.2. *The general situation.* We can now state the most important theorem of this subsection. The remaining part deals with the particular situation of Krull–Schmidt categories. But even without this assumption we are able to prove an equivalence of certain quotient categories. In the next subsection we shall explain how one can derive the usual Green correspondence from this rather abstract setting.

Theorem 2.7. [1]*(Green correspondence for adjoint functors) Let there be three additive categories $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{G}$ and functors*

$$\mathcal{D} \xrightarrow{S'} \mathcal{H} \xrightarrow{S} \mathcal{G}$$

as well as

$$\mathcal{D} \xleftarrow{T'} \mathcal{H} \xleftarrow{T} \mathcal{G}$$

where (S,T) and (S',T') will form adjoint pairs. Assume that $TS = 1_{\mathcal{H}} \oplus U$ and that for the unit $\eta : 1_{\mathcal{H}} \longrightarrow TS$ we get that $\eta_I := \eta \cdot proj_1$ is an isomorphism. Let $\mathcal{Y}$ be a full, additive subcategory of $\mathcal{H}$ such that

$$S'T'\mathcal{Y}|\mathcal{Y} \text{ and } S'T'\mathcal{Y}|U^{-1}\mathcal{Y}.$$

Then,

1. *S, T induce functors*

$$\begin{aligned} S : \mathcal{H}/S'T'\mathcal{Y} &\longrightarrow \mathcal{G}/SS'T'\mathcal{Y} \\ T : \mathcal{G}/SS'T'\mathcal{Y} &\longrightarrow \mathcal{H}/\mathcal{Y}. \end{aligned}$$

2. *For $\mathcal{Z} := (US')^{-1}\mathcal{Y}$, the restrictions of the functors S and T*

$$S : (addS'\mathcal{Z})/S'T'\mathcal{Y} \longrightarrow (addSS'\mathcal{Z})/SS'T'\mathcal{Y}$$

$$T : (addSS'\mathcal{Z})/SS'T'\mathcal{Y} \longrightarrow (addS'\mathcal{Z})/\mathcal{Y}$$

   are equivalences of categories and

$$TS : (addS'\mathcal{Z})/S'T'\mathcal{Y} \longrightarrow (addS'\mathcal{Z})/\mathcal{Y}$$

   is isomorphic to the functor induced by the identity functor.

Before we prove Theorem 2.7 we shall see what this means for the group theoretical situation.

The question if the assumptions of the theorem are satisfied in the group theoretical situation will cover subsection 2.4.5 and we postpone this question until then.

Again let R be a complete discrete valuation ring of characteristic 0 with residue field k of characteristic $p > 0$ or let R be a field of characteristic p. Let G be a finite group and let D be a $p-$subgroup of G. Let H be a subgroup of G with $D \leq N_G(D) \leq H \leq G$. We set

$$\mathcal{G} := RG - mod^0, \mathcal{H} := RH - mod^0, \mathcal{D} := RD - mod^0$$

and

$$S = ind_H^G, S' = ind_D^H, T = res_H^G, T' = res_D^H .$$

Furthermore, we set

$$\mathcal{S} := \{V \leq G | \exists g \in G \setminus H : V \leq g \cdot D \cdot g^{-1} \cap H\} .$$

Let $\mathcal{Y}$ be the full additive subcategory of $RH - mod^0$ whose objects are finite direct sums of indecomposable finitely generated RH-lattices which have vertex in $\mathcal{S}$.

We compute $S'T'\mathcal{Y}$. Let $V \in \mathcal{S}$ and $L \in Ob(RV - mod^0)$. A generating object of $S'T'\mathcal{Y}$ is of the form

$$\begin{aligned} ind_D^H res_D^H ind_V^H L &= ind_D^H (\bigoplus_{VhD \in V \backslash H/D} ind_{^hV \cap D}^D res_{^hV \cap D}^{^hV}\ {}^hL) \\ &= \bigoplus_{VhD \in V \backslash H/D} ind_{^hV \cap D}^H res_{^hV \cap D}^{^hV}\ {}^hL . \end{aligned}$$

If we now set

$$\mathcal{X} := \{V \leq D | \exists g \in G \setminus H : V \leq g \cdot D \cdot g^{-1} \cap D\}$$

we observe that the above generating modules are direct sums of modules which have vertices in $\mathcal{X}$. We set $\mathcal{U}$ the full additive subcategory of $RH - mod^0$ whose objects are finite direct sums of indecomposable finitely generated RH-lattices which have vertex in $\mathcal{X}$. $\mathcal{H}/(S'T'\mathcal{Y}) = \mathcal{H}/\mathcal{U}$ since if a morphism factors through a direct summand it also factors through the whole direct sum.

In a similar way, with somewhat more effort but still simply using Mackey's formula one proves that $\mathcal{Z}$ is the full additive subcategory of $RH - mod^0$ whose indecomposable objects are modules with vertex being a subgroup of D.

2.4.3. *The proof.* The proof of Theorem 2.7 will proceed in several steps. This will cover this subsubsection. The next subsubsection will deal with the special situation when we are given a Krull-Schmidt category.

The proof of Theorem 2.7 relies mainly on the following observation.

Proposition 2.8. *Let $\mathcal{Y}'$ be a subcategory of $\mathcal{H}$ and let $\mathcal{X}'$ be a subcategory of $\mathcal{G}$ such that $S\mathcal{Y}'|\mathcal{X}'$ and $T\mathcal{X}'|\mathcal{Y}'$. Then, S, T extend naturally to functors between $\mathcal{H}/\mathcal{Y}'$ and $\mathcal{G}/\mathcal{X}'$ and (S, T) form again an adjoint pair as functors between these quotient categories. The adjointness homomorphism α for the adjoint* *pair (S,T) of functors between $\mathcal{H}$ and $\mathcal{G}$ induces an adjointness homomorphism for the adjoint pair (S,T) of functors between $\mathcal{H}/\mathcal{Y}'$ and $\mathcal{G}/\mathcal{X}'$.*

Proof. First we prove that S extends to the quotient categories. Let there be given two objects M and N in $\mathcal{H}$ and a morphism $f \in \mathcal{H}(M,N)$ which factors through an object Y of $\mathcal{Y}'$. Then, there are $f_1 \in \mathcal{H}(M,Y)$ and $f_2 \in \mathcal{H}(Y,N)$ such that $f = f_1 f_2$. Therefore, $Sf = (Sf_1)(Sf_2)$ and Sf factors through SY. But, $S\mathcal{Y}'|\mathcal{X}'$ and therefore, there is an $X \in Ob(\mathcal{X}')$ such that $SY|X$. Hence, Sf factors through an object in $\mathcal{X}'$.

The argument that T extends to the quotient categories is absolutely analogous.

We show that for any $N \in Ob(\mathcal{H}), M \in Ob(\mathcal{G})$ the mapping $\alpha(N,M)$: $\mathcal{G}/\mathcal{X}'(SN,M) \longrightarrow \mathcal{H}/\mathcal{Y}'(N,TM)$ is an isomorphism. Let $f : N \longrightarrow TM$ be a morphism factoring through $Y \in \mathcal{Y}$. Then, there are $f_1 \in \mathcal{H}(M,Y)$ and $f_2 \in \mathcal{H}(Y,N)$ such that $f = f_1 f_2$. Now, $\alpha^{-1}(f) = \alpha^{-1}(f_1)\alpha^{-1}(f_2)$ with $Sf_1 = \alpha^{-1}(f_1) \in \mathcal{G}(SN,SY)$ and $\alpha^{-1}(f_2) \in \mathcal{G}(SY,M)$. However, $SY|X$ for an object $X \in \mathcal{X}'$. Hence, $\alpha^{-1}(f)$ factors through an object of $\mathcal{X}'$. Therefore, α^{-1} is defined over the quotient categories. Analogously, α is defined over the quotient categories. It is clear that then $\alpha(N,M)$ is a natural isomorphism.

This proves the proposition.

Corollary 2.9. *Let $\mathcal{Y}$ be a subcategory of $\mathcal{H}$.*
If $S'T'\mathcal{Y}|\mathcal{Y}$, then

1. *(S',T') is an adjoint pair as functors between $\mathcal{H}/\mathcal{Y}$ and $\mathcal{D}/T'\mathcal{Y}$. The isomorphisms γ induce adjunctions also in the quotient categories.*
2. *(S',T') is an adjoint pair as functors between $\mathcal{H}/S'T'\mathcal{Y}$ and $\mathcal{D}/T'\mathcal{Y}$. The isomorphisms γ induce adjunctions also in the quotient categories.*
3. *The functor $1_\mathcal{H}$ induces a functor $1_\mathcal{H} : \mathcal{H}/S'T'\mathcal{Y} \longrightarrow \mathcal{H}/\mathcal{Y}$ and gives rise to an isomorphism of bifunctors $(\mathcal{H}/S'T'\mathcal{Y})(S'-,-) \simeq (\mathcal{H}/\mathcal{Y})(S'-,-)$.*

If $T'TSS'T'\mathcal{Y}|T'\mathcal{Y}$, then

4. *$(SS',T'T)$ is an adjoint pair between the categories $\mathcal{G}/SS'T'\mathcal{Y}$ and $\mathcal{D}/T'\mathcal{Y}$ with adjunction induced by $\gamma\alpha$.*
5. *If moreover $S'T'\mathcal{Y}|\mathcal{Y}$, then the inverse of α induces an isomorphism functorial in both variables $L \in Ob(\mathcal{D})$, $M \in Ob(\mathcal{G})$,*

$$\mathcal{H}/S'T'\mathcal{Y}(S'L,TM) \longrightarrow \mathcal{G}/SS'T'\mathcal{Y}(SS'L,M)\,.$$

Proof:

Part 1. follows from Proposition 2.8 by just setting $\mathcal{X}' := T'\mathcal{Y}$ and $\mathcal{Y}' := \mathcal{Y}$.

Part 2. Set $\mathcal{X}' := S'T'\mathcal{Y}$ and $\mathcal{Y}' := T'\mathcal{Y}$. Then, $S'T'\mathcal{Y}|\mathcal{Y} \Longrightarrow T'S'(T'\mathcal{Y})|(T'\mathcal{Y})$ and Proposition 2.8 applies.

Part 3. We apply first 2. and then 1. to get the isomorphisms

$$\mathcal{H}/S'T'\mathcal{Y}(S'-,-) \overset{\gamma(-,-)}{\overset{\simeq}{\longrightarrow}} \mathcal{D}/T'\mathcal{Y}(-,T'-) \overset{\gamma^{-1}(-,-)}{\overset{\simeq}{\longrightarrow}} \mathcal{H}/\mathcal{Y}(S'-,-).$$

Part 4. This is an application of Proposition 2.8 with $\mathcal{X}' := SS'T'\mathcal{Y}$ and $\mathcal{Y}' := T'\mathcal{Y}$ and as functors one takes just $T'T$.

Part 5. We have

$$\mathcal{G}/SS'T'\mathcal{Y}(SS'-,-) \overset{\gamma\alpha(-,-)}{\overset{\simeq}{\longrightarrow}} \mathcal{D}/T'\mathcal{Y}(-,T'T-) \overset{\gamma^{-1}(-,-)}{\overset{\simeq}{\longrightarrow}} \mathcal{H}/S'T'\mathcal{Y}(S'-,T-)$$

where the last part is due to 2. and the first is due to 4.

We come to the actual proof of Theorem 2.7. We need a lemma.

Lemma 2.10. *Under the assumptions of Theorem 2.7 we get the following.*

$$(S'T'\mathcal{Y}|\mathcal{Y} \text{ and } S'T'\mathcal{Y}|U^{-1}(\mathcal{Y})) \Longleftrightarrow TSS'T'\mathcal{Y}|\mathcal{Y}.$$

Proof. $TS = 1_{\mathcal{H}} \oplus U \Longrightarrow TSS'T' = S'T' \oplus US'T'$ and inserting $Y \in \mathcal{Y}$ gives the result.

We can now prove Part 1 of Theorem 2.7. In fact, for S the statement is clear and for T it follows from Lemma 2.10.

Lemma 2.11. *Under the assumptions of Theorem 2.7 we get the following.*

1. *For all $L \in Ob(\mathcal{D}), B \in Ob(U^{-1}\mathcal{Y})$,*
$$S : (\mathcal{H}/S'T'\mathcal{Y})(S'L, B) \overset{\simeq}{\longrightarrow} (\mathcal{G}/SS'T'\mathcal{Y})(SS'L, SB)$$
gives an isomorphism.
2. *For all $L \in Ob((US')^{-1}(\mathcal{D})), A \in \mathcal{G}$,*
$$T : (\mathcal{G}/SS'T'\mathcal{Y})(SS'L, A) \overset{\simeq}{\longrightarrow} (\mathcal{H}/\mathcal{Y})(TSS'L, TA)$$
gives an isomorphism.

Remark 2.12. We notice that if $S'T'(US'\mathcal{D})|(US'\mathcal{D})$, then $\mathcal{Y} := (US'\mathcal{D})$ satisfies each of the equivalent conditions in part 1.

Proof of Lemma 2.11.

The conditions to apply Corollary 2.9 1.–5. are satisfied. We have $\eta : 1_{\mathcal{H}} \longrightarrow TS$ and $\eta_B \in \mathcal{H}(B, TSB)$. The following diagram is commutative.

$$\begin{array}{ccccc} \frac{\mathcal{H}}{S'T'\mathcal{Y}}(S'L,B) & \overset{\eta_B}{\longrightarrow} & \frac{\mathcal{H}}{S'T'\mathcal{Y}}(S'L,TSB) & \overset{\alpha^{-1}}{\longrightarrow} & \frac{\mathcal{G}}{SS'T'\mathcal{Y}}(SS'L,SB) \\ \downarrow 1_{\mathcal{H}} & & \simeq\downarrow 1_{\mathcal{H}} & & \\ \frac{\mathcal{H}}{\mathcal{Y}}(S'L,B) & \overset{\eta_B}{\longrightarrow} & \frac{\mathcal{H}}{\mathcal{Y}}(S'L,TSB) = \frac{\mathcal{H}}{\mathcal{Y}}(S'L,B), & & \end{array}$$

where the very left hand side vertical $1_{\mathcal{H}}$ is an isomorphism by Corollary 2.9. 3. α^{-1} is an isomorphism by Corollary 2.9. 5. The equality in the lower right corner follows from the fact that $TS = 1_{\mathcal{H}} \oplus U$ and $UB|\mathcal{Y}$. But, $\eta_{B_*} \cdot proj_1 = \eta_I$ is an isomorphism. Therefore, going down, right, up we conclude that the upper η_{B_*} is an isomorphism.

Furthermore, $\forall\, h \in \mathcal{H}(S'L, B)$, we get

$$\alpha^{-1}(\eta_B \circ h) = \alpha^{-1}(\eta_B) \circ S(h) = 1_{SB} \circ S(h) = S(h)\,,$$

where the first equation is just the functoriality, the second is the definition of η by $\eta_N = \alpha(N, TSN)^{-1}(1_{SN})$. This shows the statement for S.

The statement for the functor T is shown analogously.

We can now also prove Part 2 of Theorem 2.7. By Lemma 2.11 we see that the restriction of S to *add* $S'\mathcal{Z}/S'T'\mathcal{Y}$ and of T to *add* $SS'\mathcal{Z}/SS'T'\mathcal{Y}$ is full and faithful.

These restrictions of S and T are dense on their images.

Since $TSS'\mathcal{Z} = S'\mathcal{Z} \oplus US'\mathcal{Z}$ and since by the definition of $\mathcal{Z}$ for all $Z \in Ob(\mathcal{Z})$ we get $US'(Z)|\mathcal{Y}$, we see that *add* $TSS'\mathcal{Z}/\mathcal{Y} =$ *add* $S'\mathcal{Z}/\mathcal{Y}$.

Moreover, U causes all the occuring terms to vanish and therefore by our assumption that η_I is an isomorphism, TS is just the natural projection.

This finishes the proof of the theorem.

2.4.4. *The Krull-Schmidt situation.*

Notation 2.13. If $\mathcal{E}$ and $\mathcal{F}$ are subcategories of a common Krull–Schmidt category[7] $\mathcal{G}$, then $\mathcal{F}_{\mathcal{E}}$ denotes the full additive subcategory of $\mathcal{F}$ generated by objects $M \in Ob(\mathcal{F})$ such that no non zero direct summand of M divides $\mathcal{E}$. One should think of $\mathcal{F}_{\mathcal{E}}$ as the part of $\mathcal{F}$ which has nothing to do with $\mathcal{E}$.

Lemma 2.14. *Let $\mathcal{E}$ be a subcategory of a Krull-Schmidt category $\mathcal{F}$. Then, the identity functor induces a functor $\mathcal{F}_{\mathcal{E}} \longrightarrow \mathcal{F}_{\mathcal{E}}/\mathcal{E}$ which is full*[8]*, dense*[9] *and reflects isomorphisms*[10] *.*

Proof. $1_{\mathcal{F}}$ is clearly full and dense. Take an isomorphism $X \xrightarrow{f} Y$ in $\mathcal{F}_{\mathcal{E}}/\mathcal{E}$. Then, there is a $Y \xrightarrow{g} X$ with $gf = 1_X$ and $fg = 1_Y$ in $\mathcal{F}_{\mathcal{E}}/\mathcal{E}$. Take preimages f_0 and g_0 of f and g in $\mathcal{F}$. Then, $f_0 g_0 = 1_X + k_X$ where k_X is an endomorphism of X which factors through an object of $\mathcal{E}$. No summand of X divides $\mathcal{E}$ and therefore, $k_X \in radEnd(X)$. But, the Jacobson radical $radEnd(X)$ has the

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[7] A Krull–Schmidt category is an additive category such that every object is a finite direct sum of indecomposable objects and endomorphism rings of indecomposable objects are local. It follows then that the decomposition into direct summands is unique up to isomorphisms.

[8] $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is full, if it is surjective on the morphism sets.

[9] $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is dense, if every object in $\mathcal{C}$ is of the form FC' for a $C \in Ob(\mathcal{C}')$.

[10] $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is reflects isomorphisms, if

$F(f) \in Mor_{\mathcal{C}}(FC_1, FC_2)$ is an isomorphism iff $f \in Mor(C_1, C_2)$ is an isomorphism.

property that $1 + radEnd(X)$ is a subgroup of the unit group. Similarly, $g_0 f_0$ is invertible. Hence, f_0 is an isomorphism.

Proposition 2.15. *Let $\mathcal{Y}$ be a subcategory of $\mathcal{H}$. Then $TS : \mathcal{H} \longrightarrow \mathcal{H}/\mathcal{Y}$ satisfies*

$$TS(U^{-1}\mathcal{Y}) \leq (U^{-1}\mathcal{Y})/\mathcal{Y}$$

and

$$TS : U^{-1}\mathcal{Y} \longrightarrow (U^{-1}\mathcal{Y})/\mathcal{Y} \text{ is isomorphic to } 1_{\mathcal{H}}.$$

If $\mathcal{H}$ is a Krull-Schmidt category, then $TS : (U^{-1}\mathcal{Y})_{\mathcal{Y}} \longrightarrow (U^{-1}\mathcal{Y})_{\mathcal{Y}}/\mathcal{Y}$ is full, dense and reflects isomorphisms.

Proof. We know that η_I is an isomorphism.

$$\begin{array}{lcl} B \in U^{-1}\mathcal{Y} & \Leftrightarrow & U(B)|\mathcal{Y} \\ & \Leftrightarrow & U(B) \simeq 0 \in Ob(\mathcal{H}/\mathcal{Y}) \\ & \Rightarrow & \eta : 1_{\mathcal{H}}|_{U^{-1}\mathcal{Y}} \to TS|_{U^{-1}\mathcal{Y}} \text{ is } \eta_I \text{ which is an isomorphism.} \end{array}$$

The second statement follows immediately from Lemma 2.14.

Proposition 2.16. *Let $\mathcal{Y}$ be a full additive subcategory of the Krull-Schmidt category $\mathcal{H}$.*

1. $$S : U^{-1}(\mathcal{Y})_{\mathcal{Y}} \longrightarrow S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$$ *is dense, reflects isomorphisms, and N is indecomposable in $U^{-1}(\mathcal{Y})_{\mathcal{Y}}$ if and only if SN is indecomposable in $\mathcal{G}/T^{-1}\mathcal{Y}$.*
2. $$T : S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y} \longrightarrow U^{-1}(\mathcal{Y})_{\mathcal{Y}}/\mathcal{Y}$$ *is full, dense, reflects isomorphisms and M is indecomposable in $S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$ if and only if TM is indecomposable in $\mathcal{H}/\mathcal{Y}$.*

Proof.
S is dense by definition.
$SN_1 \simeq SN_2 \Rightarrow TSN_1 \simeq TSN_2 \Rightarrow N_1 \simeq N_2$ since TS reflects isomorphisms by Proposition 2.16.
T is dense since TS is dense.
T is full since given $TX \xrightarrow{f} TY$, then there exist X', Y' mapped to X, Y by S such that $TSX' \xrightarrow{f} TSY'$. But, TS is full, again using Proposition 2.16, hence, $f = TSf'$ for $X' \xrightarrow{f'} Y'$ and $T(Sf') = f$ and Sf is a preimage.
$TM_1 \simeq TM_2 \Rightarrow \exists_{N_1,N_2} M_i = SN_i \Rightarrow TSN_1 \simeq TSN_2 \Rightarrow N_1 \simeq N_2 \Rightarrow M_1 \simeq SN_1 \simeq SN_2 \simeq M_2$.

N is decomposable in $U^{-1}(\mathcal{Y})_{\mathcal{Y}}$ $\Rightarrow$ take $N_1|N \Rightarrow SN_1|SN$. But, $SN_1 = 0 \Rightarrow N_1 = 0$. Therefore, SN is decomposable.
Let M be decomposable in $S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$. Take $0 \neq M_1|M \Rightarrow TM_1|TM$. But, $TM_1 = 0 \in U^{-1}(\mathcal{Y})_{\mathcal{Y}}/\mathcal{Y} \Rightarrow TM_1|\mathcal{Y} \Rightarrow M_1 \in T^{-1}\mathcal{Y} \Rightarrow M_1 = 0 \in S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$. Hence, TM is decomposable.
SN decomposable $\Rightarrow TSN$ decomposable in $U^{-1}(\mathcal{Y})_{\mathcal{Y}}/T^{-1}\mathcal{Y} \Rightarrow N$ is decomposable since TS is full, dense and reflects isomorphisms.
TM is decomposable with $M \in S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y} \Rightarrow \exists_N M \simeq SN \Rightarrow TSN \simeq TM$ is decomposable $\Rightarrow N$ is decomposable since TS is full, dense, and reflects isomorphisms $\Rightarrow M = SN$ is decomposable.

All details considered, we have proved the proposition.

Corollary 2.17. *Let $\mathcal{H}$ and $\mathcal{G}$ be Krull-Schmidt categories and let $\mathcal{Y}$ be a full additive subcategory of $\mathcal{H}$.*

1. *For all indecomposable objects $N \in U^{-1}(\mathcal{Y})_{\mathcal{Y}}$ the object SN has exactly one indecomposable summand $g(N)$ which is not contained in $T^{-1}\mathcal{Y}$.*
2. *For all indecomposable objects $M \in (add\ S(U^{-1}(\mathcal{Y})_{\mathcal{Y}}))_{T^{-1}\mathcal{Y}}$ the object TM has exactly one indecomposable summand $f(M)$ that does not divide $\mathcal{Y}$.*
3. *$f(g(N)) \simeq N$.*
4. *$g(f(M)) \simeq M$.*

Remark 2.18. One should note that this establishes one part of the Green correspondence, namely, the bijective correspondence between parts of the two module categories of the group rings.

However, one should be careful with this statement. If we wanted to apply this to our group theoretical situation, we would not need that $H \geq N_G(D)$. It will become clear that if this is not the case, then *there is no indecomposable object as required.*

Proof of Corollary 2.17.

1. $SN = M_1 \oplus \cdots \oplus M_s$ for indecomposable objects M_i in $\mathcal{G}$ with $i = 1, ..., s$ and $s \in \mathbb{N}$.
   By Lemma 2.14, M_j is indecomposable or zero in $\mathcal{G}/T^{-1}\mathcal{Y}$.
   By Proposition 2.16 SN is indecomposable in $\mathcal{G}/T^{-1}\mathcal{Y}$.
   Hence, there is exactly one j_0 with M_{j_0} not being contained in $T^{-1}\mathcal{Y}$.
2. There is an indecomposable $N \in U^{-1}(\mathcal{Y})_{\mathcal{Y}}$ and some $M \in \mathcal{G}$ with $SN = M \oplus M'$. Since M is not contained in $T^{-1}\mathcal{Y}$ we get $SN \simeq M$ in $\mathcal{G}/T^{-1}\mathcal{Y}$.
   Let $TSN = TM = N_1 \oplus \cdots \oplus N_t$ for indecomposable objects N_i in $\mathcal{H}$ and $i = 1, ..., t$ and $t \in \mathbb{N}$. But we know that TSN is indecomposable in $\mathcal{H}/\mathcal{Y}$ by Proposition 2.16.
   Hence there is exactly one i_0 where N_{i_0} does not divide $\mathcal{Y}$.

3. $g(N) = SN$ in $\mathcal{G}/T^{-1}\mathcal{Y}$. $f(g(N)) \simeq TSN \simeq N$ in $\mathcal{H}/\mathcal{Y}$ since $TS \simeq 1$ as functor $U^{-1}\mathcal{Y} \longrightarrow (U^{-1}\mathcal{Y})/\mathcal{Y}$.
4.

$$\begin{array}{rcl} fgf(M) \simeq f(M) \text{ by 3.} & \Rightarrow & Tgf(M) \simeq TM \text{ in } \mathcal{H}/\mathcal{Y} \\ & \overset{Prop.2.16.2}{\Longrightarrow} & gf(M) \simeq M \text{ in } \mathcal{G}/T^{-1}\mathcal{Y} \\ & \overset{Lemma 2.14}{\Longrightarrow} & gf(M) \simeq M \text{ in } \mathcal{G}. \end{array}$$

Now we combine Theorem 2.7 and Corollary 2.17 to state the result.

Corollary 2.19. *Assume we are in the situation of Theorem 2.7 and let in addition $\mathcal{G}$ and $\mathcal{H}$ be Krull-Schmidt categories. Then*[11],

1. $\forall N \in ind(add\ S'\mathcal{Z})_{S'T'\mathcal{Y}}$, *the object* SN *has precisely one indecomposable summand* $g(N)$ *not dividing* $SS'T'\mathcal{Y}$.
2. $\forall M \in ind(add\ SS'\mathcal{Z})_{SS'T'\mathcal{Y}}$, *the object* TM *has precisely one indecomposable summand* $f(M)$ *not dividing* $\mathcal{Y}$.
3. $f(g(N)) \simeq N$.
4. $g(f(M)) \simeq M$.

2.4.5. *The situation for group rings.* Again we shall follow [1] closely.

We shall apply Theorem 2.7 to the case mentioned at the beginning of subsection 2.4. We fix the following setting.

1. Let R be a commutative Noetherian ring.
2. Let G be a finite group and let $D \leq H < G$. [12]
3. $\forall_{F \leq G}$ $ind_F^G := RG \otimes_{RF} - : RF - mod \longrightarrow RG - mod$; $res_F^G : RG - mod \longrightarrow RF - mod$ is the restriction functor.
4.

$$\begin{array}{rcl} mod(G,F) & = & add(ind_F^G(RF - mod)) \\ & = & \text{'direct summands of } RG\text{-modules induced from } F\text{'.} \end{array}$$

The objects are called relatively F-projective modules.
5. If $\mathcal{F}$ is a set of subgroups of G, then $ind_{\mathcal{F}}^G$ is the smallest full additive subcategory of $RG - mod$ containing all modules of the form $ind_F^G(V)$ with $V \in Ob(RF - mod)$ and $F \in \mathcal{F}$. $mod(G, \mathcal{F})$ is the smallest full additive subcategory of $RG - mod$ containing $mod(G, F)$ with $F \in \mathcal{F}$.
6. Let $g \in G$ and $F \leq G$. Then, ${}^gF = gFg^{-1}$ and for $M \in RF - mod$ one forms the $R\ {}^gF$-module ${}^gM$ by $f \cdot m =: gfg^{-1}m$ for all $m \in M, f \in F$.

[11] $indC$ means the class of indecomposable objects in the category C.

[12] To avoid technical difficulties we assume that $H \neq G$. Otherwise we shall have to deal with unpleasant exceptions arising from empty set discussions in our formulas.

7. We take a disjoint union

$$G = \bigcup_{i=1}^{s} Hg_iH \text{ for } g_i \in G; g_1 = 1 .$$

Then, as RH-RH-bimodule,

$$RG = \bigoplus_{i=1}^{s} RHg_iH .$$

8. $p : RG \longrightarrow RH \cdot 1 \cdot RH$ is $RH - RH$-linear and an epimorphism.
9. $i : RH \longrightarrow RH \cdot 1 \cdot RH$ is $RH - RH$-linear and a monomorphism.
10. Set $\mathcal{G} := RG$–mod, $\mathcal{H} := RH - mod$, $\mathcal{D} := RD$–mod.
 $S := ind_H^G$, $S' := ind_D^H$, $T := res_H^G$, $T' := res_D^H$.
11. For all $N \in Ob(RH - mod), M \in Ob(RG - mod)$ set

$$\alpha(N, M) : Hom_{RG}(ind_H^G(N), M) \quad \stackrel{\sim}{\longrightarrow} \quad Hom_{RH}(N, res_H^G M)$$

 by Frobenius' reciprocity, as explained in Section 2.2.
12.

$$\eta : 1_{RH-mod} \longrightarrow res_H^G \circ ind_H^G$$

 by means of

$$\eta_N = \alpha(N, ind_H^G N)(1_{ind_H^G(N)}) = (n \longrightarrow 1 \otimes n)$$

 with $N \in RH - mod, n \in N$.
13. $U := \bigoplus_{i=2}^{s} RHg_iRH \otimes_{RH} -$.
14. Since $pi = 1_{RH}$, we get $TS = 1_{RH} \oplus U$.
15. η_I is the identity, hence an isomorphism.

To apply the theorem, one has to try to construct a subcategory $\mathcal{Y}$ of $RH - mod$ such that

$$S'T'\mathcal{Y}|\mathcal{Y} \text{ and } S'T'\mathcal{Y}|U^{-1}\mathcal{Y} .$$

Notation 2.20. We fix for any set $\mathcal{S}$ of subgroups of H

$$\mathcal{X} := \{D \cap Y | Y \in \mathcal{S}\} \text{ and } \mathcal{Y} := ind_{\mathcal{S}}^H .$$

Remark 2.21. Since *ind* and *res* result to isomorphic modules when passing to conjugate subgroups, one may assume that $\mathcal{S}$ is closed under conjugation. Since *res* and *ind* are transitive, one may furthermore assume that $\mathcal{S}$ is closed under subgroups.

Proposition 2.22. 1. *$S'T'\mathcal{Y}$ is a subcategory of $ind_{\mathcal{X}}^H$. Furthermore, $ind_{\mathcal{X}}^H|S'T'\mathcal{Y}$.*
2. *$RH - mod/S'T'\mathcal{Y} = RH - mod/ind_{\mathcal{X}}^H$.*
3. *$S'T'\mathcal{Y}$ is a subcategory of $\mathcal{Y}$.*

Proof.

Part 1. Let $N \in \mathcal{Y}$. Then, $N \simeq ind_V^H W$ with $W \in Ob(RV-mod); V \in \mathcal{S}$. We apply Mackey's formula to obtain

$$T'N = res_D^H ind_V^H W = \bigoplus_{VgD \in V\backslash G/D; g_1=1} ind^D_{D\cap\, {}^gY} res^{{}^gY}_{D\cap\, {}^gY}\, {}^gW \in Ob(ind_{\mathcal{X}}^D)\,.$$

Since $ind_D^H ind_{\mathcal{X}}^D = ind_{\mathcal{X}}^H$, we get the first statement.

Let $W \in Ob(D\cap Y - mod), Y \in \mathcal{Y}$, then again using Mackey's formula,

$$res_D^H ind_{D\cap Y}^H W = ind_{D\cap Y}^D W \oplus \bigoplus_{D \neq DgD \in D\backslash H/D} ind^H_{{}^g(D\cap Y)\cap H} res^{{}^g(D\cap Y)}_{{}^g(D\cap Y)\cap H}\, {}^gW\,.$$

Therefore,

$$ind_{D\cap Y}^D W | res_D^H ind_{D\cap Y}^H W = res_D^H ind_{D\cap Y}^D W = T'(ind_{D\cap Y}^D W)$$

and with the transitivity of *ind*, one gets that

$$ind_{D\cap Y}^H W | S'T'(ind_{D\cap Y}^H W)\,.$$

Part 2. follows from 1. since by the first inclusion, every morphism factoring through an object of $S'T'\mathcal{Y}$ factors also through an object of $ind_{\mathcal{X}}^H$. On the other hand, by the second statement, a morphism factoring through an object of $ind_{\mathcal{X}}^H$ factors also through an object of $S'T'\mathcal{Y}$, the latter having the object from before as direct summand.

Part 3. follows from the transitivity of the induction and part 1.

To be able to write the result in a more concise form we introduce a new

Notation 2.23. Let $\mathcal{F}$ be a set of subgroups of H. We set

$$\mathcal{F}' := \{H \cap {}^gF | g \in G \setminus H \text{ and } F \in \mathcal{F}\}\,.$$

Lemma 2.24. *$\forall F \in \mathcal{F}$ with $F \leq H$ is*

$$U(ind_{\mathcal{F}}^H) \subseteq ind_{\mathcal{F}'}^H | U(ind_{\mathcal{F}}^H)\,.$$

Proof. The 'source' of the lemma is entirely Mackey's formula. Therefore the proof is a bit technical.

Of course, it is sufficient to prove the statements for $\mathcal{F} = \{F\}$, a set with cardinality 1. To start with we prove this statement and hence assume that we are given a $V \in RF - mod$ and set $N := ind_F^H V$.

By definition,

$$U(N) = \bigoplus_{i=2}^{s} RHg_iH \otimes_{RH} N\,.$$

Hence it is enough to prove that

$$RHgH \otimes_{RH} N \in ind_{\mathcal{F}'}^H \text{ for any } g \in G \setminus H\,.$$

We first discuss what is meant by $RHgH$. We see that gRH is isomorphic to the $R({}^gH) - RH-$bimodule ${}_gRH$ which is RH as $R-$module and on which from the left ${}^gh \in {}^gH$ acts by multiplication by ${}^gh$ on RH. Now, precisely those objects $hgh' \in HgH$ belong to $\{1\}gH$ for which $h \in H \cap {}^gH$. Therefore, $RHgH = RH \otimes_{R(H\cap {}^gH)} {}_gRH$ as bimodule. We compute

$$\begin{aligned} RHgH \otimes_{RH} N &= RH \otimes_{R(H\cap {}^gH)} {}_gRH \otimes_{RH} N \\ &= RH \otimes_{R(H\cap {}^gH)} {}^gN \\ &= ind^H_{H\cap {}^gH} res^{{}^gH}_{H\cap {}^gH} {}^gN \\ &= ind^H_{H\cap {}^gH} res^{{}^gH}_{H\cap {}^gH} ind^{{}^gH}_{{}^gF} {}^gV \\ &= ind^H_{H\cap {}^gH} \bigoplus_{(H\cap {}^gH)t\, {}^gF} ind^{[{}^gH\cap H]}_{[{}^{tg}F\cap H\cap {}^gH]} res^{[{}^{tg}F]}_{[{}^{tg}F\cap H\cap {}^gH]} {}^{tg}V . \end{aligned}$$

Furthermore, for $n := g^{-1}tg$,

$$H \cap {}^gH \cap {}^{tg}F = H \cap {}^{gn}F$$

since $x \in H \cap {}^{gn}F \Rightarrow x \in H$ and $x \in {}^{gn}F$ and ${}^{gn}F = \{gn \cdot f \cdot n^{-1}g^{-1}\} \subseteq {}^gH$, taking into account that $t \in {}^gH \Rightarrow n \in H$. Hence,

$$\begin{aligned} RHgH \otimes_{RH} N &= \bigoplus_n ind^H_{H\cap {}^{gn}F}(res^{{}^{gn}F}_{[H\cap {}^{gn}F]} {}^{gn}V) \\ &\in ind^H_{\mathcal{F}'} . \end{aligned}$$

We hence proved the first statement.

We have to prove the second statement. Let $g \in G$ and $n \in H$, let $V \in Ob(RF - mod)$ and $N := ind^H_F V$. Then, by the above calculation,

$$ind^{[{}^gH\cap H]}_{[{}^{tg}F\cap H\cap {}^gH]} res^{[{}^{tg}F]}_{[{}^{tg}F\cap H\cap {}^gH]} {}^{tg}V \mid RHgH \otimes_{RH} N .$$

We just have to show that $W = ind^H_{H\cap {}^gF}Q$ for $Q \in R(H \cap {}^gF) - mod$ is a direct summand of a module of the above form.

Set $ind^F_{{}^{g^{-1}}H\cap F} {}^{g^{-1}}Q =: V \in Ob(RF - mod)$. Then again, applying Mackey's formula,

$$\begin{aligned} & ind^H_{H\cap {}^gF} res^{{}^gF}_{H\cap {}^gF}(ind^{{}^gF}_{H\cap {}^gF}Q) \\ &\qquad = ind^H_{H\cap {}^gF}(Q \oplus \text{modules from lower subgroups}) \\ &\qquad = W \oplus \text{modules from lower subgroups}. \end{aligned}$$

Hence we have shown that W is a direct summand of a module of the above type which in turn divides $U(ind^H_F)$. We have shown the lemma.

We remember that we are given the set of subgroups $\mathcal{S}$ and we have set $\mathcal{Y} := ind^H_{\mathcal{S}}$ and $\mathcal{X} := \{V \cap D | V \in \mathcal{S}\}$.

Corollary 2.25. *If $\mathcal{X}' \subseteq \mathcal{S}$, then $S'T'\mathcal{Y}|\mathcal{Y}$ and $S'T'\mathcal{Y}|U^{-1}\mathcal{Y}$.*

Proof.

- We have just to show that $U(ind_{\mathcal{X}}^H)|ind_{\mathcal{S}}^H$, since by Proposition 2.22, $S'T'\mathcal{Y}$ is a subcategory of $\mathcal{Y}$ and hence $S'T'\mathcal{Y}|\mathcal{Y}$ automatically. Also, $S'T'\mathcal{Y}|ind_{\mathcal{X}}^H$. Hence, $U(S'T'\mathcal{Y})|U(ind_{\mathcal{X}}^H)$ and if we show that $U(ind_{\mathcal{X}}^H)|ind_{\mathcal{S}}^H = \mathcal{Y}$, then we also get the second condition.
- But, $U(ind_{\mathcal{X}}^H) \subseteq ind_{\mathcal{X}'}^H$ by Lemma 2.24.
- $ind_{\mathcal{X}'}^H|U(ind_{\mathcal{X}}^H)$ by Lemma 2.24.
- $ind_{\mathcal{X}'}^H \subseteq ind_{\mathcal{S}}^H$ since $\mathcal{X}' \subseteq \mathcal{S}$.
- Hence, $ind_{\mathcal{X}'}^H|ind_{\mathcal{S}}^H$ and even $U(ind_{\mathcal{X}}^H)|ind_{\mathcal{S}}^H$.

Remark 2.26. We immediately check two situations where we may verify the condition in Corollary 2.25.

1. If $\mathcal{S} = \{V|$ there is a $g \in G \setminus H : V \leq H \cap {}^gD\}$, then $\mathcal{X} = \{X|$ there is a $g \in G \setminus H : X \leq D \cap {}^gD\}$ and $\mathcal{X}' \subseteq \mathcal{S}$.
2. If E is a normal subgroup of H and $D \cap E$ is a normal subgroup of G, set $\mathcal{S} = \{V|V \leq E\}$. Then, $\mathcal{X} = \{X|X \leq D \cap E\}$ and $\mathcal{X}' \leq \mathcal{S}$.

The first situation leads to the classical Green correspondence whereas the second is a new application and leads to a theorem due to Auslander–Kleiner [1].

Summarizing the results we apply Theorem 2.7 to the above situation and obtain the following.

Theorem 2.27. *Let $\mathcal{Z}$ be the largest set of subgroups of D such that $\mathcal{Z}' \subseteq \mathcal{S}$. If $\mathcal{X}' \subseteq \mathcal{Y}$, then*

1. $$ind_H^G : \frac{mod(H,\mathcal{Z})}{mod(H,\mathcal{X})} \xrightarrow{\sim} \frac{mod(G,\mathcal{Z})}{mod(G,\mathcal{X})}$$
   is an equivalence of categories.
2. $$res_H^G : \frac{mod(G,\mathcal{Z})}{mod(G,\mathcal{X})} \xrightarrow{\sim} \frac{mod(H,\mathcal{Z})}{mod(H,\mathcal{S})}$$
   is an equivalence of categories.
3. *$res_H^G \circ ind_H^G$ is induced by the identity functor on $RH-mod$.*

Corollary 2.19 now translates to

Corollary 2.28. *Assume we are in the situation of Theorem 2.27 and furthermore assume that $RG-mod$ and $RH-mod$ are Krull-Schmidt categories. Given $M \in mod(G,\mathcal{Z}) \setminus mod(G,\mathcal{X})$ indecomposable and $N \in mod(H,\mathcal{Z}) \setminus mod(H,\mathcal{X})$ indecomposable.*
Then,

1. $ind_H^G(N)$ *has a unique indecomposable direct summand* $g(N)$ *in* $mod(G,\mathcal{Z}) \setminus mod(G,\mathcal{X})$.
2. $res_H^G(M)$ *has a unique indecomposable direct summand* $f(M)$ *in* $mod(H,\mathcal{Z}) \setminus mod(H,\mathcal{S})$.
3. $fg(N) \simeq N$.
4. $gf(M) \simeq M$.

An application is the definition of a Brauer correspondent. Assume that we are in the situation of Corollary 2.28.

1. The syzygy-operator Ω_G on the stable category of $RG - mod$ and the syzygy-operator Ω_H on the stable category of $RH - mod$ commutes with g and commutes with f. More precisely:
$$\Omega_G g \simeq g\Omega_H \text{ and } \Omega_H f \simeq f\Omega_G .$$
This follows since ind_H^G and res_H^G are exact and send projective modules to projective modules, hence a projective resolution to a projective resolution. Then, applying Schanuel's Lemma, we realize that syzygies are well defined up to projective direct summands. This gives the result.
2. In the situation of the first part of Corollary 2.25 we look at the various sets of subgroups of G more closely.
$$\begin{aligned} \mathcal{S} &= \{V \mid \exists g \in G\setminus H : V \leq H \cap {}^gD\} \\ \mathcal{X} &= \{X \mid \exists g \in G\setminus H : X \leq D \cap {}^gD\} \\ \mathcal{Z} &= \{Z \leq D \mid \exists g,g' \in G\setminus H : {}^{g'}Z \cap H \leq H \cap {}^gD\}. \end{aligned}$$
Since $D \leq H$, we always get $D \in \mathcal{Z}$ choosing $g = g'$. If $N_G(D)\setminus H \neq \emptyset$, then there is a $g \in N_G(D) \setminus H$. Taking this g, we conclude that $D \in \mathcal{X}$. But then $\mathcal{Z} = \mathcal{X} = \mathcal{S}$ and Theorem 2.27 establishes a bijection between the empty sets and in Corollary 2.28 there is no indecomposable module satisfying the assumptions. If $H \geq N_G(D)$, then trivially this never happens and the theorem is non trivial.
3. Let B be a block of RG with defect group D. Let $H \geq N_G(D)$. There is a Green correspondence for $\mathcal{G} = R(G\times G)$ and $\mathcal{H} = R(H\times H)$ and $\mathcal{D} = R(D\times D)$ since $N_{G\times G}(D\times D) = N_G(D)\times N_G(D)$. Furthermore, as usual the functors S, S' are the induction functors. T, T' are the corresponding restriction functors. Now,
$$res_{H\times H}^{G\times G} : RG \longrightarrow [\bigoplus_{HgH\in H\backslash G/H} RHgH] = [RH \oplus \bigoplus_{HgH\in H\backslash G/H;\ g\notin H} RHgH]$$
and so a block B of RG with defect group D has a Green correspondent b which is a direct summand of the right hand side. It is now easy to see that the Green correspondent of B is a direct summand of RH. The

Green correspondent $f(B)$ of B is a block of RH and is called the *Brauer correspondent* of B.

3. Classical theory of blocks with cyclic defect groups and Green's walk around the Brauer tree

In this section we shall present the results of Green [5], Dade [2] and Michler [13, 14].

Throughout this section we use the following notations.

1. R is a complete discrete valuation ring of characteristic 0 with residue field k of characteristic p. The field of fractions of R is K.
2. G is a finite group.
3. B is a block of RG with defect d.
4. D is a defect group of B with order $q = p^d$.
5. D_1 is the subgroup of D of order p.
6. $H = N_G(D_1) \geq N_G(D)$.
7. B' is the Brauer correspondent of B in RH.
8. $C_G(D_1) =: C$.

3.1. The theory of Dade on blocks with cyclic defect group.

Definition 3.1. (Michler [13, 14]; Feit [3])

- The number e of isomorphism classes of simple B'–modules is called the inertial index of G.
- There is a finite Galois extension $\hat{K}$ of K such that for the ring of integers $\hat{R}$ in $\hat{K}$ over R all the primitive $|G|^{th}$ roots of unity are contained in $\hat{R}/rad\ \hat{R}$. Let $\hat{B}$ be one indecomposable factor of $\hat{R} \otimes_R B'$. (The others are Galois conjugate to this.) The number e is defined to be the number of isomorphism classes of simple $\hat{B}$–modules.

Michler shows that e divides $p-1$ [13, 14].

Set $I := \{0, 1, \ldots, (e-1)\}$.

The main theorem of Dade describes the structure of the composition series of projective $k \otimes_R B$–modules in terms of combinatorial data, a *Brauer tree*. Janusz and independently Kupisch [8], [10, 11] prove that not only the composition series of the indecomposable projective modules are determined but also those of all indecomposable modules. We do not need this description for Roggenkamp's description of blocks with cyclic defect groups and so we refrain from presenting this theory as well.

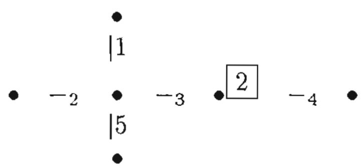
The theory of Dade on the structure of blocks with cyclic defect groups is one of the most beautiful in the theory of blocks. *It provides a complete solution to the problem of determining the module structure of blocks with cyclic defect* *groups in terms of a combinatorial description.* One of the key tools is the Green correspondence.

Theorem 3.2. *(Dade [2]) We assume that k contains all $|G|$-th roots of unity. There is a set Λ of simple KG-modules, called the* exceptional $k \otimes_R B$-modules *with the following properties.*

1. *The graph which consists of the following data is a tree:*
   - *The vertices of the graph are the isomorphism classes of the non exceptional simple $K \otimes_R B$-modules and an additional vertex; the latter representing the set of exceptional modules, called the exceptional vertex.*
   - *There is an edge between two vertices v, w if and only if there is an indecomposable projective B-module P such that $K \otimes_R P$ has the modules which correspond to the vertices v and w as direct summands.*

   The graph is called a Brauer tree *and in case $frac\ R$ is a splitting field for B, the cardinality of Λ is called the* multiplicity *of the exceptional vertex and equals $\mu = (|D| - 1)/e$.*
2. *Let P be a projective indecomposable $k \otimes_R B$-module. Then, $rad\ P/soc\ P$ is a direct sum of two uniserial modules S_P and T_P.*
3. *There is an embedding of the Brauer tree in the plane[13] such that one can get the composition series of S_P and T_P by the following algorithm. Let S_P (and T_P) correspond to a vertex v (and w). By symmetry we describe the algorithm only for S_P. Since the tree is embedded into the plane, one has an ordering of the n_P projective RG-modules Q such that KQ has composition factor KS_P by a counterclockwise numeration of the edges adjacent to v. Now, $rad^i(S_P)/rad^{i+1}(S_P) \simeq Q_i/rad(Q_i)$ for $i = 1, 2, \ldots, n(P) \cdot e(P) - 1$ where Q_i is the projective indecomposable module which is i positions after P in the counterclockwise ordering. If v is the exceptional vertex, then $e(P) = e$ and if v is not the exceptional vertex then $e(P) = 1$.*

We illustrate the algorithm by a simple example.
We are given the following Brauer tree



[13]This is just another way of saying that one imposes to each vertex v of the tree a cyclic ordering of the edges $v \overset{e}{-} w$ which are incident to the vertex v.

The second right vertex has multiplicity 2, as indicated the box. We shall give the composition series of the projective indecomposable modules for this example.

$$P_1=\begin{pmatrix}1\\2\\5\\3\\1\end{pmatrix};P_2=\begin{pmatrix}2\\5\\3\\1\\2\end{pmatrix};P_5=\begin{pmatrix}5\\3\\1\\2\\5\end{pmatrix};P_4=\begin{pmatrix}4\\3\\4\\3\\4\end{pmatrix};P_3=\begin{pmatrix} & 3 & \\ 1 & & 4\\ 2 & & 3\\ 5 & & 4\\ & 3 & \end{pmatrix}.$$

3.2. **Green's walk around the Brauer tree.** After Dade's paper, Green proved the following theorem. This theorem is of fundamental importance not only for the proof of Roggenkamp's theory of Green orders.

Let W_i; $i=0,\ldots,(e-1)$ be the projective indecomposable B-modules.

Theorem 3.3. *(Green [5]) We assume that k contains all $|G|$-th roots of unity. Let G be a finite group, let B be an $RG-$ block, let D be a cyclic defect group of B and let Γ be the Brauer tree of B.*

1. *There is a family $(A_n)_{n\in\mathbb{Z}}$ of RG-lattices and a permutation δ of $I=\{0,\ldots,(e-1)\}$ such that there exist short exact sequences of RG-modules*
$$E_{2i}:0\longrightarrow A_{2i+1}\longrightarrow W_{\delta(i)}\longrightarrow A_{2i}\longrightarrow 0$$
$$E_{2i+1}:0\longrightarrow A_{2i+2}\longrightarrow W_i\longrightarrow A_{2i+1}\longrightarrow 0$$
with $W_i\simeq W_{i+e}$ and $A_i\simeq A_{i+2e}$ for all $i\in\mathbb{Z}$.
2. *The $A_0,A_1,\ldots,A_{2e-1}$ are mutually non isomorphic.*
3. *KA_n is a vertex of Γ.*

It is possible to reconstruct the Brauer tree from the permutation δ. One forms a path

$$\bullet\xrightarrow{1}\bullet\xrightarrow{\delta(1)}\bullet\xrightarrow{2}\cdots\xrightarrow{\delta(e-1)}\bullet\xrightarrow{e}\bullet\xrightarrow{\delta(e)}\bullet$$

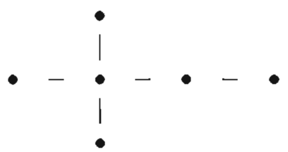
which closes to an oriented circle. Then, one glues $\bullet\xrightarrow{i}\bullet$ to $\bullet\xleftarrow{\delta(j)}\bullet$ if $i=\delta(j)$ and the result is the Brauer tree. Obviously it is a *graph* but by Dade's Theorem, it is in fact a *tree* and looking at the isomorphism classes of the vertices even the *Brauer tree*.

Conversely one may define a permutation δ of the set of edges for every *embedded* tree out of which it is possible to reconstruct the tree in the above way. This permutation depends not only on the tree but also on a starting point:

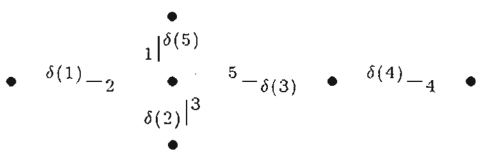
1. One starts at a certain edge e and declares this edge to be 1.

2. Take a vertex v which is adjacent to the edge taken. The edge following e in the circular ordering at v is defined to be $\delta(1)$.
3. The other extremity (not the vertex v) of $\delta(1)$ is w.
4. The edge following $\delta(1)$ in the circular ordering at w is 2.
5. To find $\delta(2)$ one proceeds as in 2.
6. One stops after having determined $\delta(e)$.

In our example

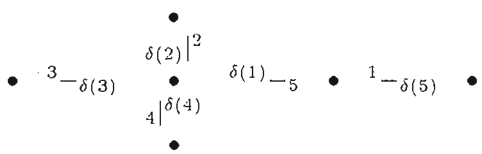


starting with the upper vertical edge one gets the assignment



and the permutation (1 2 3 5).

If one starts with the right most edge, one gets the assignment



and the permutation (1 5).

Michler generalized Theorem 3.3 to the case where there is a ring R as in the introduction, *without assuming that the residue field is large enough.*

Theorem 3.4. *(Michler [13, 14]) Let R be as in the introduction to this section. Let G be a finite group, let B be an RG – block and let D be a cyclic defect group of B.*

1. *There are precisely e pairwise non isomorphic indecomposable $k \otimes_R B$-modules M_i with source k, the trivial kD-module.*
2. *Let, for all $i = 1, \dots e$, $\overline{P}_i$ be the $k \otimes_R B$-projective cover of M_i and let*

$$0 \longrightarrow \overline{\Omega} M_i \longrightarrow \overline{P}_i \longrightarrow M_i \longrightarrow 0$$

be exact. Then, $(1-\alpha)kD$ is the source of $\overline{\Omega} M_i$.

3. *Let, for all* $i = 1, \ldots e$, $\overline{Q}_i$ *be the* $k \otimes_R B$*-projective cover of* $\overline{\Omega} M_i$ *and let*
$$0 \longrightarrow \overline{\Omega}^2 M_i \longrightarrow \overline{Q}_i \longrightarrow \overline{\Omega} M_i \longrightarrow 0$$
be exact. Then, one can find a numbering for the M_i *such that* $\overline{\Omega}^2 M_i \simeq M_{i+1}$.
4. *There are* e *pairwise non isomorphic indecomposable* B*-lattices* W_i *with source* R, *the trivial* RD*-lattice and* $k \otimes_R W_i = M_i$. *For each* i *there is, up to isomorphism, only one* B*-lattice with these properties.*
5. *Let, for all* $i = 1, \ldots e$, P_i *be the projective cover of* W_i *and let*
$$0 \longrightarrow \Omega W_i \longrightarrow P_i \longrightarrow W_i \longrightarrow 0$$
be exact. Then, $(1 - \alpha)RD$ *is the source of* ΩW_i.
6. *Let, for all* $i = 1, \ldots e$, Q_i *be the projective cover of* ΩW_i *and let*
$$0 \longrightarrow \Omega^2 W_i \longrightarrow Q_i \longrightarrow \Omega W_i \longrightarrow 0$$
be exact. Then, one can find a numbering for the W_i *such that* $\Omega^2 W_i = W_{i+1}$ *where the indices are taken modulo* e, *and* $k \otimes_R \Omega W_i = \overline{\Omega} M_i$.
7. *In the set* $\{\Omega^k W_1 | k \in \mathbb{N}\}$ *a maximal subset of pairwise non isomorphic modules has cardinality* $2e$.

Following Feit [3, Chapter VII Remark after Theorem 2.11] we define the property

> (∗) The number of characters of the group H which are afforded by irreducible $frac(R) \otimes_R B$–modules is equal to $(q-1)/\hat{e}$.

As is proved in Feit [3, Chapter VII, Corollary 6.8], the definition for a Brauer tree as in Theorem 3.2 works also for more general R satisfying condition (∗). Feit gives also an example that it is in general not enough to adjoin all q^{th} roots of unity.

As is proved in Feit [3, Chapter VII Theorem 10.6] one can prove a theorem which is analogous to Theorem 3.3 also for more general R *satisfying the assumption* (∗) from above.

Green shows Theorem 3.3 by first showing Theorem 3.5 below. He applies Green correspondence with $\mathcal{G} = kG - mod, \mathcal{H} = kH - mod$ and $\mathcal{D} = kD - mod$. Clearly, D_1 is the only minimal subgroup of D and one chooses H such that each $g \in G \setminus H$ satisfies ${}^g D_1 \cap D_1 = \{1\}$, hence, $\mathcal{X} = \{1\}$. The block B' of kH is the Brauer correspondent of B. The simple B'–modules are called $S_1, \ldots, S_e$.

Theorem 3.5. 1. *B contains e simple kG-modules V_i; $i \in I$ such that every simple kG–module in B is isomorphic to exactly one V_i. Let $\overline{W}_i$ be the projective cover of V_i as kG-module for all $i \in I$.*

2. *There is a numbering of the V_i such that*

$$Hom_{kH}(fV_j, S_i) \simeq Hom_{kG}(V_j, gS_i) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

and there is a permutation δ of I such that

$$Hom_{k\bar{H}}(S_i, fV_j) \simeq Hom_{kG}(gS_i, V_j) = \begin{cases} k & \text{if } \delta(i) = j \\ 0 & \text{if } \delta(i) \neq j \end{cases}.$$

3. *For all $i \in I$ there are non split exact sequences*

$$F_{2i} : 0 \longrightarrow \Omega_G(gS_i) \longrightarrow \overline{W}_{\delta(i)} \longrightarrow gS_i \longrightarrow 0$$

and

$$F_{2i+1} : 0 \longrightarrow gS_{i+1} \longrightarrow \overline{W}_i \longrightarrow \Omega_G(gS_i) \longrightarrow 0 .$$

Let us interpret Theorem 3.5. The theorem says in other words that the permutation δ can be obtained by the Green correspondents of the simple B'–modules. In fact, the Green correspondent $g(S_i)$ has the property

$$top(g(S_i)) = V_{\delta(i)} \text{ and } soc(g(S_i)) = V_i \text{ for all } i = 1, \ldots, e$$

of course after a renumeration. By the discussion of the permutation δ one gets the tree back from the permutation. Therefore, the Brauer tree as abstract tree is determined by the Green correspondence.

We shall prove Theorem 3.3 in detail in the following subsections.

3.3. **Dade's description for blocks with normal cyclic defect groups.** As illustration on the degree of completeness of the description of the module structure as well as preparation for the proof of Theorem 3.3 we give Dade's results for the special case of a normal cyclic defect group D of of the block B in this subsection.

Then, the Brauer tree is a star and the exceptional vertex is in the centre. This is the subject of the following lemma.

We introduce some notation before. As above, the Brauer correspondent of B in kH is called B'.

Lemma 3.6. *(Dade) We assume that k contains all $|G|$-th roots of unity. B' contains e simple modules $S_0, \ldots, S_{e-1}$ with projective covers $T_0, \ldots, T_{e-1}$.*

1. *There is a multiplicative isomorphism $\bar{\ }$: $D \longrightarrow Centre(kC)$ such that taking a generator α of D and defining $a := \overline{\alpha} - 1$ the only composition series, which is also the radical series of each T_i with $i = 1, \ldots e$ is*

$$T_i > T_i \cdot a > T_i \cdot a^2 > \cdots > T_i a^q = 0 .$$

2. *Every indecomposable kH–module is isomorphic to one of the $T_{i,\nu} = T_i/(T_i a^\nu)$; $i = 0, 1, \ldots, (e-1); \nu = 0, 1, \ldots, (q-1)$.*

3. *There is a $kC_G(D_1)$-block b such that $kH \otimes_{kC_G(D_1)} b = B'$ and all such blocks are conjugate in H. Moreover, the stabilizer of b in H is of the form $C_G(D_1) \rtimes E$ for a subgroup E of $N_G(D)$. The group E operates on D_1 by conjugation and $^-$ is E-linear.*

We take $\alpha_1 = \alpha^{p^{d-1}}$.

Then, $D_1 =< \alpha_1 >$. Since $H = N_G(D_1)$ for all $h \in H$ there is a number $n(h)$ defined uniquely modulo p such that,

$$h^{-1} \cdot \alpha_1 h = \alpha_1^{n(h)}.$$

$$\begin{array}{rcl} \psi : H & \longrightarrow & (k \setminus \{0\}, \cdot) \\ h & \longrightarrow & n(h) \end{array}$$

is a homomorphism and gives rise to a one dimensional module. Since $C = C_G(D_1)$, we have $C \in ker(\psi)$ and hence $\psi^{|H:C|} = 1$.

We do similar computations with E and D. We define $\forall_{z \in E} z^{-1}\alpha z = \alpha^{n(z)}$. We use the same symbols for D_1 as well as for D since we used a compatible choice for the generators of D and D_1.

We compute

$$\overline{\alpha}^z \overset{by\ 3.}{=} \overline{\alpha^z} \overset{by\ Def.}{=} \overline{\alpha^{n(z)}} \overset{by\ 3.}{=} \overline{\alpha}^{n(z)}.$$

Lemma 3.7. *(Green)*

1. *$S_{i,\nu} := T_{i,\nu}/T_{i,(\nu+1)} \simeq \psi^\nu \otimes_k S_i$.*
2. *For all $n \in \mathbb{Z}$ set $S_n := S_{0,n}$ and then $\{S_0, S_1, \ldots, S_{e-1}\}$ is a complete set of representatives of isomorphism classes of simple $kH-$modules.*
3. *The composition factors of T_i are $S_i, S_{i+1}, \ldots, S_{i+q-1} \simeq S_i$.*

Proof.

Part 1. $\forall_{t \in T_i; z \in E} : t \cdot a^\nu \cdot z = t \cdot z \cdot (\overline{\alpha}^{n(z)} - 1)^\nu = t \cdot z \cdot a^\nu \cdot (1 + \overline{\alpha} + \overline{\alpha}^2 + \cdots + \overline{\alpha}^{n(z)-1})^\nu \equiv t \cdot z \cdot a^\nu \cdot n(z)^\nu$ since $(1 + \overline{\alpha} + \overline{\alpha}^2 + \cdots + \overline{\alpha}^{n(z)-1}) \equiv n(z)$ modulo $T_1 \cdot a$.

Part 2. Since $\psi^{|H:C|} = 1 \Rightarrow S_m \simeq S_n$ if $m \equiv n \mod |H:C|$. But, $|H:C| \mid (p-1)$, then follows by part 1 that all S_n are composition factors of T_0. Hence, S_n all belong to B'.

Let S be a simple $B'-$module. Then, there is a sequence $i_0, i_1, \ldots, i_r \in I$ such that $S_{i_0} \simeq S_0$ and $S_{i_r} \simeq S$ and $S_{i,j}$ is a composition factor of $T_{i_{j-1}}$.[14] We know that $S_{i_j} \simeq \psi^{\nu_j} \otimes S_{i_{j-1}}$ for all j. Hence, there is an integral number x

[14] This is an alternative method of describing blocks. In fact, we need only the necessity. If there was not such a sequence, then we could divide the projective indecomposables into two disjoint sets $\mathcal{P}_1$ and $\mathcal{P}_2$ such that for all $P_1 \in \mathcal{P}_1$ and all $P_2 \in \mathcal{P}_2$ $Hom(P_1, P_2) = Hom(P_2, P_1) = 0$ and hence $B' = End(\oplus_{P_1 \in \mathcal{P}_1} P_1 \oplus \oplus_{P_2 \in \mathcal{P}_2} P_2) = End(\oplus_{P_1 \in \mathcal{P}_1} P_1) \oplus End(\oplus_{P_2 \in \mathcal{P}_2} P_2)$ decomposes.

such that $S \simeq \psi^x \otimes S_0$. We know, that there are precisely e simple modules and therefore we found all of them.

Part 3. follows from Part 1, Part 2. and Lemma 3.6.

Corollary 3.8. *(Green) Let $i \in I$; $\nu \in \{1, \ldots, q\}$.*

1. *$T_{i,\nu}$ is projective if and only if $\nu = q$.*
2. *There are non split exact sequences*
$$0 \longrightarrow T_{i+1,q-\nu} \longrightarrow T_{i,q} \longrightarrow T_{i,\nu} \longrightarrow 0\,.$$
3. $\forall_{1 \leq \nu \leq q-1} \Omega_H(T_{i,\nu}) \simeq T_{i+\nu,q-\nu}$.
4. $\Omega_H(\Omega_H S_i) \simeq S_{i+1}$.

The proof is clear.

3.4. **Definition of the 'walk' δ.** In this subsection we follow closely Green [5]. We shall prove in this subsection Theorem 3.5.

In the following we first examine the situation over k and pass then, in the next subsection, over to R.

For the proof of Theorem 3.5 we proceed in several lemmata.

Let $\{V_j\}$ be a complete set of simple kG–modules. For proving Part 1 of the theorem we have to show that there is a bijection between I and J.

Claim 3.9. *fV_j is indecomposable and non projective and belongs to B'. gS_i is indecomposable and non projective and belongs to B.*

Proof. The only thing one has to show is that fV_j and gS_i belong to the blocks as claimed. Since $fg(N) = N$ and $gf(M) = M$ for all N and M, we just have to show one of the statements.

We have the Green correspondence with $\mathcal{G} = kG - mod$ and $\mathcal{H} = kH - mod$ and $\mathcal{D} = kD - mod$ on the level of the modules. The Brauer correspondence is a Green correspondence with $\mathcal{G} = k(G \times G) - mod$ and $\mathcal{H} = k(H \times H) - mod$ and $\mathcal{D} = k(D \times D) - mod$. Since $G \times G \longrightarrow G \times 1 \simeq G$ is an epimorphism, we can view each kG–module as $k(G \times G)$–module. The analogous holds for H and D. The Green correspondent for a kG–module V is the same as the Green correspondent of V as $k(G \times G)$–module. This proves the statement since belonging to a block means for a module that the corresponding idempotent of the block acts as identity on the module. Using the functoriality we obtain the statement.

We now turn to prove Part 2 of the theorem.

For this purpose we prove that

$$Hom_{kH}(S_i, fV_j) = (Hom_{kH}/mod(H,1))(S_i, fV_j)$$

and similarly

$$Hom_{kG}(gS_i, V_j) = (Hom_{kG}/mod(G,1))(gS_i, V_j) .$$

More generally, let X be a non projective indecomposable module, then

$$Hom_{kH}(S_i, X) = (Hom_{kH}/mod(H,1))(S_i, X) .$$

Let $\phi : S_i \longrightarrow X$ be a map which is zero on the right side of the equation. Then, ϕ factors through a projective module. However, group rings are selfinjective algebras[15]. Since S_i is simple, the projective module over which the mapping factors has as direct summand the injective hull P of S_i and the mapping actually factors over P. But, $soc(P) = S_i$ and therefore, if the mapping is not zero, it is injective. However, then the injective module P is a submodule and hence even a direct summand of X. This leads to a contradiction.

We proved

Lemma 3.10.

$$\begin{aligned} Hom_{RH}(S_i, fV_j) &\overset{above}{=} (Hom_{RH}/mod(H,1))(S_i, fV_j) \\ &\overset{Green corr.}{=} (Hom_{RG}/mod(G,1))(gS_i, V_j) \\ &\overset{above}{=} Hom_{RG}(gS_i, V_j) \end{aligned}$$

and analogously

$$Hom_{RH}(fV_j, S_i) = Hom_{RG}(V_j, gS_i) .$$

Lemma 3.11. *There is a bijection $h : J \longrightarrow I$ such that*

$$\forall_{i\in I, j\in J} h(j) = i \iff Hom_{RH}(fV_j, S_i) \neq 0 .$$

Proof. $fV_j \simeq T_{h(j),\nu(j)}$ for $h(j) \in I, \nu(j) \in \{1, \ldots, q-1\}$ by its indecomposability. But then, it is even uniserial and

$$Hom_{kH}(fV_j, S_i) = \begin{cases} k & \text{if } h(j) = i \\ 0 & \text{if } h(j) \neq i \end{cases}$$

---

[15] An R–algebra A is called selfinjective if each projective A–module is injective. Group rings are selfinjective since there is a linear map $\lambda : A \longrightarrow R$ such that $ker\lambda$ contains no non zero left nor right ideal and $\forall_{a,b\in A}\lambda(ab) = \lambda(ba)$. Such algebras are called symmetric, which is a slightly stronger condition. A group algebra RG is symmetric by setting $\lambda(\sum_{g\in G} r_g g) := r_1$. Since $\lambda((\sum_{g\in G} r_g g)g^{-1}) = r_g$ there is no ideal in $ker\lambda$. Taking

$$\begin{aligned} {}_AA &\longrightarrow Hom_R(A_A, R) \\ a &\longrightarrow b \longrightarrow \lambda(ab)) \end{aligned}$$

we realize that this mapping is injective since an element in the kernel would induce an ideal in the kernel of λ generated by this element. Going to the residue field of R we see that this mapping is also surjective, hence an isomorphism. Thus injective modules are also projective and vice versa. Projective modules for symmetric artinian algebras over a field have the property that the socle and the head are isomorphic.

by Schur's Lemma.[16] Given $i \in I$ and $S|soc(gS_i)$. Since S is a B–module, there is a $j \in J$ such that $V_j \simeq S$. Hence, $Hom_{kG}(V_j, gS_i) \neq 0$ and therefore, $Hom_{kH}(fV_j, S_i) \neq 0$ by Lemma 3.10. This proves that $h(i) = j$ and h is surjective.

Given $j, j' \in J$ with $h(j) = h(j') = i$. Then, we may assume without loss of generality, interchanging j and j' if necessary that $f(V_j) = T_{i,\nu}$ and $f(V_{j'}) = T_{i,\nu'}$ for some $1 \leq \nu' \leq \nu \leq q-1$. Hence, there is an epimorphism

$$T_{i,\nu} \longrightarrow\!\!\!\!\rightarrow T_{i,\nu'} .$$

If this mapping would factor through a projective module it would factor through T_i which is the projective cover of $T_{i,\nu'}$. Hence,[17] $top(T_{i,\nu})$ is mapped to a subquotient of $rad(T_{i,\nu})$ unless $\nu = 0$ what we excluded. Therefore, the mapping was not surjective and this leads to a contradiction. We conclude

$$\begin{aligned} 0 \neq (Hom_{kH}/mod(H,1))(fV_j, fV_{j'}) &\overset{Green\ corr.}{=} (Hom_{kG}/mod(G,1))(V_j, V_{j'}) \\ &\overset{Schur}{=} Hom_{kG}(V_j, V_{j'}) \\ &\Rightarrow \quad j = j'. \end{aligned}$$

Hence, h is also injective which finishes the proof of Lemma 3.11.

From now on, we take $I = J$ and $h = id_I$ and have $fV_j = T_{j,\nu(j)}$ for all $j \in I$ and certain $\nu(j) \in \{1, \ldots, q-1\}$.

Now we use the same proof as in the lemma in the situation $Hom_{kH}(S_i, fV_j)$ instead of $Hom_{kH}(fV_j, S_i)$ to obtain a bijection $\delta : I \longrightarrow I$, which is a permutation, such that $Hom_{kH}(S_i, fV_j) = k$ if and only if $\delta(i) = j$ and 0 else.

This completes the proof of Part 2. of the theorem.

We are going to show Part 3.

By Part 2. we get $soc(gS_i) \simeq V_i$ and $top(gS_i) \simeq V_{\delta(i)}$. Therefore, there is a short exact sequence

$$0 \longrightarrow \Omega g S_i \longrightarrow \overline{W}_{\delta(i)} \longrightarrow gS_i \longrightarrow 0 .$$

Since $\overline{W}_{i+1}$ is also injective and since $soc(gS_i) \simeq V_i$ there is a short exact sequence

$$0 \longrightarrow gS_{i+1} \longrightarrow \overline{W}_i \longrightarrow V \longrightarrow 0$$

---

[16] Schur's Lemma says that given a ring A and simple A–modules S and T, then

$$Hom_A(S,T) = \begin{cases} \text{a skewfield} & \text{if } S \simeq T \\ 0 & \text{else} \end{cases} .$$

The proof is easy since a kernel and an image under an A-isomorphism are ideals which are either 0 or the whole module by the simplicity of S and T.

[17] We use the terminus 'top' synonymous to 'head'

with some kG–module V. But,

$$gS_{i+1} \simeq g\Omega^2 S_i \simeq \Omega^2 gS_i$$

and we see that there is a non split exact sequence

$$0 \longrightarrow gS_{i+1} \longrightarrow \overline{W} \longrightarrow \Omega gS_i \longrightarrow 0$$

with a projective kG–module $\overline{W}$. Applying Schanuel's Lemma gives $V \simeq \Omega gS_i$.

We have also proven Part 3. of the theorem.

Remark 3.12. The same proof, and statement, works for a stable equivalence between two selfinjective k–algebras A and B such that A is serial.

3.5. **Turning to characteristic** 0. We now prove the main result of this section. For the convenience of the reader we state it here again.

Theorem 3.13. *(Green) Let G be a finite group, let B be an RG – block, let D be a cyclic defect group of B and let Γ be the Brauer tree of B.*

1. *There is a family $(A_n)_{n\in\mathbb{Z}}$ of RG-lattices and a permutation δ of $I = \{0,\ldots,(e-1)\}$ such that there exist short exact sequences of RG-modules*
$$E_{2i} : 0 \longrightarrow A_{2i+1} \longrightarrow W_{\delta(i)} \longrightarrow A_{2i} \longrightarrow 0$$
$$E_{2i+1} : 0 \longrightarrow A_{2i+2} \longrightarrow W_i \longrightarrow A_{2i+1} \longrightarrow 0$$
where $W_i \simeq W_{i+e}$ are projective indecomposable B-modules and $A_i \simeq A_{i+2e}$ for all $i \in \mathbb{Z}$.
2. *The $A_0, A_1, \ldots, A_{2e-1}$ are mutually non isomorphic.*
3. *KA_n is a vertex of Γ.*

Proof. We can lift the projective indecomposable modules $\overline{W}_i$ to projective indecomposable RG–modules W_i such that $k \otimes_R W_i \simeq \overline{W}_i$ for all $i \in I$. We may extend this definition to $i \in \mathbb{Z}$ by requiring that $W_i \simeq W_{i+e}$.

In the situation of Theorem 3.5 we define,

$$B_{2i} := gS_i \text{ and } B_{2i+1} := \Omega gS_i\,.$$

Lemma 3.14. *Let $m \in \mathbb{Z}$ and M be an RG–lattice such that $k \otimes_R M \simeq B_m$. Then there are RG–lattices A_n with $A_m \simeq M$ and short exact sequences E_n such that $k \otimes_R E_n \simeq F_n$ for all $n \in \mathbb{Z}$.*

$$E_n : 0 \longrightarrow A_{n+1} \longrightarrow W_n \longrightarrow A_n \longrightarrow 0\,.$$

Proof. Start with $A_m = M$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{m+1} & \longrightarrow & W_m & \longrightarrow & A_m & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow k\otimes_R - & & \downarrow k\otimes_R - & & \\ 0 & \longrightarrow & B_{m+1} & \longrightarrow & \overline{W}_m & \longrightarrow & B_m & \longrightarrow & 0 \end{array}$$

where A_{m+1} is just defined to be the kernel of $W_m \longrightarrow A_m$. ϕ is defined by the universal property of the kernel. Hence, there is an A_{m+1} which lifts B_{m+1} and inductively one gets all A_n for $n \geq m$.

For $n \leq m$ one uses the following strategy. We apply $Hom_R(-, R)$ to the first row and $Hom_k(-, k)$ to the second. We get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^*_{m-1} & \longrightarrow & Hom_R(W_{m-1}, R) & \longrightarrow & Hom_R(A_m, R) & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow k \otimes_R - & & \downarrow k \otimes_R - & & \\ 0 & \longrightarrow & Hom_k(B_{m-1}, k) & \longrightarrow & Hom_k(\overline{W}_{m-1}, k) & \longrightarrow & Hom_k(B_m, k) & \longrightarrow & 0 \end{array}$$

where A^*_{m-1} is just defined to be the kernel of $Hom_R(W_{m-1}, R) \longrightarrow Hom_R(A_m, R)$ and ϕ is defined by the universal property of the kernel. Dualizing again,

$$0 \longrightarrow A^*_{m-1} \longrightarrow Hom_R(W_{m-1}, R) \longrightarrow Hom_R(A_m, R) \longrightarrow 0$$

we get that

$$0 \longrightarrow A_m \longrightarrow W_{m-1} \longrightarrow Hom_R(A_{m-1}, R) \longrightarrow 0$$

is exact and inductively we get the statement.

Lemma 3.15. *If KM is a vertex of Γ, then all KA_n are vertices and $KA_n \simeq KA_{n+2e}$ for all $n \in \mathbb{Z}$ and $KA_n \simeq KA_{n+2e}$.*

Proof. We get that

$$KW_n = KA_n \oplus KA_{n+1} \ \forall n \in \mathbb{Z}\,.$$

Hence, all KA_n are vertices. In the Grothendieck group $K^0(KG)$ we take

$$\sum_{i=n}^{n+2e} (-1)^i [KW_i] = 0 = (-1)^n \cdot ([KA_n] - [KA_{n+2e}])\,.$$

Lemma 3.16. *Let A_n and M be as above. then $A_n \simeq A_{n+2e}$.*

Proof. Given $n \in \mathbb{Z}$. We have a unique decomposition $KW_n = Y_n(1) \oplus Y_n(2)$ where $Y_n(1)$ and $Y_n(2)$ are both vertices of Γ. Define

$$X_n(i) := W_n \cap Y_n(i) \ ; i = \{1, 2\}\,.$$

These are R-pure submodules of W_n since

$$W_n/(W_n \cap Y_n(i)) \simeq (W(n) + Y_n(i))/Y_n(i) \leq KW(n)/Y_n(i)$$

the latter being R-torsion free. Furthermore, they are the only two R-pure submodules $\tilde{X}$ of W_n with $K\tilde{X}$ being a vertex in Γ. In fact, let $\tilde{X}$ be a counterexample with $K\tilde{X} = Y_n(i)$. Then surely $\tilde{X} \leq X_n(i)$. The following diagram is then commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{X} & \longrightarrow & W_n & \longrightarrow & Q_n & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & X_n(i) & \longrightarrow & W_n & \longrightarrow & R_n & \longrightarrow & 0 \end{array}$$

where by the serpent lemma the right most vertical map is surjective. Also, we assumed Q_n to be R-free and again by the serpent lemma also the cokernel of $\tilde{X} \longrightarrow X_n(i)$ is torsion free. But then, $K\tilde{X} \neq X_n(i)$ and we reach a contradiction.

This proves the lemma.

Now the problem is just reduced to trying to find an M to start with. For the principal block we just take $M = R$, the trivial module.

The general case is more complicated and uses a construction of Dade.

It remains to prove Part 2.

We have:

$$k \otimes_R A_{2n+1} \simeq \Omega g S_n$$

$$k \otimes_R A_{2n} \simeq g S_n \,.$$

If $q = |D| \geq 2$ then $S_i = T_{i,1} \not\simeq T_{i,q-1} = \Omega S_i$ and clearly $S_i \not\simeq S_j$ for $i \neq j \ mod\ e$. If $q = 2$ then $A_0 \simeq A_1$ and since $e|(q-1)$ we get $e = 1$. But, $KW_0 \simeq Y_0(1) \oplus Y_0(2)$ which are non isomorphic and also $KW_0 \simeq KA_0 \oplus KA_1$ where both are vertices of Γ. This leads to a contradiction.

Hence, we proved now the whole theorem.

4. Blocks with cyclic defect groups are Green orders

We maintain the notation of Section 3.

In this section we shall define 'Green orders' due to Roggenkamp and prove that a block of the group ring RG with cyclic defect group D is a Green order. Moreover, we shall describe the structure of Green orders in great detail.

We follow the exposition in [16] and mention that all the material is contained in [16].

4.1. **A small example.** Let p be a rational prime number. We shall discuss the integral group ring of the dihedral group D_p of order $2p$. We remark that D_p fits well in Green's framework at the prime p, where we mean that $\mathbb{Z}_p D_p$ is a group ring which satisfies all the assumptions of Theorem 3.3.

Let $D_p =< a, b | a^p = b^2 = baba = 1 >$ be a presentation of D_p. Then, $< a >$ is a cyclic normal subgroup of index 2 in D_p. Hence, we get a surjective ring homomorphism

$$\mathbb{Z} D_p \longrightarrow \mathbb{Z} C_2 \,.$$

This is induced by multiplication by the central idempotent

$$e = \frac{1}{p}\sum_{i=1}^{p} a^i \in \mathbb{Q}D_p .$$

Hence, one gets a pullback diagram

$$\begin{array}{ccc} \mathbb{Z}D_p & \xrightarrow{\quad e\cdot \quad} & \mathbb{Z}D_p e \\ {\scriptstyle \cdot(1-e)}\downarrow & & \downarrow \\ \mathbb{Z}D_p(1-e) & \longrightarrow & \mathbb{Z}D_p e/(\mathbb{Z}D_p e \cap \mathbb{Z}D_p) \end{array}$$

which becomes

$$\begin{array}{ccc} \mathbb{Z}D_p & \xrightarrow{e\cdot} & \mathbb{Z}C_2 \\ {\scriptstyle \cdot(1-e)}\downarrow & & \downarrow \\ \Lambda & \longrightarrow & \mathbb{F}_p C_2 \end{array}$$

where $\mathbb{F}_p$ is the prime field of characteristic p and the right hand vertical mapping is just reduction modulo p. In fact, $pe \in \mathbb{Z}D_p$ and $\mathbb{Z}D_p e \cap \mathbb{Z}D_p = p\mathbb{Z}D_p e$.

We have to determine Λ. Multiplication of $\mathbb{Z}D_p$ by $(1-e)$ means that a acts on $\mathbb{Z}[\zeta_p]$ as multiplication by ζ_p, where ζ_p is a primitive p^{th} root of unity. In fact, $1 + a + a^2 + \cdots + a^{p-1}$ acts as 0 and this is the only relation among the elements of $\mathbb{Z} < a >$. However, b inverts a and acts therefore as Galois automorphism $\zeta_p \longrightarrow \zeta_p^{-1}$. The element a acts as ζ_p and this means that over the fixed ring $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ the element a satisfies the minimal polynomial

$$X^2 - (\zeta_p + \zeta_p^{-1})X + 1 .$$

With basis $\{1, \zeta_p\}$ the representation can be described by the accompanying matrices

$$a \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & \zeta_p + \zeta_p^{-1} \end{pmatrix} \text{ and } b \longrightarrow \begin{pmatrix} 1 & \zeta_p + \zeta_p^{-1} \\ 0 & -1 \end{pmatrix} .$$

Conjugating by $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ from the left one gets the representations

$$a \longrightarrow \begin{pmatrix} \zeta_p + \zeta_p^{-1} - 1 & 1 \\ \zeta_p + \zeta_p^{-1} - 2 & 1 \end{pmatrix} \text{ and } b \longrightarrow \begin{pmatrix} -1 & 0 \\ \zeta_p + \zeta_p^{-1} - 2 & 1 \end{pmatrix}$$

where $\pi_p := \zeta_p + \zeta_p^{-1} - 2$ generates the unique prime ideal above p in $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ (see [6]).

After constructing the standard idempotents in the matrix ring, one gets

$$\Lambda = \begin{pmatrix} \mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \\ \pi_p\mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \end{pmatrix}$$

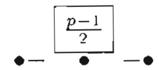
and the mapping to $\mathbb{F}_p C_2$ equals reduction modulo

$$J = \begin{pmatrix} \pi_p\mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \\ \pi_p\mathbb{Z}[\pi_p] & \pi_p\mathbb{Z}[\pi_p] \end{pmatrix} .$$

We get

$$\mathbb{Z}D_p = \{(u, \begin{pmatrix} x & y \\ z & w \end{pmatrix}, v) \in \mathbb{Z} \times \begin{pmatrix} \mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \\ \pi_p\mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \end{pmatrix} \times \mathbb{Z} \quad | \quad x - u \in \pi_p\mathbb{Z}[\pi_p],$$
$$u - w \in \pi_p\mathbb{Z}[\pi_p],$$
$$u - v \in 2\mathbb{Z}\} .$$

Localizing at the prime p we obtain the Brauer tree



where the exceptional vertex in the centre has multiplicity $(p-1)/2$. We shall show that this structure has a feature which is common for all blocks of finite groups with cyclic defect groups.

4.2. **Defining Green orders.** We shall define a class of orders with a structure like in the above example. These orders are introduced by Roggenkamp [16] who called them *Green orders*.

Throughout this subsection let R be a local Dedekind domain.

- Let Γ be a tree[18] embedded in the plane[19].
- Choose a local R–torsion R–algebra k finitely generated as R–module.
- Associate to each vertex v of Γ a pair (Ω_v, f_v) where Ω_v is a local R–order in a semisimple algebra A_v, and where f_v is a surjective ring homomorphism $f_v : \Omega_v \longrightarrow k$ with kernel being a principal ideal $a_v\Omega_v$.
- If $v \overset{e}{-} w$ is an edge, then put $\nu_e(v) := w$ and $\nu_e(w) := v$ the mapping giving the other extremity of the edge e. Of course, ν_e is an involution.
- If e is an edge incident to a vertex v of the graph Γ, then set $\alpha_v(e)$ the edge which follows e in the cyclic ordering at the vertex v. We set $\alpha_v^i(e) = \alpha_v^{i-1}(\alpha_v(e))$ and $\alpha_v^1 = \alpha_v$.
- Set $n_v := \#\{\alpha_v^i(e) | i \in \mathbb{N}; e \text{ is an edge incident to } v\}$ for any vertex v of the graph Γ.

  A leaf of Γ is a vertex v with $n_v = 1$.

---

[18] A tree is a finite, connected, undirected graph without cycles.

[19] This is equivalent to saying that one imposes to each vertex a cyclic ordering of the edges incident to the vertex.

- Attach to each vertex v of Γ the order

$$\Lambda_v := \begin{pmatrix} \Omega_v & \dots & \dots & \dots & \Omega_v \\ (a_v) & \Omega_v & & & \vdots \\ \vdots & (a_v) & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a_v) & \dots & \dots & (a_v) & \Omega_v \end{pmatrix}_{n_v \times n_v} .$$

- We denote the pullback

$$(*) \qquad \begin{array}{ccc} \Omega_v - \Omega_w & \longrightarrow & \Omega_v \\ \downarrow & & \downarrow f_w \\ \Omega_w & \xrightarrow{f_v} & k \end{array} .$$

- We form the iterated pullback of the orders Λ_v for each vertex v under the following iterative procedure. Set $\Lambda_g := \bigoplus_{v \in \Gamma_{vertex}} \Lambda_v$.
  1. Fix a leaf[20] v. The edge leaving v is e and $w := \nu_e(v)$. Form the pullback $(*)$ between[21] $(\Lambda_w)_{(1,1)}$ and Ω_v. Set Λ_g the subring of the old Λ_g given by this pullback.
  2. For each $i = 2, \dots, n_w$ form the subring of Λ_g by the pullback $(*)$ between $(\Lambda_w)_{(i,i)}$ and $(\Lambda_{\nu_{\alpha_w^{i-1}(e)}(w)})_{(1,1)}$. Put Λ_g the new subring formed by these pullbacks. Call the vertices $\nu_{\alpha_w^{i-1}(e)}(w)$ reached. Call w saturated.
  3. If there is no vertex which is not yet saturated, then we define the *generic Green order to the tree* Γ *with data* (Ω_v, f_v) *to be* Λ_g. Stop the algorithm!
  4. Else there is a vertex v which is reached and not saturated. Since v is reached, Λ_g contains a pullback between $(\Lambda_v)_{(1,1)}$ and a Ω_w. Set e to be the edge $v - w$. Proceed with 2.

Definition 4.1. The resulting order Λ_g which occurs after executing point 3 in the algorithm is called *generic Green order to the tree* Γ *with data* $(\Omega_v, f_v)_{v \in \Gamma_{vertex}}$.

The reader might like to construct the generic Green order to the tree in Section 3.1.

We remark that the isomorphism type of the generic Green order depends only on the embedded graph and the data. This can be proved since Λ_v contains

[20] A leaf of a graph is a vertex v with $n_v = 1$.

[21] The notation $M_{(i,i)}$ means the (i,i)-entry of the matrix M.

an automorphism conjugation by

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ a_v & 0 & \dots & \dots & 0 \end{pmatrix}_{n\times n}.$$

This induces a cyclic permutation of the diagonal entries, the last becoming the first.

In the following section we shall elaborate on the orders Λ_v.

Definition 4.2. (Roggenkamp [16]) Let R be a Dedekind domain with field of fractions K. An R-order Λ in a separable K-algebra A is called a *Green order* if there is a finite connected tree with vertices $\{v_i\}_{i=0}^n$ and edges $\{e_k\}_{k=1}^n$.

1. The vertices $\{v_i\}_{i=0}^n$ correspond to (not necessarily primitive) central idempotents $\{\eta_i\}_{i=1}^n$ of A with $1 = \sum_{i=0}^n \eta_i$.
2. The edges $\{e_k\}_{k=1}^n$ correspond to a full set of indecomposable projective Λ-lattices $\{P_k\}_{k=1}^n$.
3. The tree and a starting vertex determine[22] a permutation δ of $\{1,\dots,e\}$ and there is a set of Λ-lattices $\{A_i\}_{i=1}^n$ such that
   (a) $KA_i \simeq A\eta_i$ for all $i = 0, 1, \dots, n$
   (b) for all $i = 0, \dots, n$ there are short exact sequences

$$E_{2i} : 0 \longrightarrow A_{2i+1} \longrightarrow P_{\delta(i)} \xrightarrow{\eta_{\delta(i)}} A_{2i} \longrightarrow 0$$

$$E_{2i+1} : 0 \longrightarrow A_{2i+2} \longrightarrow P_i \xrightarrow{\eta_i} A_{2i+1} \longrightarrow 0\,.$$

The term *generic Green order* is used since in Theorem 4.3 (see also the proof of Lemma 3.16) it will be proven that all Green orders are Morita equivalent to generic Green orders.

Example. Let G be a finite group and let R be a complete discrete valuation ring of characteristic 0 with residue field of characteristic p containing all the $|G|^{th}$ roots of unity. Let B be a block of RG with cyclic defect group D. By Theorem 3.3 B is a Green order.

Theorem 4.3. *(Roggenkamp)* [16] *Let Λ be a Green order with tree Γ. Then Λ is Morita equivalent to a generic Green order with tree Γ.*

[22] in the sense described in the discussion in Section 3.2.

We shall give Roggenkamp's proof of Theorem 4.3 in the sequel. For this purpose we shall introduce in the next section another type of orders, which Roggenkamp calls *isotypic orders* in [17]. These are the orders Λ_v in the definition of a generic Green order.

4.3. **The rational components; isotypic orders.** Throughout this subsection let R be a Dedekind domain with field of fractions K. Let Λ be an R-order in a separable K-algebra A.

Definition 4.4. (Roggenkamp) [16] The order Λ is called *isotypic order* provided there is a twosided Λ-ideal J such that

1. $K \cdot J = A$,
2. J is projective as left Λ-module,
3. Λ/J is a direct product of local R-algebras,
4. Λ is nilpotent modulo the Higman ideal $H(\Lambda)$.[23]

Then, J is called associated to Λ or defining ideal of the isotypic order.

One first property is almost immediate:

Λ is isotypic if and only if $\hat{R}_\wp \otimes \Lambda$ is isotypic for all prime ideals $\wp$ of R. Here we denote by $\hat{R}_\wp$ the completion of R at $\wp$.

Proof. If Λ is isotypic, then $\hat{R}_\wp \otimes \Lambda$ is isotypic. In fact, 1. and 3. are clear. 2. and 4. are consequences of the 'change of rings' theorem.

If $\hat{R}_\wp \otimes \Lambda$ is isotypic for all prime ideals $\wp$ of R, then we use the following general property for orders Λ in a separable algebra A over a Dedekind domain R.

If M and L are full R-lattices in the K-vector space V, then $M_\wp = L_\wp$ almost everywhere. Furthermore, for each $\wp$ let there be given for each $\wp$ a full $R_\wp$-lattice $X(\wp)$ such that $X(\wp) = M_\wp$. Then $N := \bigcap X(\wp)$ has the property that $N_\wp = X(\wp)$ for all $\wp \in Spec(R)$.

We apply this to J. We have ideals $J(\wp)$ for all $\wp$ by the definition of isotypic orders. Set $J(\wp) = \Lambda_\wp$ whenever $\Lambda_\wp$ is a maximal order. We form $J := \bigcap_{\wp \in Spec(R)} J(\wp)$. Now, J is projective since $J(\wp) = J_\wp$ is projective for all $\wp$. (This is observed most easily by seeing that $Hom_\Lambda(J, -)$ is exact. This in turn is seen by the 'change of rings theorem'. $KJ = A$ since this holds locally and $\Lambda_\wp/J_\wp$ is $R_\wp$-torsion since $K\Lambda_\wp = KJ_\wp$. Hence,

$$\Lambda/J \simeq \prod_\wp R_\wp \otimes \Lambda/J \simeq \prod_\wp (R_\wp \otimes \Lambda)/(R_\wp \otimes J) \simeq \prod_\wp \Lambda_\wp/J_\wp$$

the latter being a direct product of local algebras by assumption.

This finishes the proof of the observation.

---

[23] The Higman ideal of an R-order Λ is the R-annihilator of $Ext^1_{\Lambda \otimes_R \Lambda^{op}}(\Lambda, -)$. For orders Λ in a separable algebra we have $K \cdot H(\Lambda) = 0$. [18, V. 3.5]

Because of this observation we may assume, to clarify the structure of isotypic orders, that R is a complete discrete valuation domain with residue field $\mathbb{F}$ and radical πR.

Assuming this, the Higman ideal of Λ is a power of πR. By a general property of Jacobson radicals for artinian algebras we obtain that J is nilpotent modulo πR, if and only if $J \leq rad\ \Lambda$. However, J is nilpotent modulo πR if and only if J is nilpotent modulo a certain power of πR. (Just multiply the nilpotency degree by the power which was fixed at the beginning.) So Condition 4. translates to $J \leq rad\ \Lambda$.

Theorem 4.5. *(Roggenkamp [16]) Let R be a complete discrete valuation domain. Assume that Λ is a basic*[24] *isotypic R-order with associated ideal J. Let furthermore Λ be indecomposable as ring.*

Then, there is a local R-order Ω and a regular non unit $a \in \Omega$ such that $a\Omega = \Omega a =: (a)$ and $\Omega/(a\Omega)$ is a local algebra, and a natural number n such that

$$\Lambda \simeq \Lambda_0 = \Lambda_0(\Omega, a, n) := \begin{pmatrix} \Omega & \dots & \dots & \dots & \Omega \\ (a) & \Omega & & & \vdots \\ \vdots & (a) & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a) & \dots & \dots & (a) & \Omega \end{pmatrix}_{n \times n}.$$

Conversely, every such order is isotypic.

Remark 4.6. 1. For Λ_0 one can take the associated ideal

$$J = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ a & 0 & \dots & \dots & 0 \end{pmatrix}_{n \times n} \cdot \begin{pmatrix} \Omega & \dots & \dots & \dots & \Omega \\ (a) & \Omega & & & \vdots \\ \vdots & (a) & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a) & \dots & \dots & (a) & \Omega \end{pmatrix}_{n \times n}$$

which is principal. We call the generating element above ω.

2. The product of local algebras as in 3. in the definition of 'isotypic order' ranges over a set of pairwise isomorphic local algebras $\Omega/(a)$.

Proof.
Λ_0 is isotypic. In fact,

[24] A Noetherian ring is basic if for each decomposition into a direct sum of indecomposable left projective modules $\Lambda = \oplus_{i \in I} P_i$, no two projective summands P_i and P_j with $i \neq j; i, j \in I$ are isomorphic.

- a is regular, hence Part 1. of the definition of an isotypic order follows.
- J is principal, generated by a regular element, hence free and Part 2. follows.
- $\Lambda/J = \prod_{i=1}^{n} \Omega/(a)$ is a direct product of local orders, hence we get also Part 3.

a is not a unit. Since Ω is local, J is contained in the radical of Λ. The radical of Λ is the ideal with $rad\Omega$ in the main diagonal and in the lower triangular matrix, Ω in the rest of the entries.

We have to prove the converse. We may order a complete set of projective indecomposable modules $P_1, P_2, \ldots, P_n$ such that $rank_R P_i \leq rank_R P_{i+1}$ for all $i = 1, 2, \ldots, n-1$.

Since for any i the module P_i is projective, there is a Q_i and an integer $m(i)$ such that $P_i \oplus Q_i \simeq \Lambda^{m(i)}$. But then, $J \otimes_\Lambda P_i \oplus J \otimes_\Lambda Q_i \simeq J^m$ which is projective since J is projective. Hence, $J \otimes_\Lambda P_i$ is projective. Then, for any i the set

$$\mathcal{P}_i := \{ \text{ isomorphism classes of the modules } J^j \otimes_\Lambda P_i | j \in \mathbb{N}\}$$

consists of projective modules. The cardinality of this set is $k(i)$.

Claim 4.7. $J^{k(1)} \otimes_\Lambda P_1 \simeq P_1$. *Furthermore,* $k(1) = n$ *and hence* $\mathcal{P}_1$ *is a complete set of isomorphism classes of projective modules.*

Proof. Since $KJ = A$, we get $rank_R P_1 = rank_R J^j \otimes_\Lambda P_1$ and we may order the projective indecomposable modules such that $J^j \otimes_\Lambda P_1 = P_{1+j}$. Let $Q = \{Q_1, Q_2, \ldots, Q_s\}$ be a set of representatives of isomorphism classes of indecomposable projective modules such that the isomorphism class of no element of Q is contained in $\mathcal{P}_1$.

Since Λ is indecomposable, there exists Q_i with $Hom_\Lambda(Q_i, P_{1+j}) \neq 0$. If not, the endomorphism ring of Λ would be the direct product of the endomorphism rings of the direct sum of modules in Q and that of $\oplus_{i=0}^{k(1)} P_{1+i}$.

We now reduce to artinian algebras. Take $0 \neq \phi \in Hom_\Lambda(Q_i, P_{1+j})$. Then, there exists $\nu \in \mathbb{N}$ such that the composition

$$Q_i \xrightarrow{\phi} P_{1+j} \longrightarrow (P_{1+j})/(\pi^\nu P_{1+j})$$

is non zero. Since $J \leq rad\,\Lambda$, there exists μ such that $J^\mu \leq \pi^\nu \Lambda$. But, for all k, J^k/J^{k+1} is a projective Λ/J module. However, this is a direct product of local algebras. Hence,

$$J^k \cdot P_{1+j}/J^{k+1} \cdot P_{1+j}$$

is a local Λ/J-module and has all composition factors isomorphic to $P_{1+j}/rad\ P_{1+j}$. Hence, $P_{1+j}/\pi^\nu P_{1+j}$ has composition factors all isomorphic to $P_k/rad\ P_k$ for $k = 1, \ldots, k(1)$. But, the top of Q_i is isomorphic to one of them, hence the isomorphism class of Q_i is in $\mathcal{P}_1$, which is a contradiction and Q is empty.

It remains to prove that $J^{k(1)} \otimes_\Lambda P_1 \simeq P_1$.

If $J^{k(1)} \otimes_\Lambda P_1 \simeq P_k$ for some k, then one forms the set $\mathcal{P}_k$ instead of $\mathcal{P}_1$ and $\{P_1, \ldots, P_{k-1}\}$ would be in Q. The analogous arguments as above lead to a contradiction and the claim is proven.

By Morita theory, $\oplus_{i=1}^n P_i$ gives a Morita bimodule inducing a Morita equivalence between Λ and $End_\Lambda(\oplus_{i=1}^n P_i)$. Since we have the Krull–Schmidt Theorem for Λ and since Λ is basic, $\Lambda \simeq End_\Lambda(\oplus_{i=1}^n P_i)$.

Claim 4.8. *$End_\Lambda(\oplus_{i=1}^n P_i) \simeq \Lambda_0$ for $\Omega := End_\Lambda(P_1)$ and $(a) := Hom_\Lambda(P_1, J^n \otimes_\Lambda P_1)$.*

Proof. Since KP_1 is simple by construction, the isomorphism $P_1 \longrightarrow J^n P_1$ is multiplication by a regular element $a \in \Omega$. Hence,

$$Hom_\Lambda(P_1, J^n \otimes_\Lambda P_1) = \Omega \cdot a\,.$$

Any endomorphism $\phi \in \Omega$ can be extended to an endomorphism of $J^i \otimes_\Lambda P_1$ by $id_{J^i} \otimes \phi$. Hence,

$$\Omega \leq End_\Lambda(P_2) \leq \cdots \leq End_\Lambda(P_{n-1}) \leq End_\Lambda(P_n) \leq End_\Lambda(J^n \otimes_\Lambda P_1) = a^{-1} \cdot \Omega \cdot a\,.$$

But then, all the endomorphism rings are equal, Ω being noetherian.

Since when tensored by K over R, all the P_i are isomorphic and simple, any non zero mapping $\phi : P_i \longrightarrow P_{i+j}$ for $i = 1, \ldots, n$ and $j = 1-i, \ldots, n-i$ is injective. Looking at the tops of the modules $P_{i+j}/J^k P_{i+j}$ by the arguments given in the proof of the preceding claim, $Hom_\Lambda(P_i, P_j) = \Omega$ if $j \leq i$ and $Hom_\Lambda(P_i, P_j) = \Omega \cdot a$ if $j > i$.

This proves the claim and also the theorem.

4.4. **Structure theorem for Green orders.** Proof of Theorem 4.3. We assume that Λ is basic. The proof will be done in several steps.

Lemma 4.9. *Let Λ be a basic Green order with connected tree Γ and let v be a leaf. Let e be the edge joining v with some vertex w. The projective indecomposable module associated with e is P_0. Then, $End_\Lambda(\Lambda/P_0)$ is a Green order with tree Γ' where $\Gamma'_{vertex} = \Gamma_{vertex} \setminus \{v\}$ and $\Gamma'_{edge} = \Gamma_{edge} \setminus \{e\}$.*

Proof. Without loss of generality we may set $A_0 = \Lambda\eta_0$. Set $P := \Lambda/P_0$. Apply $F := Hom_\Lambda(P, -)$ to E_m for all $m = 0, \ldots 2n-1$. Since P is projective, this functor is exact. However, since there is only one projective indecomposable module, namely P_0, $F(A_0) = 0$. But then $F(E_{2i})$ for $i = 1, \ldots, n$ and $F(E_{2i+1})$ for $i = 0, \ldots, n-1$ are the required exact sequences.

This finishes the proof of the lemma.

Lemma 4.10. *Let Λ be a basic R-order in the separable algebra A. Let e be a central idempotent of A and let $P|\Lambda$ with $P \cdot e = 0$. Put $End_\Lambda(\Lambda/P) =: \Lambda_0$ as* *a subring of* Λ *(not with the same unit!). Then,*

$$\Lambda \cdot e = \Lambda_0 \cdot e \text{ and } \Lambda_0 \cdot e \cap \Lambda = \Lambda_0 \cdot e \cap \Lambda_0 .$$

Proof. To prove the first statement one observes that

$$\begin{aligned}
\Lambda e &= e\Lambda e \\
&= e End_\Lambda((\Lambda/P) \oplus P)e \\
&= e \begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P), P) \\ Hom_\Lambda(P, \Lambda/P) & End_\Lambda(P) \end{pmatrix} e \\
&= \begin{pmatrix} e End_\Lambda(\Lambda/P)e & 0 \\ 0 & 0 \end{pmatrix} \\
&= \Lambda_0 e .
\end{aligned}$$

The second statement is proved as follows:
By the first statement,

$$\Lambda_0 \cap (\Lambda_0 e) = \Lambda_0 \cap (\Lambda e)$$

and clearly

$$\Lambda \cap (\Lambda e) \supseteq \Lambda_0 \cap (\Lambda e) .$$

Hence we have to show that

$$\Lambda \cap (\Lambda e) \subseteq \Lambda_0 \cap (\Lambda e) .$$

This is done by the following computation:

$$\begin{aligned}
\Lambda \cap (\Lambda e) &= \begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P), P) \\ Hom_\Lambda(P, \Lambda/P) & End_\Lambda(P) \end{pmatrix} \\
&\quad \cap e \begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P), P) \\ Hom_\Lambda(P, \Lambda/P) & End_\Lambda(P) \end{pmatrix} e \\
&= \begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P), P) \\ Hom_\Lambda(P, \Lambda/P) & End_\Lambda(P) \end{pmatrix} \\
&\quad \cap \begin{pmatrix} e End_\Lambda(\Lambda/P)e & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} End_\Lambda(\Lambda/P) & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} e End_\Lambda(\Lambda/P)e & 0 \\ 0 & 0 \end{pmatrix} \\
&= \Lambda_0 \cap (\Lambda_0 e) .
\end{aligned}$$

Claim 4.11. *Let* Λ *be a basic Green order with tree* Γ. *For the idempotent* η *corresponding to a vertex* v *the ring* $\Lambda/(\Lambda \cap \Lambda\eta)$ *is a direct product of local algebras.*

Proof. We use induction on the number of vertices.

If the Green order has only 2 vertices, then the statement is clear since there is just one projective indecomposable module and the Green order is local.

Let then v be a leaf of the tree Γ. Let e be the edge of the tree that links v with the rest of the tree. Let P_0 be the projective indecomposable which corresponds to e. Set $P = \Lambda/P_0$ and $\Lambda_0 = End_\Lambda(P)$. Then, by Lemma 4.10 the tree Γ' defined by

$$\Gamma_{vertex} \setminus \{v\} =: \Gamma'_{vertex}; \Gamma_{edge} \setminus \{e\} =: \Gamma'_{edge}$$

defines a Green order structure on Λ_0.

Let $v \overset{e}{-} w$ and let v' with $w \neq v' \neq v$ be a vertex of Γ. Then, for v' the statement is true by induction.

We need a proof for w and its central idempotent η_w only. Let η_v be the central idempotent corresponding to v.

$\epsilon := 1 - \eta_v - \eta_w$. We get the following pullback diagram.

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda(\epsilon + \eta_v) \\ \downarrow & & \downarrow \\ \Lambda\eta_w & \longrightarrow & \Lambda(\epsilon + \eta_v)/(\Lambda \cap \Lambda(\epsilon + \eta_v)) \,. \end{array}$$

We see that

$$\Lambda(\epsilon + \eta_v) = \Lambda\epsilon \oplus \Lambda\eta_v$$

and

$$\Lambda(\epsilon + \eta_v)/(\Lambda \cap \Lambda(\epsilon + \eta_v)) = [\Lambda\epsilon/\Lambda \cap \Lambda\epsilon] \oplus [\Lambda\eta_v/\Lambda \cap \Lambda\eta_v] \,.$$

But, by Lemma 4.10

$$\Lambda\epsilon/(\Lambda \cap \Lambda(\epsilon)) = \Lambda_0\epsilon/(\Lambda_0 \cap \Lambda_0(\epsilon))$$

and since ϵ is also a central idempotent of $K\Lambda_0$, which gives rise to a pullback diagram itself with quotient $\mathcal{A} := \Lambda_0\epsilon/(\Lambda_0 \cap \Lambda_0(\epsilon))$, the ring $\mathcal{A}$ is a direct product of local R-algebras by induction.

Now we use the fact that v is a leaf. If $\Lambda\eta_v$ was not local, it would have two non isomorphic simple modules. Since $\Lambda\eta_v$ is an image of Λ, Λ itself has these two non isomorphic modules. However, there is only one simple Λ-module, namely the top of P_0, on which $Hom_\Lambda(P_0, -)$ is non zero. The two simple Λ-modules constructed above however have this property, and hence they cannot exist. We conclude, $\Lambda\eta_v$ is local.

Since $\Lambda\eta_v$ is local, $\Lambda\eta_v/(\Lambda\eta_v \cap \Lambda)$ is local.

Lemma 4.12. *Let Λ be a basic Green order with tree Γ. Let v be a vertex of Γ with corresponding central idempotent η_v. Then, $\Lambda\eta_v \cap \Lambda$ is a free $\Lambda\eta_v$-module.*

Proof. Again we use induction on the number of vertices of Γ.

Assume that there are only two vertices. Hence, there are short exact sequences

$$0 \longrightarrow A_1 \longrightarrow \Lambda \xrightarrow{\eta_v} A_0 \longrightarrow 0$$
$$0 \longrightarrow A_0 \longrightarrow \Lambda \xrightarrow{1-\eta_v} A_1 \longrightarrow 0\,.$$

Thus, $\Lambda(1-\eta_v) \cap \Lambda \simeq A_0 \simeq \Lambda\eta_v$.

Assume we have more than two vertices.

If v is a leaf with idempotent η_v, let $v \overset{e}{-} w$ be the edge of the tree linking v with the rest of Γ. The idempotent associated with w is denoted by η_w. Hence, there is an indecomposable projective Λ-module P_0, and there are short exact sequences

$$0 \longrightarrow A_0 \longrightarrow P_0 \xrightarrow{\eta_w} A_{-1} \longrightarrow 0$$
$$0 \longrightarrow A_1 \longrightarrow P_0 \xrightarrow{\eta_v} A_0 \longrightarrow 0\,.$$

Let $P_0 = \Lambda e_0$ for an idempotent e_0 of Λ. Since v is a leaf, $\eta_v P_0 = \eta_v \Lambda$; the proof is the same as proving that $\Lambda\eta_v$ is local. Now, since $\eta_w P_0 = (1-\eta_v)P_0$,

$$\Lambda\eta_v \cap \Lambda = \eta_v P_0 \cap \Lambda = \eta_v P_0 \cap P_0 \simeq A_0 = \eta_v P_0\,.$$

If v_2 is a vertex different from v and w, then let η_{v_2} be the central idempotent associated to v_2. Then, since

$$\Lambda\eta_{v_2} \cap \Lambda = \Lambda_0\eta_{v_2} \cap \Lambda_0 \text{ and } \Lambda\eta_{v_2} = \Lambda_0\eta_{v_2}\,,$$

for $\Lambda_0 = End_\Lambda(\Lambda/P_0)$, by Lemma 4.10, the statement is true by the induction hypothesis.

We have to prove the statement for w. Let $\{e_1, e_2, \ldots, e_n\}$ be the set of edges adjacent to w. Let $\{P_1, P_2, \ldots, P_n\}$ be the corresponding projective indecomposable modules. Set $\Delta := End_\Lambda(\bigoplus_{i=1}^n P_i)$. Then, again by Lemma 4.10

$$\Delta\eta_w = \Lambda\eta_w \text{ and } \Delta\eta_w \cap \Delta = \Lambda\eta_w \cap \Lambda\,.$$

Hence we may prove the statement for Γ being a star and w in the centre and number the edges in their cyclic ordering. Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be the idempotents corresponding to the vertices adjacent to $\{e_1, e_2, \ldots, e_n\}$ but unequal to w. The idempotent corresponding to w is η. By the definition of a Green order we get short exact sequences

$$0 \longrightarrow A_1 \longrightarrow P_1 \xrightarrow{\eta_1\cdot} A_0 \longrightarrow 0$$
$$0 \longrightarrow A_2 \longrightarrow P_2 \xrightarrow{\eta\cdot} A_1 \longrightarrow 0$$
$$0 \longrightarrow A_3 \longrightarrow P_2 \xrightarrow{\eta_2\cdot} A_2 \longrightarrow 0$$
$$0 \longrightarrow A_4 \longrightarrow P_3 \xrightarrow{\eta\cdot} A_3 \longrightarrow 0$$
$$0 \longrightarrow A_5 \longrightarrow P_3 \xrightarrow{\eta_3\cdot} A_4 \longrightarrow 0$$

$$\cdots\cdots$$
$$0 \longrightarrow A_{2n-1} \longrightarrow P_n \xrightarrow{\eta_n} A_{2n-2} \longrightarrow 0$$
$$0 \longrightarrow A_0 \longrightarrow P_1 \xrightarrow{\eta} A_{2n-1} \longrightarrow 0\,.$$

For computing $\Lambda\eta \cap \Lambda$ we have to sum up every second kernel and get that

$$\Lambda\eta \cap \Lambda \simeq \bigoplus_{i=1}^{n} A_{2i} = \bigoplus_{i=1}^{n} \eta P_i = \eta \bigoplus_{i=1}^{n} P_i$$

is a progenerator. If we furthermore assume that the Green order is basic, $\Lambda\eta\cap\Lambda$ is a free $\Lambda\eta$–module.

This proves the lemma.

To prove the theorem we just have to assemble the different parts.

We see that for each vertex v the order $\Lambda_v := \Lambda\eta_v$ is an isotypic order with defining ideal $J_v := \Lambda\eta_v \cap \Lambda$.

In fact,

- $K \otimes_R (\Lambda\eta_v \cap \Lambda) = K \otimes_R \Lambda\eta_v$,
- J_v is a free $\Lambda\eta_v$–ideal by Lemma 4.12,
- and $\Lambda\eta_v/(\Lambda\eta_v \cap \Lambda)$ is a direct product of local R–algebras by Lemma 4.9.
- The ideal J_v for all vertices v is contained in the radical of Λ_v: Let S be a simple Λ_v–module. One has to show that J_v acts as 0 on S. Since $\Lambda \longrightarrow\!\!\!\!\rightarrow \Lambda_v$, each simple Λ_v–module is also a simple Λ–module. But, we get n simple Λ–modules just by the local algebras which we get in the quotients Λ_v/J_v, where n is the number of edges of Γ. All these are annihilated by all the J_v. S is one of them.

We get the structure of Λ_v by Theorem 4.5. The Ω_v in the main diagonals of the matrices correspond to the projective indecomposable modules, as one sees from the proof of Theorem 4.5. For Green orders these are just the $\eta_v P$ for η_v being the idempotent corresponding to v and for P being a projective indecomposable module corresponding to an edge incident to v. The pullbacks linking the different Λ_v are as constructed in the example since this is the way the exact sequences corresponding to the tree are built.

Remark 4.13. In [19, 9] it is proven that two Green orders having the same data (Ω_v, f_v) but not necessarily the same underlying tree have equivalent bounded derived module categories. An explicit twosided tilting complex which provides this derived equivalence is given in [20].

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ON STABLE AND DERIVED EQUIVALENCES OF BLOCKS AND ALGEBRAS

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In the representation theory of finite groups, the concepts of derived and stable categories turned out to be of considerable relevance during the last years. After recalling some basic properties of algebras in section 1, we give the definitions and fundamental properties of stable and derived categories in the sections 2 and 3, respectively. Section 4 is then devoted to applications to the theory of blocks of finite groups and in section 5, we go deeper in the study of invariants of stable equivalences. Finally, section 6 sketches some results and conjectures in the theory of blocks of finite groups.

1. Generalities on algebra

Throughout these notes we fix a prime p and a complete discrete valuation ring $\mathcal{O}$ having a residue field k with characteristic p and a quotient field K. The case $K = \mathcal{O} = k$ is not excluded, unless stated explicitly otherwise. Remind that $\mathcal{O}$ has a unique maximal ideal, namely its radical $J(\mathcal{O})$, and this is a principal ideal; we denote by π an element of $\mathcal{O}$ such that $J(\mathcal{O}) = \pi\mathcal{O}$.

In most applications to finite groups however $\mathcal{O}$ will have characteristic zero, since $\mathcal{O}$ will be the "link" between representations of a finite group in zero and non zero characteristic.

By an $\mathcal{O}-$algebra we mean an associative unitary algebra A which is $\mathcal{O}-$free of finite rank as $\mathcal{O}-$module; an $A-$module is always a unitary finitely generated left module (not necessarily $\mathcal{O}-$free), unless stated otherwise. We denote by $J(A)$ the Jacobson radical of A; that is, the annihilator of all simple $A-$modules. Remind that $J(\mathcal{O})A \subset J(A)$; thus any simple $A-$module can be viewed as simple module over the $k-$algebra $k \underset{\mathcal{O}}{\otimes} A$ (and, of course, vice versa). If U is an

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A–module, we set $rad(U) = J(A)U$, called the *radical of* U; this is the intersection of all maximal submodules of U, or, equivalently, the smallest submodule of U for which the quotient module $U/rad(U)$ is semisimple (i.e. a direct sum of simple modules). Dually, we denote by $soc(U)$ the sum of all minimal (i.e. simple) submodules of U; this is the largest semisimple submodule of U, called the *socle of* U. This notion will play a rôle only in the context of k–algebras, since if U is $\mathcal{O}$–free and $\mathcal{O} \neq k$, then U has no simple submodule: indeed, for any non zero submodule V of U, multiplication by π is an isomorphism from V to its proper submodule $J(\mathcal{O})V$.

We denote by $A^{\times}$ the group of invertible elements in A and we will say that two elements (or sometimes two subsets) a, a' in A are *conjugate in* A, if there is $u \in A^{\times}$ such that $uau^{-1} = a'$.

If A, B are $\mathcal{O}$–algebras, an $A - B$–bimodule M is a bimodule whose left and right $\mathcal{O}$–module structure coincide and which hence may be regarded as $A \underset{\mathcal{O}}{\otimes} B^0$–module, where B^0 is the algebra obtained by endowing B with the opposite product.

We remind that an A–module U is called *projective* if it is isomorphic to a direct summand to a free A–module $A^n \cong A \underset{\mathcal{O}}{\otimes} \mathcal{O}^n$ for some positive integer n and it is called *relatively* $\mathcal{O}$*–projective* if it is isomorphic to a direct summand of $A \underset{\mathcal{O}}{\otimes} V$ for some (not necessarily free) $\mathcal{O}$–module V. Equivalently, U is relatively $\mathcal{O}$–projective if and only if the module $k \underset{\mathcal{O}}{\otimes} U$ viewed as module over the k–algebra $k \underset{\mathcal{O}}{\otimes} A$ is projective.

For an $\mathcal{O}$–algebra A we denote by $Mod(A)$ the category of (finitely generated unitary left) A–modules. Note that if U is an A–module, then its $\mathcal{O}$–dual $U^* = Hom_{\mathcal{O}}(U, \mathcal{O})$ becomes a right A–module through $(f.a)(u) = f(au)$ for any $f \in U^*$, $a \in A$ and $u \in U$. Similarly, if A, B are $\mathcal{O}$–algebras and M is an $A - B$–bimodule, its $\mathcal{O}$–dual becomes a $B - A$–bimodule.

If A is a k–algebra, k–duality maps projective modules to injective right modules and vice versa. For $\mathcal{O}$–algebras, if $\mathcal{O} \neq k$, this is no longer true since there are no non zero injective modules but only relatively $\mathcal{O}$–injective modules.

We quote now some standard properties of idempotents and projective covers. An idempotent in an $\mathcal{O}$–algebra A is a nonzero element i in A satisfying $i^2 = i$; two idempotents i, j in A are *orthogonal* if $ij = 0 = ji$, and i is called *primitive*, if it is not the sum of two orthogonal idempotents. A *primitive decomposition of an idempotent* e *in* A is a set I of pairwise orthogonal primitive idempotents in A satisfying $\sum_{i \in I} i = e$.

Theorem 1.1. *Let A, B be $\mathcal{O}$−algebras, $f : A \longrightarrow B$ be a surjective algebra homomorphism, i, j be primitive idempotents in A and e be any idempotent in A.*

(i) The projective indecomposable A−modules Ai, Aj are isomorphic if and only if i, j are conjugate in A;

(ii) The correspondence sending the projective indecomposable A−module Ai to the A−module $S_i = Ai/J(A)i$ induces a bijection between the sets of isomorphism classes of projective indecomposable and simple A−modules;

(iii) Any two primitive decompositions of e in A are conjugate;

(iv) Either $f(i) = 0$ or $f(i)$ is again a primitive idempotent in B: moreover, if both $f(i)$ and $f(j)$ are non zero, then i, j are conjugate in A if and only if $f(i)$, $f(j)$ are conjugate in B.

As a consequence of 1.1(ii) we have:

Proposition 1.2. *Let A be an $\mathcal{O}$−algebra. For any A−module U there is up to isomorphism a unique pair (P_U, π_U), called minimal projective cover of U, consisting of a projective A−module P_U and a surjective A−homomorphism $\pi_U : P_U \longrightarrow U$ which induces an isomorphism $P_U/rad(P_U) \cong U/rad(U)$, and then $\Omega_A(U) = ker(\pi_U)$ is contained in $rad(P_U)$.*

Indeed, by 1.1(ii) there is up to isomorphism a unique projective A−module P_U such that $P_U/rad(P_U) \cong U/rad(U)$, and by the projectivity of P_U, any such isomorphism lifts to an A−homomorphism $\pi_U : P_U \longrightarrow U$, which is surjective by Nakayama's Lemma.

The operator Ω_A defined by taking the kernel of a minimal projective cover as in 1.2 is called *Heller operator of A*; if $A = \mathcal{O}G$ for some finite group G, we write Ω_G instead of Ω_A and even sometimes just Ω, if A is clear from the context. In order to use the Heller operator we always assume implicitly a choice of projective covers, but 1.2 asserts, that the Heller operator is up to isomorphism independent of such a choice.

Using the fact that k−duality maps projective modules to injective modules, we have a dual version of 1.2, namely

Proposition 1.3. *Let A be a k−algebra. For any A−module U, there is up to isomorphism a unique pair (I_U, ι_U), called minimal injective envelope of U, consisting of an injective A−module I_U and an injective A−homomorphism $\iota_U : U \longrightarrow I_U$ mapping $soc(U)$ onto $soc(I_U)$, and then we set $\Omega_A^{-1}(U) = coker(\iota_U)$.*

Note that 1.1 (iii) implies the Krull-Schmidt Theorem:

Theorem 1.4. *Let A be an $\mathcal{O}$−algebra and U an A−module. If $U = U_1 \oplus .. \oplus U_n = V_1 \oplus .. \oplus V_m$ are two decompositions of U as direct sums of indecomposable modules U_i, V_j, then $n = m$ and there is a permutation σ of $\{1, 2, .., n\}$ such that $U_i \cong V_{\sigma(i)}$ for $1 \leq i \leq n$.*

Indeed, a decomposition of U as a direct sum of indecomposable modules corresponds naturally to a primitive decomposition of Id_U in $End_A(U)$, and so 1.4 follows from 1.1 (iii).

If A is an $\mathcal{O}$–algebra, U an A–module and α an automorphism of A, we denote by ${}_\alpha U$ the module obtained from U by "twisting" with α: that is, as $\mathcal{O}$–module we have ${}_\alpha U = U$ and $a \in A$ acts on ${}_\alpha U$ as left multiplication by $\alpha(a)$. Similarly for right modules and bimodules. Clearly "twisting by α" extends to a functor on $Mod(A)$, namely to the functor ${}_\alpha A \underset{A}{\otimes} -$, and this is an equivalence with inverse functor ${}_{\alpha^{-1}}A \underset{A}{\otimes} -$. This is a particular case of a more general concept:

Theorem 1.5. *If A, B are $\mathcal{O}$–algebras with equivalent module categories $Mod(A) \cong Mod(B)$, there is a $B-A$–bimodule M and an $A-B$–bimodule N such that $N \underset{B}{\otimes} M \cong A$ as $A-A$–bimodules and $M \underset{A}{\otimes} N \cong B$ as $B-B$–bimodules.*

Clearly in the situation of 1.5, the functors $M \underset{A}{\otimes} - : Mod(A) \longrightarrow Mod(B)$ and $N \underset{B}{\otimes} - : Mod(B) \longrightarrow Mod(A)$ are mutually inverse equivalences, and we will say that *M and N induce a Morita equivalence between A and B.*

Most of the time we will deal with particular classes of algebras:

Definition 1.6.

(i) An $\mathcal{O}$–algebra A is called symmetric if $A \cong A^*$ as $A-A$–bimodules.

(ii) A k–algebra A is called selfinjective if every projective A–module is also injective.

Again, since k–duality maps projective modules to injective modules, it is easy to see that a symmetric k–algebra is selfinjective. The reason why symmetric algebras come in is that group algebras of finite groups are symmetric: indeed, if G is a finite group, it is elementary to check, that the map sending $\varphi \in (\mathcal{O}G)^*$ to $\sum_{x \in G} \varphi(x^{-1})x \in \mathcal{O}G$ is an isomorphism of $\mathcal{O}G - \mathcal{O}G$–bimodules.

We recall the following properties of selfinjective algebras:

Proposition 1.7. *Let A be a selfinjective k–algebra.*

(i) The right and left annihilator of $J(A)$ in A both coincide with $soc(A)$ and then $J(A)$ is the left and right annihilator of $soc(A)$; in particular, $soc(A)$ annihilates no non zero projective module.

(ii) For any projective indecomposable A–module W, $soc(W)$ is simple, and the map sending $W/rad(W)$ to $soc(W)$ induces a permutation on the set of isomorphism classes of simple A–modules; in particular, every simple A–module is isomorphic to a direct summand of $soc(A)$ as left A–module.

(iii) If A is moreover symmetric, then $soc(W) \cong W/rad(W)$ for any projective indecomposable A–module W.

Remark. Detailed proofs of the material of this section can be found in Thévenaz [27].

2. Stable categories and equivalences

The *stable category* $\overline{\mathcal{C}}$ of an additive category $\mathcal{C}$ is the category whose objects are the objects of $\mathcal{C}$ and such that the morphism space $Mor_{\overline{\mathcal{C}}}(U, V)$ is the quotient of $Mor_{\mathcal{C}}(U, V)$ by the subspace of all morphisms from U to V which factor through a projective object in $\mathcal{C}$, where U, V are any objects in $\mathcal{C}$.

If we are dealing with $\mathcal{O}$-algebras, we need a slightly different concept:

Definition 2.1. *Let A be an $\mathcal{O}$-algebra. The $\mathcal{O}$-stable category $\overline{Mod}(A)$ of A is the $\mathcal{O}$-linear category whose objects are the objects of $Mod(A)$ such that for any two A-modules U, V, the morphism space $\overline{Hom}_A(U, V)$ in $\overline{Mod}(A)$ is the quotient space of $Hom_A(U, V)$ by the subspace of all A-homomorphisms from U to V which factor through a relatively $\mathcal{O}$-projective A-module.*

The above definition amounts to saying that we identify to zero all relatively $\mathcal{O}$-projective modules. Clearly, if $\mathcal{O} = k$ we obtain just the usual notion of a stable category. In order to deal with stable categories, it is necessary to have criteria, when a homomorphism between two modules does factor through a (relatively $\mathcal{O}$-) projective module:

Proposition 2.2. *Let A be an $\mathcal{O}$-algebra, U, V be indecomposable non projective A-modules and $\varphi \in Hom_A(U, V)$. If φ factors through a relatively $\mathcal{O}$-projective A-module, then $Im(\varphi) \subset rad(V)$; in particular, φ is not surjective.*

Proof. We clearly may assume that $\mathcal{O} = k$. If the image of φ is not contained in $rad(V)$, there is a simple quotient S of V such that the composition $U \xrightarrow{\varphi} V \xrightarrow{\tau} S$ is surjective, where τ is the canonical surjection. Moreover, this composition $\tau\varphi$ factors through a projective module and hence factors through a minimal projective cover $\pi_S : P_S \longrightarrow S$. Since $ker(\pi_S)$ is the unique maximal submodule of P_S, this is only possible, if U maps onto P_S. Since any surjective map from U to P_S splits, this contradicts the hypotheses, and the proposition follows.

For selfinjective algebras we get a dual statement:

Proposition 2.3. *Let A be a selfinjective k-algebra, U, V be indecomposable non projective A-modules and $\varphi \in Hom_A(U, V)$. If φ factors through a projective A-module, then $Im(\varphi) \subset rad(V)$ and $soc(U) \subset ker(\varphi)$. In particular, φ is neither surjective nor injective, and hence, if one of U or V is simple, then $\overline{Hom}_A(U, V) \cong Hom_A(U, V)$.*

Definition 2.4. *An $\mathcal{O}$-stable equivalence between two $\mathcal{O}$-algebras A, B is an equivalence of the $\mathcal{O}$-stable categories $\overline{Mod}(A) \cong \overline{Mod}(B)$; in that case, we will say that A and B are stably equivalent.*

One of the most prominent examples of a stable equivalence arises from the Heller operator of a self-injective algebra (see e.g.[1, ch. 20]):

Proposition 2.5. *Let A be a self-injective k-algebra. Then Ω_A and Ω_A^{-1} extend to mutually inverse stable equivalences on $\overline{Mod}(A)$.*

Proof. We consider diagrams of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega U & \longrightarrow & P_U & \xrightarrow{\pi_U} & U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \Omega V & \longrightarrow & P_V & \xrightarrow[\pi_V]{} & V & \longrightarrow & 0
\end{array}$$

where U, V are A-modules. Given α, using the projectivity of P_U, it is easy to see that there are β and γ which make the above diagram commutative.

Conversely, given γ, using the injectivity of P_V, it is again easy to see, that there are β and α which make this diagram commutative.

Moreover, it is not hard to check that if this diagram is commutative, α factors through a projective module if and only γ does so. Thus the correspondence sending α to γ induces an isomorphism $\overline{Hom}_A(U, V) \cong \overline{Hom}_A(\Omega U, \Omega V)$, and 2.5 follows.

In view of possible generalizations of 2.5 to $\mathcal{O}$-algebras, we observe the following: if A is a k-algebra and U an A-module, applying k-duality to the exact sequence $0 \longrightarrow \Omega_A(U^*) \longrightarrow P_{U^*} \longrightarrow U^* \longrightarrow 0$ yields an exact sequence $0 \longrightarrow U \longrightarrow (P_{U^*})^* \longrightarrow (\Omega_A(U^*))^* \longrightarrow 0$ which, compared to the exact sequence $0 \longrightarrow U \longrightarrow I_U \longrightarrow \Omega_A^{-1}(U) \longrightarrow 0$ shows that I_U is isomorphic to a direct summand of $(P_{U^*})^*$ by the uniqueness of minimal injective envelopes and the fact that $(P_{U^*})^*$ is injective as it is the dual of a projective module. It follows that $(\Omega_A(U^*))^*$ is isomorphic to the direct sum of $\Omega_A^{-1}(U)$ plus possibly some injective A-module; actually, if A is self-injective, necessarily $\Omega_A^{-1}(U) \cong (\Omega_A(U^*))^*$ since any injective direct summand of the latter module is also projective and hence splits off in the corresponding exact sequence above. If A is an $\mathcal{O}$-algebra and U an $\mathcal{O}$-free A-module, we may take $\Omega_A^{-1}(U) = (\Omega_A(U^*))^*$ as a definition and it is not difficult to see, that the arguments used in the proof of 2.5 may be generalized to obtain an $\mathcal{O}$-stable equivalence on the subcategory of $Mod(A)$ of $\mathcal{O}$-free A-modules. Using suitable bimodules, it is possible to show, that this can be extended to the whole category $Mod(A)$; see 2.7 below.

$\mathcal{O}-$stable equivalences arise frequently between blocks of finite groups; the most well-known examples are blocks with cyclic defect groups (see section 4 below). In general an equivalence between stable categories need not be induced by a functor between the considered categories themselves. In the context of blocks of finite groups however, all known $\mathcal{O}-$stable equivalences are actually induced by exact functors between the module categories, namely by tensoring with suitable bimodules. This motivates the following definition of a particular class of $\mathcal{O}-$stable equivalences, due to M. Broué:

Definition 2.6. *(Broué [6, section 5]) Let A, B be $\mathcal{O}-$algebras, M a $B-A-$bimodule and N an $A-B-$bimodule. We say that M, N induce a stable equivalence of Morita type between A and B, if M and N are projective both as left and right modules, and if*

$$N \underset{B}{\otimes} M \cong A \oplus X$$

as $A-A-$bimodules, where X is a projective $A-A-$bimodule, and

$$M \underset{A}{\otimes} N \cong B \oplus Y$$

as $B-B-$bimodules, where Y is a projective $B-B-$bimodule.

Clearly, in that situation, the functors $M \underset{A}{\otimes} -$ and $N \underset{B}{\otimes} -$ induce mutually inverse equivalences between the $\mathcal{O}-$stable categories $\overline{Mod}(A)$ and $\overline{Mod}(B)$. Another motivation for this definition comes from Broué's observation, that if A and B are symmetric $\mathcal{O}$-algebras, a derived equivalence $D^b(A) \cong D^b(B)$ induced by a two-sided tilting complex of $A-B-$bimodules which are projective as left and right modules, induces actually a stable equivalence of Morita type. Note that M, N induce a Morita equivalence precisely if X and Y are zero; in fact, if one of them is zero, so is the other.

The next proposition is the expected generalization of 2.5:

Proposition 2.7. *Let A be an $\mathcal{O}-$algebra such that $k \underset{\mathcal{O}}{\otimes} A$ is self-injective. For any non negative integer n, the $A-A-$bimodules $\Omega^n_{A\otimes A^0}(A)$ and $(\Omega^n_{A\otimes A^0}(A^*))^*$ induce a stable equivalence of Morita type on A.*

As mentioned above, in block theory, there occur many examples of stable equivalences of Morita type, though, in most cases, it is not known whether they "lift" to derived equivalences. We treat the main examples in section 4 below.

We give now the basic properties of stable equivalences of Morita type (cf. [17],section 2 or [18], section 3).

Theorem 2.8. *Let A, B be $\mathcal{O}$–algebras such that $k \underset{\mathcal{O}}{\otimes} A$, $k \underset{\mathcal{O}}{\otimes} B$ are self-injective indecomposable non-simple k–algebras whose semi-simple quotients are separable. Let M be a $B-A$–bimodule and N an $A-B$–bimodule such that M, N induce a stable equivalence of Morita type between A and B.*

(i) The $B-A$–bimodule M has, up to isomorphism, a unique indecomposable non projective direct summand M', and then $k \underset{\mathcal{O}}{\otimes} M'$ is, up to isomorphism, the unique indecomposable non projective direct summand of $k \underset{\mathcal{O}}{\otimes} M$ as $k \underset{\mathcal{O}}{\otimes} B - k \underset{\mathcal{O}}{\otimes} A$–bimodule.

(ii) If M is indecomposable as $B-A$–bimodule, for every simple A–module S the B–module $M \underset{A}{\otimes} S$ is indecomposable and non projective as $k \underset{\mathcal{O}}{\otimes} B$–module.

(iii) The functor $M \underset{A}{\otimes} -$ from $Mod(A)$ to $Mod(B)$ is an equivalence if and only if, for any simple A–module S, the B–module $M \underset{A}{\otimes} S$ is again simple.

Proof. Let X be a projective $A-A$–bimodule and Y be a projective $B-B$–bimodule such that $N \underset{B}{\otimes} M \cong A \oplus X$ and $M \underset{B}{\otimes} N \cong B \oplus Y$.

(i) (Rouquier[26]) If $M = M' \oplus M''$ for some $B-A$–bimodules M', M'', then $A \oplus X \cong N \underset{B}{\otimes} M \cong (N \underset{B}{\otimes} M') \oplus (N \underset{B}{\otimes} M'')$. This shows that we may assume that $N \underset{B}{\otimes} M''$ is projective as $A-A$–bimodule. Tensoring with $M \underset{A}{\otimes} -$ again yields that M'' is projective, hence (i) follows.

(ii) We clearly may assume that $\mathcal{O} = k$. Since the semi-simple quotients of A, B are separable we have $soc(A \underset{k}{\otimes} B^0) = soc(A) \otimes soc(B^0)$, and therefore, if M has no non zero projective direct summand as $B-A$–bimodule, we have $\{0\} = soc(B)Msoc(A) = soc(B)(M \underset{A}{\otimes} soc(A))$, thus $M \underset{A}{\otimes} soc(A)$ has no non zero projective direct summand as B–module, since B is self-injective. Since A is self-injective, any simple A–module is isomorphic to a direct summand of $soc(A)$, which implies (ii).

(iii) If , for any simple A–module S, the B–module $M \underset{A}{\otimes} S$ is simple, then $S \cong N \underset{B}{\otimes} M \underset{A}{\otimes} S$ by (ii), thus $S \cong S \oplus (X \underset{A}{\otimes} S)$, which forces $X = \{0\}$. But then $M \cong M \underset{A}{\otimes} A \cong M \underset{A}{\otimes} N \underset{B}{\otimes} M \cong (B \oplus Y) \underset{B}{\otimes} M \cong M \oplus (Y \underset{B}{\otimes} M)$ and so $Y = \{0\}$. The result follows.

If a $B-A$–bimodule M induces an $\mathcal{O}$–stable equivalence between A and B it is not clear, whether there is always an $A-B$–bimodule N such that M, N induce a stable equivalence of Morita type between A and B. At least for symmetric algebras this is true:

Proposition 2.9. *Let A, B be indecomposable symmetric $\mathcal{O}$–algebras and M a $B-A$–bimodule which is projective as left and right module. If the functor*

The complex $\mathcal{P} = \bigoplus_{i\geq 0} P_i$ with differential $d = (d_n)_n$ is then a right bounded complex of projective $A-$modules whose homology is zero in all non zero degrees and isomorphic to U in degree zero; more precisely, if we consider U as complex concentrated in degree zero, the map π defines a homomorphism of complexes

$$\begin{array}{ccccccccc} .. \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 & \longrightarrow .. \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & \\ .. \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & U & \longrightarrow & 0 & \longrightarrow .. \end{array}$$

which is clearly a quasi-isomorphism, since we have $U = Im(\pi) \cong P_0/ker(\pi) \cong P_0/Im(d_1) = H_0(\mathcal{P})$.

This example has the following generalization (the proof is somewhat technical and left to the reader):

Proposition 3.2. *Let A be an $\mathcal{O}-$algebra. For any right bounded complex $\mathcal{C}$ there is a right bounded complex $\mathcal{P}$ of projective $A-$modules and a quasi-isomorphism $f : \mathcal{P} \longrightarrow \mathcal{C}$, and then the pair $(\mathcal{P}, f)$ is unique up to unique homotopy equivalence.*

Definition 3.3. *Let A be an $\mathcal{O}-$algebra. The bounded derived category of A is the category $D^b(A)$ whose objects are the right bounded complexes $\mathcal{P}$ of projective $A-$modules with bounded homology, and whose morphisms are homotopy equivalence classes of morphisms of complexes.*

Remark. Let A be an $\mathcal{O}-$algebra and denote by $K^b(A)$ the homotopy category of bounded complexes of $A-$modules. It follows from 3.2 that we have a functor

$$K^b(A) \longrightarrow D^b(A)$$

which maps any quasi-isomorphism in $K^b(A)$ to an isomorphism in $D^b(A)$, and it turns out that $D^b(A)$ with this functor is universal with this property. Thus it is possible to show, that an equivalent definition of $D^b(A)$ goes as follows:

- the objects are the bounded complexes of $A-$modules,
- the morphisms between two complexes $\mathcal{C}$, $\mathcal{C}'$ are suitable equivalence classes of symbols $\frac{f}{s}$, where $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a morphism of complexes , and $s : \mathcal{C}' \longrightarrow \mathcal{D}$ is a quasi-isomorphism.

In other words, $D^b(A)$ is obtained from $K^b(A)$ by a procedure formally similar to the localization of a commutative ring at a prime ideal; that is, by making invertible the quasi-isomorphisms, except that technical complications arise from the fact, that this situation is "non commutative". See e.g. [13] for technical details.

Proposition 3.4. *Let A be an $\mathcal{O}-$algebra. The map sending an $A-$module U to a projective resolution $\mathcal{P}$ of U induces a full embedding $Mod(A) \longrightarrow D^b(A)$.*

Even though by the preceding proposition the module category of an $\mathcal{O}-$algebra embeds into its bounded derived category, this does not mean, that a derived equivalence $D^b(A) \cong D^b(B)$ of two $\mathcal{O}-$algebras A, B implies a Morita equivalence $Mod(A) \cong Mod(B)$, since such a derived equivalence need not map the natural image of $Mod(A)$ in $D^b(A)$ to that of $Mod(B)$ in $D^b(B)$. We have the following connection with stable categories:

Proposition 3.5. *Let A be an $\mathcal{O}-$algebra such that $k \underset{\mathcal{O}}{\otimes} A$ is self-injective. The canonical functor $Mod(A) \longrightarrow \overline{Mod}(A)$ extends through the embedding $Mod(A) \longrightarrow D^b(A)$ to a functor $D^b(A) \longrightarrow \overline{Mod}(A)$ making the following diagram commutative:*

$$\begin{array}{ccc} Mod(A) & \longrightarrow & D^b(A) \\ & \searrow & \downarrow \\ & & \overline{Mod}(A) \end{array}$$

If A is a self-injective $k-$algebra, the proof is due to J. Rickard; the idea is as follows: if $\mathcal{P}$ is an object of $D^b(A)$ with differential $d = (d_i)_i$, there is an integer n such that $H_m(\mathcal{P}) = 0$ for all $m \geq n$ since $\mathcal{P}$ has bounded homology. This means that $ker(d_{m+1}) = \Omega(Im(d_{m+1})) = \Omega(ker(d_m))$, and therefore, the map sending $\mathcal{P}$ to $\Omega^{-m}(ker(d_m))$ for some $m \geq n$ does not depend on m and is easily seen to extend to a functor $D^b(A) \longrightarrow \overline{Mod}(A)$.

If A is an $\mathcal{O}-$algebra such that $k \underset{\mathcal{O}}{\otimes} A$ is self-injective, one has just to check that "Ω^{-1}" still makes sense; by the remark following 2.5, for an $\mathcal{O}-$free $A-$module U we can take $(\Omega(U^*))^*$ as $\Omega^{-1}(U)$ and this is sufficient in this context, as $ker(d_m)$ is certainly $\mathcal{O}-$free (since it is a submodule of an $\mathcal{O}-$free module). See also 2.7 above.

Clearly a Morita equivalence implies a derived equivalence, and by the preceding proposition, a derived equivalence implies an $\mathcal{O}-$stable equivalence. The converse implications do not hold in general. However, similarly to what happens in the case of a Morita equivalence, J. Rickard proved under suitable hypotheses, that if two $\mathcal{O}-$algebras are derived equivalent, there are complexes of bimodules inducing mutually inverse derived equivalences. More precisely:

Theorem 3.6. *(Rickard) Let A, B be $\mathcal{O}-$algebras such that $k \underset{\mathcal{O}}{\otimes} A$, $k \underset{\mathcal{O}}{\otimes} B$ are self-injective. We have an equivalence of derived categories $D^b(A) \cong D^b(B)$ if and only if there are bounded complexes T of $B-A-$bimodules, U of $A-B-$bimodules whose terms are projective as left and right modules such that the complex $U \underset{B}{\otimes} T$ is homotopy equivalent to A as complex of $A-A-$bimodules and $T \underset{A}{\otimes} U$ is homotopy equivalent to B as complex of $B-B-$bimodules.*

Remarks.

(1) Clearly the functors $U \underset{A}{\otimes} -$, $T \underset{B}{\otimes} -$ induce mutually inverse derived equivalences.

(2) We recall that the tensor product $U \underset{B}{\otimes} T$ in the above theorem is the complex of $A - A-$modules which is in degree n equal to $\bigoplus_{i+j=n} U_i \underset{B}{\otimes} T_j$ with differential given by the maps $U_i \underset{B}{\otimes} T_i \longrightarrow U_{i-1} \underset{B}{\otimes} T_j \oplus U_i \underset{B}{\otimes} T_{j-1}$ sending $u \otimes t$ to $d(u) \otimes t + (-1)^i u \otimes e(t)$, where d and e are the differentials of U, T, respectively.

(3) By truncating the projective resolutions of U, T, it is possible to choose U , T in such a way that all terms of U, T are projective as bimodules except possibly in one degree.

(4) Rickard's Theorem holds actually in a more general setting (see [24]); we chose this slightly less general version, since it is sufficient in the context of finite groups and since it allows us to avoid the use of the derived functors of the tensor product in the statement.

The next proposition is Broué's remark, that a derived equivalence given by suitable complexes of bimodules induces a stable equivalence of Morita type:

Proposition 3.7. *Let A, B be $\mathcal{O}-$algebras such that A (resp. B) has non non zero projective direct summand as $A-A-$bimodule (resp. $B-B-$bimodule). Let T, U be bounded complexes of $B-A-$bimodules, $A-B-$bimodules, respectively. Suppose that all terms of T, U except T_0, U_0 are projective as bimodules and that the complexes $U \underset{B}{\otimes} T$, $T \underset{A}{\otimes} U$ are homotopy equivalent to A, B, respectively. Then T_0, U_0 induce a stable equivalence of Morita type between A and B.*

Proof. On one hand, calculating the tensor product $U \underset{B}{\otimes} T$ shows that its degree zero term is isomorphic to the direct sum of $U_0 \underset{B}{\otimes} T_0$ plus a projective $A - A-$bimodule, and on the other hand, since $U \underset{B}{\otimes} T$ is homotopy equivalent to A, its degree zero term must have A as direct summand as $A - A-$bimodule but no other non projective summand.

The following theorem of Rouquier shows that under certain circumstances, it is possible to "lift" a stable equivalence of Morita type to a derived equivalence:

Theorem 3.8. *(Rouquier[26]) Let A, B be symmetric $\mathcal{O}-$algebras with separable semi-simple quotients, such that $\mathcal{K} \underset{\mathcal{O}}{\otimes} A$, $\mathcal{K} \underset{\mathcal{O}}{\otimes} B$ are split semi-simple. Let M be a $B - A-$bimodule such that M, M^* induce a stable equivalence of Morita type between A and B, let P be a direct summand of a projective cover P_M of M as $B - A-$bimodule and let π be the restriction to P of a surjective $B - A-$homomorphism from P_M to M. Suppose that as left $B-$modules or as right $A-$modules, P and P_M/P have no non zero isomorphic direct summands.*

and that the complex $P \xrightarrow{\pi} M$ *induces an isometry between the Grothendieck groups of* $K \underset{\mathcal{O}}{\otimes} A$*-modules and* $K \underset{\mathcal{O}}{\otimes} B$*-modules. Then the complex* $P \xrightarrow{\pi} M$ *and its dual induce mutually inverse derived equivalences* $D^b(A) \cong D^b(B)$.

4. Stable equivalences for certain blocks of finite groups

In this section we first recall the basic notions and properties of blocks of finite groups and show then, that stable equivalences of Morita type frequently occur in this context.

Definition 4.1. *Let G be a finite group. A block of G is a primitive idempotent b in the center $Z(\mathcal{O}G)$ of the group algebra $\mathcal{O}G$, and the algebra $\mathcal{O}Gb$ is then called the block algebra of the block b.*

Thus the block algebra $\mathcal{O}Gb$ is an indecomposable direct summand of $\mathcal{O}G$ as $\mathcal{O}G-\mathcal{O}G-$bimodule, and clearly, if U is an $\mathcal{O}G-$module, then the $\mathcal{O}Gb-$module bU, viewed as $\mathcal{O}G-$module, is a direct summand of U. In particular, if U is indecomposable, there is a unique block b of G such that $bU = U$, and then we say that *the module U belongs to the block b*. The unique block b_0 of G to which the trivial $\mathcal{O}G-$module (abusively denoted by $\mathcal{O}$ again) belongs, is called the *principal block of G*.

One of the most fundamental invariants of a block are the so-called defect groups, introduced by R. Brauer in the early 1940's (see also Brauer's semi-expository work [3]):

Definition 4.2. *Let b be a block of a finite group G. A defect group of b is a subgroup P of G which is minimal with respect to the property, that the $\mathcal{O}Gb-\mathcal{O}Gb-$bimodule homomorphism $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} \mathcal{O}Gb \longrightarrow \mathcal{O}Gb$ mapping $a \otimes a'$ to aa', where $a, a' \in \mathcal{O}Gb$, has a section.*

Note that if P is a defect group of b, $\mathcal{O}Gb$ is isomorphic to a direct summand of $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} \mathcal{O}Gb$ as $\mathcal{O}Gb-\mathcal{O}Gb-$bimodule; that is, the identity functor $\mathcal{O}Gb \underset{\mathcal{O}Gb}{\otimes} -$ is isomorphic to a direct summand of the functor "restriction from $\mathcal{O}Gb$ to $\mathcal{O}P$" followed by "induction from $\mathcal{O}P$ to $\mathcal{O}Gb$", and hence, in particular, any $\mathcal{O}Gb-$module U is isomorphic to a direct summand of $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} U$ (a module with this property is called *relatively $P-$projective*).

It is easy to see, that the defect groups of the principal block b_0 of a finite group G are precisely the Sylow$-p-$subgroups of G. In general, the defect groups of a block b of G form a unique conjugacy class of $p-$subgroups of G.

There are various reformulations of the definition of a defect group involving two fundamental tools, the *relative trace map* and the *Brauer homomorphism*, which we recall now.

Observe that any subgroup H of a finite group G acts on $\mathcal{O}G$ by conjugation, and we denote by $(\mathcal{O}G)^H$ the subalgebra of elements $a \in \mathcal{O}G$ which are H-stable; that is, which satisfy $ah = ha$ for all $h \in H$. Furthermore, we denote by $[G/H]$ a set of representatives in G of the set of cosets G/H of H in G. For any subgroup L of H we define the *relative trace map* (sometimes called "transfer")

$$Tr_L^H : (\mathcal{O}G)^L \longrightarrow (\mathcal{O}G)^H$$

by $Tr_L^H(a) = \sum_{x \in [H/L]} xax^{-1}$ for any $a \in (\mathcal{O}G)^L$. Clearly this definition makes sense and does not depend on the choice of $[H/L]$ in H. We set $(\mathcal{O}G)_L^H = Im(Tr_L^H)$; this is easily seen to be an ideal in $(\mathcal{O}G)^H$.

Among the various more or less elementary properties of relative traces (see Green [11]) we recall the *Mackey formula*:

if H, K are subgroups of a finite group G, for any $a \in (\mathcal{O}G)^H$ we have

$$Tr_H^G(a) = \sum_x Tr_{K \cap {}^xH}^K({}^xa)$$

where x runs over a set of representatives of the $K-H$-double cosets in G.

Clearly, the above definition of relative traces can be generalized to the situation of a finite group G acting on an $\mathcal{O}$-algebra A by algebra automorphisms (such an algebra endowed with an action of G is called a G-*algebra*).

For any p-subgroup P of G we define the *Brauer homomorphism*

$$Br_P : (\mathcal{O}G)^P \longrightarrow kC_G(P)$$

as the restriction to $(\mathcal{O}G)^P$ of the $\mathcal{O}$-linear projection $\mathcal{O}G \longrightarrow kC_G(P)$ mapping $x \in C_G(P)$ to its image in $kC_G(P)$ and $x \in G - C_G(P)$ to zero.

The kernel of Br_P is the ideal $\sum_{Q<P} (\mathcal{O}G)_Q^P + J(\mathcal{O})(\mathcal{O}G)^P$ and hence Br_P is a surjective algebra homomorphism, and again, using this description of the kernel of Br_P we may generalize this definition to G-algebras (see [19, section 1]).

Applying the lifting theorem for idempotents to Br_P and the characterization 4.5 of defect groups of a block allow to prove a correspondence of blocks (a version of Brauer's first main Theorem in [3]; we refer to [2] for details):

Theorem 4.3. *(Brauer correspondence, version given by Alperin-Broué [2])*

Let G be a finite group, P a p-subgroup of G and H a subgroup of G containing $N_G(P)$. For any block b of G with P as a defect group there is a unique block c of H with P as defect group such that $Br_P(b) = Br_P(c)$, and the map sending b to c is then a bijection between the sets of blocks of G and H with P as a defect group.

Proof. If $x \in G - H$ then $P \cap {}^xP$ is a proper subgroup of P, and hence, the Mackey formula applied to $H = K = P$ shows that $Tr_P^G(a) = Tr_P^H(a)$ for all $a \in (\mathcal{O}G)^P$. Thus we have $Br_P((\mathcal{O}G)_P^G) = Br_P((\mathcal{O}G)_P^H) = Br_P((\mathcal{O}H)_P^H)$. Now the blocks of G with P as a defect group are precisely the primitive idempotents in $(\mathcal{O}G)_P^G$ which are not contained in $ker(Br_P)$ (see 4.5 below) and therefore correspond via Br_P bijectively to the primitive idempotents contained in $Br_P((\mathcal{O}G)_P^G) = Br_P((\mathcal{O}H)_P^H)$, which in turn, by the same argument, correspond bijectively to the blocks of H with P as a defect group.

There is a useful reformulation of the definition of a defect group using the relative trace map :

Lemma 4.4. *Let G be a finite group, P a subgroup of G and b a block of G. The following are equivalent:*

(i) The bimodule homomorphism $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} \mathcal{O}Gb \longrightarrow \mathcal{O}Gb$ mapping $a \otimes a'$ to aa', where $a, a' \in \mathcal{O}Gb$, has a section.

(ii) There is $d \in (\mathcal{O}Gb)^P$ such that $b = Tr_P^G(d)$.

Proof. If (ii) holds, it is easy to verify, that the map sending $a \in \mathcal{O}Gb$ to the element $\sum_{x \in [G/P]} axs \otimes x^{-1}$ is a section for the bimodule homomorphism in (i). Conversely, if there is such a section, it maps b to a G-stable element of the form $\sum_{x \in [G/P]} xs \otimes x^{-1}$ for some $s \in (\mathcal{O}Gb)^P$, and 4.4 follows.

Using the above lemma and Rosenberg's Lemma [27, Ch. I (4.9)] it is not hard to show the following equivalences:

Proposition 4.5. *Let G be a finite group, P a subgroup of G and b a block of G. The following are equivalent:*

(i) The group P is a defect group of b.

(ii) The group P is minimal such that $b \in (\mathcal{O}G)_P^G$.

(iii) The group P is a maximal p-subgroup of G such that $Br_P(b) \neq 0$.

In that case, for any idempotent $e \in (\mathcal{O}Gb)^P$ satisfying $Br_P(e) \neq 0$, the algebras $\mathcal{O}Gb$ and $e\mathcal{O}Ge$ are Morita equivalent.

We state now a result on stable equivalences of Morita type in "TI-like" situations ("TI" stands for "trivial intersection"):

Theorem 4.6. *Let G be a finite group, b a block of G with defect group P, H a subgroup of G satisfying $P \cap P^x = 1$ for any $x \in G - H$ and c the block of H corresponding to b. Then the $\mathcal{O}Gb - \mathcal{O}Hc$-bimodule $b\mathcal{O}Gc$ and the $\mathcal{O}Hc - \mathcal{O}Gb$-bimodule $c\mathcal{O}Gb$ induce a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.*

The "prototype" of the situation treated in 4.6 are blocks with cyclic defect groups: if b is a block of a finite group G having a non trivial cyclic defect group P and if H is the normalizer in G of the unique subgroup of order p of P, it is easy to see, that the hypotheses of 4.6 are fulfilled. Another such situation arises, if $p = 2$, P is a generalized quaternionian 2−group and H is the centralizer in G of the unique subgroup of order 2 of P.

We do not know in general, whether this stable equivalence of Morita type "lifts" in general to a derived equivalence (we expect it does). The answer is positive if P is cyclic, since, as Rouquier showed in[26], it is then possible to apply his Theorem 3.8 above.

The proof of 4.6 requires the following technical result:

Lemma 4.7. *With the notation and hypotheses of 4.6, for any non trivial subgroup Q of P we have $Br_Q(b) = Br_Q(c)$; in particular, $c - bc \in (\mathcal{O}G)_1^H$.*

Proof. Let Q be a non trivial subgroup of P and $a \in (\mathcal{O}G)^P$. We have $Tr_P^G(a) = \sum_x Tr_{P^x\cap Q}^Q(a^x)$, where x runs over a set of representatives of the double cosets $P\backslash G/Q$ in G. As $P^x \cap Q = 1$ if $x \in G - H$, applying Br_Q yields $Br_Q(Tr_Q^G(a)) = Br_Q(Tr_P^H(a))$, thus $Br_Q((\mathcal{O}G)_P^G) = Br_Q((\mathcal{O}G)_P^H) = Br_Q((\mathcal{O}H)_P^H)$, where the last equality holds since $C_G(P) \subset H$. Consequently, $Br_Q(b)$ and $Br_Q(c)$ are primitive idempotents in the ideal $Br_Q((\mathcal{O}G)_P^G)$ of the commutative subalgebra $Br_Q((\mathcal{O}G)^G)$ of $kC_G(Q)$ and therefore are either equal or orthogonal. Since $Br_P(Br_Q(b)) = Br_P(b) = Br_P(c) = Br_P(Br_Q(c))$ they cannot be orthogonal. As $c - bc \in (\mathcal{O}G)_P^H$ and $c - bc \in ker(Br_Q)$ for any non trivial subgroup Q of P, the last statement follows easily.

Proof of 4.6. Clearly $b\mathcal{O}Gc$ and $c\mathcal{O}Gb$ are projective as left and right modules. We show first, that the $\mathcal{O}Hc - \mathcal{O}Hc$−bimodule $c\mathcal{O}Gb \underset{\mathcal{O}Gb}{\otimes} b\mathcal{O}Gc \cong c\mathcal{O}Gbc$ has $\mathcal{O}Hc$ as unique non projective indecomposable direct summand, up to isomorphism. Since $\mathcal{O}H$ is a direct summand of $\mathcal{O}G$ as $\mathcal{O}H - \mathcal{O}H$−bimodule, clearly $\mathcal{O}Hc$ is a direct summand of $c\mathcal{O}Gc$ as $\mathcal{O}Hc - \mathcal{O}Hc$−bimodule.

If $x \in G - H$ we have $P \cap P^x = 1$; thus it follows easily that the $P \times P$−permutation module $\mathcal{O}[HxH]$ is projective as $P \times P$−module and therefore $c\mathcal{O}[HxH]c$ is projective as $\mathcal{O}Hc - \mathcal{O}Hc$−bimodule, since it is relatively $P \times P$− projective and projective as $P \times P$−module. Consequently, $c\mathcal{O}Gc$ has $\mathcal{O}Hc$ as unique non projective indecomposable direct summand, up to isomorphism. Now $c\mathcal{O}Gc = c\mathcal{O}Gbc \oplus c\mathcal{O}G(c - bc)$, and we only have to observe that $\mathcal{O}Hc$ is not a direct summand of $c\mathcal{O}G(c - bc)$. This however follows easily, using Higman's criterion, as $c - bc \in (\mathcal{O}G)_1^H$ by 4.7, and therefore $c\mathcal{O}G(c - cb)$ is projective as $\mathcal{O}Hc - \mathcal{O}Hc$−bimodule. It remains to show that the $\mathcal{O}Gb - \mathcal{O}Gb$−bimodule $b\mathcal{O}Gc \underset{\mathcal{O}Hc}{\otimes} c\mathcal{O}Gb$ has $\mathcal{O}Gb$ as unique indecomposable non projective direct summand, up to isomorphism. Since $Br_P(b) = Br_P(c)$, by 4.5, multiplication by c induces a Morita equivalence between $\mathcal{O}Gb$

and $c\mathcal{O}Gbc$. It suffices thus to show that $c\mathcal{O}Gbc \underset{\mathcal{O}Hc}{\otimes} c\mathcal{O}Gbc$ has $c\mathcal{O}Gbc$ as unique non projective indecomposable direct summand, up to isomorphism, as $c\mathcal{O}Gbc - c\mathcal{O}Gbc$–bimodule. Now multiplication induces a surjective bimodule homomorphism $c\mathcal{O}Gbc \underset{\mathcal{O}Hc}{\otimes} c\mathcal{O}Gbc \longrightarrow c\mathcal{O}Gbc$, which has a section as $\mathcal{O}Hc - c\mathcal{O}Gbc$–bimodule homomorphism, namely the map sending $a \in c\mathcal{O}Gbc$ to $bc \otimes a$. Thus it has a section as a $c\mathcal{O}Gbc - c\mathcal{O}Gbc$–bimodule homomorphism, since any $c\mathcal{O}Gbc - c\mathcal{O}Gbc$–bimodule is relatively $\mathcal{O}Hc - c\mathcal{O}Gbc$–projective; in particular, $c\mathcal{O}Gbc$ is a direct summand of $c\mathcal{O}Gbc \underset{\mathcal{O}Hc}{\otimes} c\mathcal{O}Gbc$. There is no other non projective direct summand, since even as $\mathcal{O}Hc - \mathcal{O}Hc$–bimodule, $\mathcal{O}Hc$ is, up to isomorphism, the unique non projective direct summand of $c\mathcal{O}Gbc$, thus of $c\mathcal{O}Gbc \underset{\mathcal{O}Hc}{\otimes} c\mathcal{O}Gbc$, too.

Observe that the bimodules $b\mathcal{O}Gc$ and $c\mathcal{O}Gb$ need not be indecomposable, since the idempotent cb need not be primitive in $(\mathcal{O}G)^H$.

Proposition 4.8. *With the notation of 4.6, let f be a primitive idempotent in $(\mathcal{O}Gb)^H_P$ such that $Br_P(f) = Br_P(b)$.*

(i) Multiplication by f induces a Morita equivalence between $\mathcal{O}Gb$ and $f\mathcal{O}Gf$.

(ii) The $\mathcal{O}Gb - \mathcal{O}Hc$–bimodule $b\mathcal{O}Gf$ and the $\mathcal{O}Hc - \mathcal{O}Gb$–bimodule $f\mathcal{O}Gb$ are indecomposable and induce a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

(iii) Multiplication by f on $\mathcal{O}Hc$ is an injective homomorphism $\mathcal{O}Hc \longrightarrow f\mathcal{O}Gf$ of interior H–algebras which induces a stable equivalence of Morita type between $\mathcal{O}Hc$ and $f\mathcal{O}Gf$.

Proof. Since $Br_P(f) = Br_P(b)$, (i) follows from 4.5. Statement (ii) follows easily from 4.6 and the observation that $b\mathcal{O}Gf$ and $f\mathcal{O}Gb$ are, up to isomorphism, the unique indecomposable non projective direct summands of $b\mathcal{O}Gc$ and $c\mathcal{O}Gb$, since f is primitive in $(\mathcal{O}Gb)^H$ and not contained in $(\mathcal{O}Gb)^H_1$. Statement (iii) is an immediate consequence of (i) and (ii).

The idempotent f in 4.8 need not be unique, but it is unique up to conjugacy in $(\mathcal{O}Gb)^H$.

Remark. Further applications of stable equivalences of Morita type to finite groups can be found in [17]: we determine there all stable equivalences of Morita type between finite p–groups and the automorphism groups of blocks with cyclic defect groups (see also [18]).

5. Invariants of stable, derived and Morita equivalences

Recall that for any algebra A there is a canonical isomorphism $Z(A) \cong End_{A \underset{\mathcal{O}}{\otimes} A^0}(A)$ mapping $z \in Z(A)$ to left multiplication by z on A whose inverse maps any $A \underset{\mathcal{O}}{\otimes} A^0$–endomorphism φ of A to $\varphi(1)$.

Definition 5.1. *Let A be an $\mathcal{O}-$algebra. The projective ideal of $Z(A)$ is the ideal $Z^{pr}(A)$ consisting of all elements z of $Z(A)$ whose image in $End_{A\otimes A^0}(A)$ factors through a projective $A\underset{\mathcal{O}}{\otimes}A^0-$module, and the stable center of A is the corresponding quotient $\overline{Z}(A) = Z(A)/Z^{pr}(A)$.*

Observe that $\overline{Z}(A) \cong \overline{End}_{A\otimes A^0}(A)$.

We recall the following standard results:

Proposition 5.2. *Let A, B be $\mathcal{O}-$algebras, M a $B-A-$bimodule, N an $A-B-$bimodule such that M, N induce a Morita equivalence between A and B. Then*

$$Z(A) \cong Z(B).$$

Proof. The map sending $\varphi \in End_{A\otimes A^0}(A)$ to $Id_M \otimes \varphi \otimes Id_N \in End_{B\otimes B^0}(M \underset{A}{\otimes} A \underset{A}{\otimes} N) \cong End_{B\otimes B^0}(B)$ is easily seen to be an isomorphism whose inverse is obtained by exchanging the rôles of A and B.

Proposition 5.3. *Let A, B be $\mathcal{O}-$algebras such that $D^b(A) \cong D^b(B)$. Then $Z(A) \cong Z(B)$.*

Proof. This follows from the well-known fact, that $Z(A)$ is naturally isomorphic to the *center* of the category $D^b(A)$ (which is the algebra of natural transformations of the identity functor on $D^b(A)$).

In the case of a stable equivalence of Morita type, we are unfortunately not able to prove an isomorphism of the centers (even though in all examples known to us at present the centers are isomorphic in that case). Still, we have:

Proposition 5.4. *(Broué [6, 5.4]) A stable equivalence of Morita type between two $\mathcal{O}-$algebras A and B induces an isomorphism of the stable centers $\overline{Z}(A) \cong \overline{Z}(B)$.*

Proof. Let M be a $B-A-$bimodule, N an $A-B-$bimodule such that $M \underset{A}{\otimes} N \cong B \oplus Y$ for some projective $B-B-$bimodule Y. Tensoring with M on the left and N on the right induces homomorphisms $\overline{End}_{A\otimes A^0}(A) \longrightarrow \overline{End}_{B\otimes B^0}(M \underset{A}{\otimes} A \underset{A}{\otimes} N) \cong \overline{End}_{B\otimes B^0}(B \oplus Y) \cong \overline{End}_{B\otimes B^0}(B)$, where the last isomorphism comes from the fact that Y is projective as $B\underset{\mathcal{O}}{\otimes}B^0-$module. Exchanging the rôles of A and B shows then easily that the induced map $\overline{Z}(A) \longrightarrow \overline{Z}(B)$ is an isomorphism.

We do not know what is the center of $\overline{Mod}(A)$ in general.

The next theorem describes a link between derived equivalences and ordinary characters of finite groups.

Let G be a finite group; assume that $\mathcal{K}$ is "large enough" and that $char(\mathcal{K}) = 0$. We have then

$$\mathcal{K}G = \sum_{\chi} M_{\chi(1)}(\mathcal{K})$$

where in this sum χ runs over the set $Irr_{\mathcal{K}}(G)$ of ordinary irreducible characters of G; thus in particular, $Z(\mathcal{K}G) = \sum_{\chi} \mathcal{K}e(\chi)$, where $e(\chi)$ is the unit element of $M_{\chi(1)}(\mathcal{K})$ (note that $e(\chi)$ is a block of $\mathcal{K}G$). If b is a block of $\mathcal{O}G$, we set $Irr_{\mathcal{K}}(G,b) = \{\chi \in Irr_{\mathcal{K}}(G) | \chi(b) \neq 0\}$. We have then clearly $b = \sum_{\chi \in Irr_{\mathcal{K}}(G)} e(\chi)$ and therefore

$$\mathcal{K}Gb = \bigoplus_{\chi \in Irr_{\mathcal{K}}(G,b)} M_{\chi(1)}(\mathcal{K}).$$

If U is an $\mathcal{O}$-free $\mathcal{O}G$-module, we denote by $[U]$ its character (or, equivalently, its image in the Grothendieck group of $\mathcal{K}G$-modules). If T is a bounded complex of $\mathcal{O}$-free $\mathcal{O}G$-modules, we set

$$[T] = \sum_{i \in \mathbb{Z}} (-1)^i [T_i].$$

Observe that if T is homotopic to zero we have $[T] = 0$, since in that case, T is a finite direct sum of complexes of the form $U \xrightarrow{Id_U} U$. Finally, if χ is a character of G, we denote by χ^* its conjugate character (i.e., $\chi^*(g) = \chi(g^{-1})$ for $g \in G$), and if η is a character of another finite group H, we denote by $\eta\chi^*$ the character of $H \times G$ mapping $(h,g) \in H \times G$ to $\eta(h)\chi^*(g)$.

Theorem 5.5. *Let G, H be finite groups, b, c blocks of G, H, respectively, and let T be a bounded complex of left and right projective $\mathcal{O}Hc - \mathcal{O}Gb$-bimodules such that the complex $T^* \underset{\mathcal{O}Hc}{\otimes} T$ is homotopy equivalent to $\mathcal{O}Gb$ as complex of $\mathcal{O}Gb - \mathcal{O}Gb$-bimodules, and such that $T \underset{\mathcal{O}Gb}{\otimes} T^*$ is homotopy equivalent to $\mathcal{O}Hc$. Suppose that $\mathcal{K}$ is a splitting field for G and H.*

Then there is a bijection $Irr_{\mathcal{K}}(G,b) \longrightarrow Irr_{\mathcal{K}}(H,c)$ mapping χ to η_χ, and, for any $\chi \in Irr_{\mathcal{K}}(G,b)$, a sign $\epsilon_\chi \in \{1,-1\}$ such that

$$[T] = \sum_{\chi \in Irr_{\mathcal{K}}(G,b)} \epsilon_\chi \eta_\chi \chi^*$$

and then the isomorphism $Z(\mathcal{K}Gb) \longrightarrow Z(\mathcal{K}Hc)$ mapping $e(\chi)$ to $e(\eta_\chi)$ induces an isomorphism

$$Z(\mathcal{O}Gb) \cong Z(\mathcal{O}Hc).$$

Remark. A "bijection with signs" between the sets of ordinary irreducible characters of blocks of finite groups as in the above theorem is called a *perfect isometry*, a concept, which has been introduced by M. Broué (see e.g. [4]).

Proof. Since the "character" of a zero homotopic complex vanishes, the hypotheses imply that

$$[T^* \underset{\mathcal{O}Hc}{\otimes} T] = [\mathcal{O}Gb] = \sum_{\chi \in Irr_K(G,b)} \chi\chi^*$$

and then the result follows from an explicit computation of the character of $T^* \underset{\mathcal{O}Hc}{\otimes} T$ using the standard orthogonality relations.

6. Complements

At this stage it is important to point out, that so far, we studied blocks of finite groups as $\mathcal{O}$–algebras. But this is only one half of the picture, since a block b of a finite group G has many invariants which cannot, a priori, be deduced from the $\mathcal{O}$–algebra structure of $\mathcal{O}Gb$ such as the defect groups of b. It is, for instance, not known, whether two blocks of finite groups which are Morita equivalent have isomorphic defect groups.

Another type of such invariants is the so-called *local structure* of a block, which can, for instance, be described in terms of *Brauer pairs* (see Alperin-Broué[2]) or in terms of *local pointed groups* (see Puig [19]).

Any of the three levels of equivalences mentioned so far has then a refinement taking into account the local structure of the considered blocks:

Morita equivalences lead to *Puig equivalences*; that is, to isomorphisms between the *source algebras* of the considered blocks (see e.g. [20] for a definition and fundamental properties of the local structure of block source algebras and [16], sections 2 - 7 for a short overview on some of Puig's concepts without proofs; a very detailed and comprehensive presentation of important parts of Puig's work is contained in Thévenaz' book [27]).

Derived equivalences lead to *Rickard equivalences*, called also *splendid derived equivalences*; here the major reference is Rickard's paper [25].

In a similar way, one gets a refinement for stable equivalences of Morita type.

At the level of ordinary characters of finite groups, Broué's notion of a *perfect isometry* refines to what he calls an *isotypy* (see Broué [4] and also[25] for a connection between isotypies and Rickard equivalences)

We describe now briefly (without any attempt of completeness) for which finite p–groups P the structure of the block algebras $\mathcal{O}Gb$ of a block b having P as a defect group is known (at least, up to Morita equivalence, but we point out, that in many of the cases mentioned below, even the source algebras are known; see [16], [18]). For simplicity, we assume that $\mathcal{O}$ is large enough:

$P = 1$: In that case, $\mathcal{O}Gb$ is a separable $\mathcal{O}$–algebra, whence a matrix algebra over $\mathcal{O}$ by our assumption on $\mathcal{O}$.

P is a Klein four group (and necessarily $p = 2$) : The algebra $\mathcal{O}Gb$ is Morita equivalent to either $\mathcal{O}P$, $\mathcal{O}A_4$ or $\mathcal{O}A_5b_0$ (cf. [16]). Moreover, $\mathcal{O}A_4$ and $\mathcal{O}A_5b_0$ are derived equivalent (this has first been proved by Rickard; see also Rouquier

[26]). The classification over $\mathcal{O}$ relies heavily on Erdmann's classification over k of these blocks (see [9]).

P is cyclic : The Morita equivalence class of $\mathcal{O}Gb$ is determined by the so-called *Brauer tree* of the block (this is a tree with some additional combinatorial information). The source algebras are determined in [18] (relying again on previous work of many authors). Moreover, any two blocks with isomorphic cyclic defect groups and the same number of isomorphism classes of simple modules are derived equivalent (this has first been proved by Rickard in [20] over k, then generalized in [15] over $\mathcal{O}$ and the construction of an explicit two-sided tilting complex realizing such a derived equivalence is due to Rouquier [26], as we mentioned earlier).

By Erdmann's work (see her book [9] for a detailed account and further references), the Morita equivalence classes over k are known for P being a 2−group with *tame representation type* (in some cases up to certain scalars occurring in the relations which describe the algebra structure).

Furthermore, many results are available for blocks of certain classes of finite groups (symmetric groups, finite groups of Lie type, p−solvable groups) as well as for blocks with a certain given local structure (e.g. nilpotent blocks [7], [21] and their extensions [14]).

We finish this section by some of the most prominent

Conjectures.

Alperin's Conjecture relates the number $l(G)$ of isomorphism classes of simple kG−modules, where G is a finite group, to the local structure of G, claiming that

$$l(G) = \sum_{Q} k_0(N_G(Q)/Q)$$

where Q runs over a set of representatives of the G−conjugacy classes of p−subgroups in G and $k_0(H)$ is the number of blocks of defect zero of a finite group H. This conjecture can also be formulated for blocks.

Broué's Conjecture claims that if b is a block of a finite group G having an abelian defect group P and if c is the corresponding block of $N_G(P)$, then $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ are derived equivalent.

Broué's Conjecture implies Alperin's Conjecture in the case of blocks with abelian defect groups.

Donovan's Conjecture says that given a finite p−group P, there are only finitely many Morita equivalence classes of blocks (over k) of finite groups having a defect group isomorphic to P.

Puig's Conjecture refines the preceding conjecture, asserting that there are only finitely many isomorphism classes of source algebras of blocks of finite groups with defect groups isomorphic to a given p−group P.

Puig's Conjecture holds for blocks with cyclic defect groups; Donovan's Conjecture holds also for blocks with a Klein four defect group, but it is not yet proved, whether Puig's Conjecture holds in this case.

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CLIFFORD THEORY AND A COUNTEREXAMPLE TO THE ISOMORPHISM PROBLEM FOR INFINITE GROUPS

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ABSTRACT. We introduce the reader to Clifford theory; i. e. how to obtain integral (p-adic) representations of the group G from representations of a normal subgroup, whose order is a unit in the base ring. We formulate a version of Clifford theory for automorphisms and apply this to construct a group and a non inner automorphism of it, which becomes inner in an integral group-ring. This is then used to show that there are two (infinite) non-isomorphic poly-cyclic groups, which have isomorphic integral group-rings.

1. CLIFFORD THEORY

In ordinary representation theory CLIFFORD THEORY is used to construct characters of the finite group G from the characters of a normal subgroup.

In integral representation theory – R is the ring of algebraic integers in a p-adic number field K – it is used to construct irreducible RG-lattices from those of a normal subgroup N with $(|N|, p) = 1$; it can even be used to describe certain blocks of RG from those of RN.

Clifford theory is a very powerful tool to describe parts of the group-ring RG, and it is used heavily in CONSTRUCTING A COUNTEREXAMPLE TO THE ZASSENHAUS CONJECTURE (cf. the article [Ro; 96,III]).

In case K is a splitting field[1] for G and all of its subgroups, the argument is due to P. Schmidt [Sch; 88 1, Sch; 83, Sch; 88 2][2]. However, the version we prove here holds more generally.

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[1] This is for example satisfied if K contains a primitive $|G|$-th root of unity.

[2] I have learnt it from L. L. Scott.

Before we can state the main result, we have to introduce some more NOTATIONS.

We now FIX FOR THE REST OF THIS SECTION an indecomposable RN-lattice M which corresponds to a block B of RN, $B \simeq Mat_n(S)$, where $End_{RN}(M) = S$, and S – with field of fractions L – is a finite unramified extension of R, since $|N|$ is a unit in R (cf. [Ro; 96,I], Example 3.16, 2.).

M can also be viewed as SN-module, which we shall denote by M_S. If we view M as SN-module, we get the ordinary Clifford theory; however, when we view M as RN-module, then the situation is quite different. Note also that

$$End_{SN}(M_S) = S.$$

Definition 1.1. 1. *For $g \in G$ we have the* CONJUGATE MODULE

${}^gM$, which is M as R-module with the action $n \cdot_{{}^gM} m = {}^gn \cdot m$ [3].

We note that ${}^gM \simeq_{RN} M$ and ${}^gM_S \simeq_{SN} M_S$ if $g \in N$ [4].

2. *The* INERTIA GROUP *of M as RN-module is defined as*

$$I(M) = \{g \in G : {}^gM \simeq_{RN} M\}.$$

3. *The* INERTIA GROUP *of M_S as SN-module is defined as*

$$I_S(M) = \{g \in G : {}^gM_S \simeq_{SN} M_S\}.$$

$I(M)$ and $I_S(M)$ are surely subgroups of G; moreover, $I(M) \geq I_S(M_S)$, since $R \subseteq S$.

We are now in the position to formulate the main result:

Theorem 1.2. *Assume that*

1. *N is a normal subgroup of the finite group G,*
2. *with $|N| \cdot R = R$ and $(|N|, |G : N|) = 1$.*
3. *Let M be an irreducible RN-lattice with $End_{RN}(M) = S$ a finite unramified extension of R.*
4. *Put $n = dim_S(M)$.*
5. *Let $I(M)$ and $I_S(M)$ be the inertia groups of M as RN-module and SN-module resp. Then $I_S(M)$ is normal in $I(M)$, and the quotient $T(M)$ acts as group of R-Galois automorphisms on S.*
6. *Assume furthermore that $I_S(M)$ has a complement $T_0(M)$ in $I(M)$; i. e.*

$$T(M) \simeq T_0(M) \leq I(M) \text{ and } I_S(M) \rtimes T_0(M) = I(M).$$

7. *Denote by $S(I_S(M)/N)^\circ$ the $RI(M)$-module, which is as S-module the group-ring $S(I_S(M)/N)$, and the action is given as follows: $I_S(M)$ acts by left multiplication and $T_0(M)$ acts by conjugation.*

[3] ${}^gn = g \cdot n \cdot g^{-1}$.

[4] Send $n \cdot m$ to m.

8. S° *is the* $RT_0(M)$*-module, where* $T_0(M)$ *acts via* $T(M)$ *as Galois automorphisms.*

Then the group-ring RG *contains a ring direct summand of the form*

$$B := Mat_{|G:I_S(M)|\cdot n}(H^0(T_0(M), R(I_S(M)/N)^\circ) \otimes_R S^\circ),$$

where $H^0(-,-)$ *is the fixed points functor (cf.* [Ro; 96,I], *Definition 5.1, 1.).*

The proof will proceed in several steps.

Definition 1.3. *Let* $G = \{g_i\}_{1\leq i\leq s}$.

1. *For each* $g_i \in I_S(M_S)$ *we fix an* SN*-isomorphism*
$$\phi(g_i) \ : {}^{g_i}M_S \ \to \ M_S .$$
Then $\phi(g_i) \in End_S(M)$, *and we have*
$$\phi(g_i)({}^{g_i}n \cdot m) = n \cdot \phi(g_i)(m) \text{ for all } n \in N \text{ and all } m \in M ,$$
i. e.
$$\phi(g_i)\,{}^{g_i}n\ \phi(g_i)^{-1} = \ n \text{ for each } n \in N ,$$
viewing n *in its action on* M_S *; i. e. interpreting* $n \in N$ *as an element in* $End_S(M)$. *Since* M_S *is irreducible,* $\phi(g_i)$ *is uniquely determined up to scalar multiples in* S .
2. *For each* $g_i \in I(M)$ *we fix an* RN*-isomorphism*
$$\psi(g_i) \ : {}^{g_i}M \ \to \ M .$$
Then $\psi(g_i) \in End_R(M)$, *and we have*
$$\psi(g_i)({}^{g_i}n \cdot m) = n \cdot \psi(g_i)(m) \text{ for all } n \in N\ ,\ m \in M ,$$
i. e.
$$\psi(g_i)\,{}^{g_i}n\ \psi(g_i)^{-1} = \ n \text{ for each } n \in N ,$$
viewing n *in its action on* M *: i. e. interpreting* $n \in N$ *as an element in* $End_S(M)$. *Since* M *is irreducible,* $\psi(g_i)$ *is uniquely determined up to scalar multiples in* S .

We choose these homomorphisms such that

$$\text{for } g_i \in I_S(M) \text{ we have } \phi(g_i) = \psi(g_i) . \tag{1}$$

We point out that for $g_i \in I(M) \setminus I_S(M)$ the homomorphism $\psi(g_i)$ IS DEFINITELY NOT AN SN-ISOMORPHISM.

We refer now to section 1.1 for an example concerning the change of the ring structure, if one extends the coefficient domain.

Note 1.4. Since the above maps ϕ and ψ are only determined up to scalar multiples, they are "representatives of projective representations" and hence determine a 2-cocycle (cf. [Ro; 96,I], Definition 5.3).
We shall elaborate on this next:
Assume for the moment that $G = I(M)$ and put

$$\mu(g_i, g_j) = \psi(g_i)^{-1} \cdot \psi(g_j)^{-1} \cdot \psi(g_i \cdot g_j).$$

Note that this measures how far ψ is from being a homomorphism[5]. Let $g_j \cdot g_i = g_k$. Then for every $n \in N$ we have

$$\begin{aligned} \mu(g_j, g_i) \cdot {}^{g_k}n \cdot \mu(g_j, g_i)^{-1} &= \psi(g_j)^{-1} \cdot \psi(g_i)^{-1} \cdot \psi(g_k) \cdot {}^{g_k}n \cdot \psi(g_k)^{-1} \cdot \psi(g_i) \cdot \psi(g_j) \\ &= \psi(g_j)^{-1} \cdot \psi(g_i)^{-1} \cdot n \cdot \psi(g_i) \cdot \psi(g_j) \\ &= \psi(g_j)^{-1} \cdot {}^{g_i}n \cdot \psi(g_j) = {}^{g_j \cdot g_i}n = {}^{g_k}n. \end{aligned}$$

Thus $\mu(g_i, g_j)$ centralizes N in its action on the irreducible RN-module M, and hence it is scalar multiplication with a diagonal matrix with entries in S^*, the group of units in S – recall that $S = End_{RN}(M)$.

Lemma 1.5. *The quotient-group $T(M) = I(M)/I_S(M)$ injects naturally into $Gal(S/R)$, the group of R-automorphisms of S. Note that S is a Galois extension of R with cyclic Galois group, S being unramified over R*[6].

THUS $T(M)$ IS A CYCLIC GROUP.

By R_0 we denote the subring of S fixed by T. Via the action of T on the R-module S it becomes an $RI(M)$-module. Since T also acts on the abelian group S^* of units in S, we can view S^* as a $\mathbb{Z}I(M)$-module.
The action is given – as the proof will show – by

$${}^{g}s = \psi(g) \cdot s \cdot \psi(g)^{-1}, \text{ where } \psi \text{ is defined in Definition 1.1.}$$

Proof: The RN-isomorphism $\psi(g) : {}^{g}M \longrightarrow M$ induces an isomorphism

$$\rho(g) : End_{RN}({}^{g}M) \longrightarrow End_{RN}(M) \text{ via}$$

$$\alpha \in End_{RN}({}^{g}M) \longrightarrow \psi(g) \cdot \alpha \cdot \psi(g)^{-1} \in End_{RN}(M).$$

Since $End_{RN}({}^{g}M) = S = End_{RN}(M)$, the map $\rho(g)$ can be interpreted as an automorphism – also denoted by $\rho(g) : S \longrightarrow S$.

In order to complete the argument we show:

---

[5] Since we have written maps on the left, ψ is a homomorphism if and only if $\psi(gh) = \psi(h)\psi(g)$.

[6] The Galois group is isomorphic to that of the field extensions corresponding to the residue fields of R and S and is therefore generated by the FROBENIUS HOMOMORPHISM (cf. [Has; 49]).

Claim 1.6. *The map*

$$\rho : I(M) \longrightarrow Aut(S),$$
$$\rho(g) : s \longrightarrow \psi(g) \cdot s \cdot \psi(g)^{-1}$$

is a homomorphism of groups with kernel $I_S(M)$ *, and its image lies in* $Gal(S/R)$ *.*

Proof: Since $\psi(g)$ is an RN-isomorphism, the automorphism $\rho(g)$ is surely R-linear. Moreover, if $g \in I_S(M)$, then $\psi(g)$ is S-linear, and then $\rho(g) = id_S$ is the identity. Conversely, if $\rho(g) = id_S$ we have that $\psi(g)$ commutes with all elements in S and thus is S-linear.

It remains to show that ρ is a homomorphism of groups. However, for

$\alpha \in End_{RN}(^{gh}M)$ we have (cf. Note 1.4)

$$\begin{aligned} \rho(gh)(\alpha) &= \psi(gh) \cdot \alpha \cdot \psi(gh)^{-1} \\ &= \psi(h) \cdot \psi(g) \cdot \mu(g,h) \cdot \alpha \cdot \mu(g,h)^{-1} \cdot \psi(g)^{-1} \cdot \psi(h)^{-1} . \end{aligned}$$

But, α is scalar multiplication with an element $s_0 \in S$, and since $\mu(-,-)$ also lies in the commutative ring S , we conclude $\rho(gh) = \rho(h) \cdot \rho(g)$ and ρ is a group homomorphism. q.e.d. Claim 1.6

We now return to the proof of Lemma 1.5: Since $I_S(M)$ is the kernel of the homomorphism ρ , the group $I_S(M)$ is normal in $I(M)$ and the quotient $T = I(M)/I_S(M)$ is cyclic, since $Gal(S/R)$ is cyclic.
q.e.d. Lemma 1.5

The "cocycle" μ from Note 1.4 will play a crucial role in the structure of the block of RG associated to M :

Proposition 1.7. *Assume that* $I(M) = G$ *. As above we put*

$$\mu(g_j, g_i) := \psi(g_j)^{-1} \cdot \psi(g_i)^{-1} \cdot \psi(g_j \cdot g_i) .$$

Then

$$\mu : G \times G \to S^*$$

is a 2-cocycle of G with coefficients in the G-module S^* *(cf. Section 5 in* [Ro; 96,I], *Definition 5.3), which is uniquely determined up to 2-coboundaries. It vanishes on N and hence defines a unique element in* $H^2(G/N, S^*)$ *.*
Moreover, if $(|I_S(M)/N|, |N|) = 1$ *, then* μ *restricted to* $I_S(M)$ *is a 2-coboundary, and consequently, M can be extended to an* $SI_S(M)$*-module. In particular, we may assume that* μ *is the identity on* $I_S(M)$ *.*

Proof: By Lemma 1.5, G acts on S^* by conjugation with $\psi(g)$, and thus we have to show that μ from Note 1.4 is a multiplicative 2-cocycle. Recall that

$$\mu(g,h) = \psi(g)^{-1} \cdot \psi(h)^{-1} \cdot \psi(gh) \in S^*$$

and

$$^{\rho(g)}s = \psi(g) \cdot s \cdot \psi(g)^{-1}, s \in S .$$

We thus have to show:

$$^{g^{-1}}\mu(h,k) \cdot \mu(g,hk) = \mu(g,h) \cdot \mu(gh,k) \text{ [7]} .$$

However,

$$\begin{aligned} \psi(g \cdot (hk)) &= \psi(hk) \cdot \psi(g) \cdot \mu(g,hk) \\ &= \psi(k) \cdot \psi(h) \cdot \mu(h,k) \cdot \psi(g) \cdot \mu(g,hk) \\ \psi((gh) \cdot k) &= \psi(k) \cdot \psi(gh) \cdot \mu(gh,k) \\ &= \psi(k) \cdot \psi(h) \cdot \psi(g) \cdot \mu(g,h) \cdot \mu(gh,k). \end{aligned}$$

But $\psi(g \cdot (hk)) = \psi((gh) \cdot k)$ and so we obtain

$$\mu(h,k) \cdot \psi(g) \cdot \mu(g,hk) = \psi(g) \cdot \mu(g,h) \cdot \mu(gh,k) ,$$

since we can cancel.

The uniqueness is shown as follows:
Take another RN-isomorphism $\chi(g) : {}^gM \longrightarrow M$. Then $\chi(g) = \psi(g) \cdot s_g$ for an automorphism

$$s_g \in End_{RN}(M) = S ,$$

and one gets an associated 2-cocycle

$$\begin{aligned} \nu(h,g) &= \chi(h)^{-1} \cdot \chi(g)^{-1} \cdot \chi(gh) \\ &= s_h^{-1} \cdot \psi(h)^{-1} \cdot s_g^{-1} \cdot \psi(g)^{-1} \cdot \psi(gh) \cdot s_{gh} \\ &= s_h^{-1} \cdot {}^{\psi(h)^{-1}}(s_g^{-1}) \cdot \psi(h)^{-1} \cdot \psi(g)^{-1} \cdot \psi(gh) \cdot s_{gh} \\ &= s_h^{-1} \cdot {}^{\psi(h)^{-1}}(s_g^{-1}) \cdot \mu(h,g) \cdot s_{gh} ; \end{aligned}$$

however, $\mu(g,h) \in S$ commutes with s_{gh} , and so

$$\nu(h,g) = s_g^{-1} \cdot {}^{\psi(g)^{-1}}(s_h^{-1}) \cdot s_{gh} \cdot \mu(h,g) .$$

Hence μ and ν differ by a 2-coboundary, and so μ gives rise to a unique element in $H^2(G,S^*)$.

In order to continue with the proof, we note:

Claim 1.8. *μ gives rise to a unique element $\overline{\mu}$ in $H^2(G/N,S^*)$ defined by*

$$\overline{\mu}(Ng,Nh) =: \mu(g,h) .$$

[7] Note that our multiplication is contravariant.

Proof: We have to show that this is well defined. Let

$$G = \bigcup N g_i$$

be the decomposition into cosets. First we can arrange the isomorphisms ψ such that $\psi(ng_i) = \psi(g_i) \cdot \psi(n)$ for all $n \in N$ and all i. In fact we have

$$\psi(n)(m) = n^{-1}m$$

for all $m \in M, n \in N$ (cf. Definition 1.1, 1. footnote), but then $\psi(n)$ also induces an isomorphism from ${}^{ng_i}M$ to ${}^{g_i}M$ and thus $\psi(g_i) \cdot \psi(n)$ is an RN-isomorphism from ${}^{ng_i}M$ to M.

By definition we have the relation

$$\psi(g) \cdot {}^{g}n = n \cdot \psi(g)$$

for all $g \in G, n \in N$. Let us now compute for $n_1, n_2 \in N$:

$$\begin{aligned}
\mu(n_1 g_i, n_2 g_j) &= \psi(n_1 g_i)^{-1}\psi(n_2 g_j)^{-1}\psi(n_1 g_i n_2 g_j) \\
&= \psi(n_1)^{-1}\psi(g_i)^{-1}\psi(n_2)^{-1}\psi(g_j)^{-1}\psi(n_1\, {}^{g_i}n_2 g_i g_j) \\
&= \psi(n_1)^{-1}\psi(g_i)^{-1}\psi(n_2)^{-1}\psi(g_j)^{-1}\psi(g_i g_j)\psi(n_1\, {}^{g_i}n_2) \\
&= \psi(n_1)^{-1}({}^{g_i}\psi(n_2))^{-1}\mu(g_i, g_j)\psi({}^{g_i}n_2)\psi(n_1) \\
&= \mu(g_i, g_j),
\end{aligned}$$

since $\psi({}^{g_i}n_2) = {}^{g_i}\psi(n_2)$ is multiplication by ${}^{g_i}n_2^{-1}$ and since μ has values in S.

This proves the claim. q.e.d.

Note 1.9. If $\overline{\mu}$ is a coboundary, then so is μ as one sees from the definition.

In order to complete the proof, we recall from Definition 1.3, that for $g \in I_S(M)$, $\psi(g) = \phi(g)$ (cf. Equation 1) is an SN-isomorphism; in particular, $\phi(g)$ centralizes S.

Recall that we did assume in Theorem 1.2 that $(|G/N|, |N|) = 1$ and the assumption of Proposition 1.7 was that $G = I(M)$. Hence, if we put $E = I_S(M)/N$, then $(|N|, |E|) = 1$ and we have for $x, y \in E$ – note that μ vanishes on N –

$$\mu(x, y) = \psi(x)^{-1} \cdot \psi(y)^{-1} \cdot \psi(x \cdot y).$$

We interpret this as an equation of S-linear maps on M. Taking determinants we get with $d = dim_L(LM)$, where L is the quotient field of S:

$$\mu(x, y)^d = det(\psi(x)) \cdot det(\psi(y)) \cdot det(\psi(x \cdot y))^{-1}.$$

Now all factors lie in S^* , and the right hand side is exactly the condition for μ^d being a 2-coboundary as one sees from the definition above. (Note that $\phi(g)$ centralizes S .)

On the other hand, the cohomology group $H^2(E, S^*)$ is annihilated by $|E|$ (cf. [Ro; 96,I], Remark 3.15, 1.) Since the degrees of the characters (irreducible complex representations) divide the group order, the dimension d divides $|N|$. However, $|E|$ and $|N|$ are relatively prime, and so μ must be a coboundary, since it is annihilated by $|N|$ and $|E|$. Say for $g, h \in I_S(M)$ we have $\mu(g,h) = s_g \cdot s_h \cdot s_{gh}^{-1}$ – note that $\phi(g)$ centralizes S. If we now replace $\psi(g)$ by $\chi(g) := \psi(g) \cdot s_g$, then $\chi(g)$ is an SM-isomorphism from ${}^gM$ to M , and an easy calculation – as above – shows that $\chi : I_S(M) \to End_S(M)$ is a homomorphism; i. e. M extends to an $SI_S(M)$-module. The modified cocycle associated to χ vanishes on $I_S(M)$. q.e.d.

Claim 1.10. *Let $\{g_i\}_{1\le i\le t}$ be a set of coset representatives of $I_S(M)$ in $I(M)$. Then the RG-homomorphisms $\{\psi(g_i)\}_{1\le i\le t}$ are in fact R_0-isomorphisms, where R_0 is the fixed ring of S under $T(M)$ acting via ρ . Moreover, the SN-module ${}^{g_i}M$ is the Galois conjugate module to M under $\rho(g_i)$. In addition, as $SI(M)$-module we have an isomorphism*

$$S \otimes_{R_0} M \simeq_{SI(M)} M \uparrow_{I_S(M)}^{I(M)} .$$

Proof: By the definition of $\psi(g_i) : {}^{g_i}M \to M$ we have for $s \in S$ the relation

$$s \cdot \psi(g_i) = \psi(g_i) \cdot {}^{\psi(g_i)^{-1}}s \ ;$$

however, ${}^{\psi(g_i)^{-1}}s = {}^{g_i^{-1}}s$ is the Galois action. In particular, if $s \in R_0$, then this shows that $\psi(g_i)$ is an R_0N-isomorphism.

CONSEQUENTLY, WE DO NOT LOOSE ANYTHING, IF WE ASSUME FOR THE TIME BEING, THAT

$$R = R_0 .$$

Recall that M is an RN-module corresponding to the simple component $B = (S)_n$.

Then $S \otimes_R M$ decomposes into the Galois conjugate modules ${}^{\rho(g_i)}M$ corresponding to the t simple components of

$$S \otimes_R B \simeq \prod_1^t (S)_n, \text{ with } t = |T(M)| ;$$

note that $S \otimes_R B$ is separable with t non-isomorphic modules, each occuring with multiplicity n .

These Galois conjugate modules are all RN-isomorphic to M ; but as SN-modules they are non isomorphic. Now, the conjugate SN-modules $\{{}^{g_i}M\}_{1\le i\le t}$

have the same property. But the group-ring SN has exactly t non isomorphic modules, which become isomorphic as RN-modules, namely one column from each of the t copies of $(S)_n$. Our construction of ρ then shows that ${}^{\rho(g_i)}M \simeq_{SN} {}^{g_i}M$, as claimed. This also shows that

$$S \otimes_R M \simeq_{SI(M)} M \uparrow_{I_S(M)}^{I(M)} .$$

q.e.d.

Lemma 1.11. *We keep the assumptions of Lemma 1.5, and assume that M is an $I_S(M)$-module. Assume furthermore that $G = I(M)$. Then M can be extended to an RG-module, also denoted by M, and the induced module $M \uparrow_{I_S(M)}^G$ decomposes as RG-module into t copies of M where $t = |I(M) : I_S(M)|$.*

Proof: Let R_0 be the fixed ring of S under the Galois action of T. We have seen in Claim 1.10 that – up to now – there is no loss of generality, if we ASSUME that $R = R_0$.

Because of Claim 1.10 and its proof, the SG-lattice $M \uparrow_I^G$ is irreducible[8] Moreover, by Frobenius reciprocity (cf. [Ro; 96,I], Proposition 4.2), we have

$$End_{SG}(M \uparrow_I^G) \simeq Hom_{SI}(M, M \uparrow_I^G \downarrow_I^G),$$

where the restriction of the induced module M_I^G to I is by Mackey's formula given as

$$M \uparrow_I^G \downarrow_I^G \simeq_{SI} \bigoplus_{i=1}^{t} {}^{g_i}M ,$$

the direct sum of the conjugate modules ${}^{g_i}M$ for coset representatives $\{g_i\}$ of I in G – note that $I = I_S(M)$ is normal in $I(M) = G$. According to the definition, ${}^{g_i}M \simeq_{SI} M$ if and only if $g_i = 1$. Hence $End_{SG}(M \uparrow_I^G) = S$, since $End_{SI}(M) = S$ – recall that $\phi(g)$ is S-linear. On the other hand,

$$End_{R_0G}(M \uparrow_I^G) \simeq Hom_{R_0I}(M, M \uparrow_I^G \downarrow_I^G), \text{ with}$$

$$M \uparrow_I^G \downarrow_I^G \simeq_{R_0I} \oplus_{i=1}^t {}^{g_i}M$$

the direct sum of the conjugate modules ${}^{g_i}M$ for coset representatives $\{g_i\}$ of I in G. According to the definition, ${}^{g_i}M \simeq_{R_0I} M$ for all $1 \leq i \leq t$. Hence $\Lambda := End_{R_0G}(M \uparrow_I^G) = S^t$.

By the change of rings theorem we have for R_0G-lattices X and Y:

$$S \otimes_{R_0} Hom_{R_0G}(X, Y) \simeq Hom_{SG}(S \otimes_{R_0} X, S \otimes_{R_0} Y).$$

Thus

$$S \otimes_{R_0} S^t \simeq End_{SG}((S \otimes_{R_0} M) \uparrow_I^G).$$

[8] This also follows from ordinary Clifford theory.

Claim 1.12.

$$(S \otimes_{R_0} M) \uparrow_I^G \simeq \sum_1^t M \uparrow_I^G \quad \textit{as } SN\textit{-modules},$$

and hence $End_{SG}((S \otimes_{R_0} M) \uparrow_I^G) \simeq (S)_t$, *and* $S \otimes_{R_0} \Lambda \simeq (S)_t$.

Proof: We have seen in Claim 1.10 that the 'group conjugate' module ${}^{g_i}M$ is SN-isomorphic to the 'Galois conjugate' module ${}^{\rho(g_i)}M$. Since $S \otimes_{R_0} M$ is RN-isomorphic to the direct sum of the Galois conjugate modules $\{M_i\}_{1 \le i \le t}$, we conclude,

$$S \otimes_{R_0} M \simeq_{SN} M \uparrow_I^G .$$

In particular, $(M_i) \uparrow_I^G \simeq_{SG} M \uparrow_I^G$. Thus $S \otimes M \uparrow_I^G \simeq \sum_1^t M \uparrow_I^G$, and Claim 1.12 is proved. q.e.d.

Recall, $\Lambda = End_{RG}(M \uparrow_I^G)$, and $S \otimes_{R_0} \Lambda \simeq (S)_t$, where S is an unramified extension of R_0 . Thus $S \otimes_{R_0} rad(\Lambda) = rad(S \otimes_{R_0} \Lambda)$. Since $rad(S \otimes_{R_0} \Lambda) \simeq S \otimes_{R_0} \Lambda$ as modules $S \otimes_{R_0} \Lambda$, ($S \otimes_{R_0} \Lambda$ being isomorphic to the full t by t matrix ring over S), we can invoke the Noether-Deuring Theorem [Ro; 96,I] 2.2 to conclude $rad(\Lambda) \simeq \Lambda$. But then Λ is hereditary (cf. Proposition [Ro; 96,I] 3.5)[9]. If now P is an indecomposable projective left Λ-module, then $P \simeq rad(\Lambda)$, since this holds when extended to S . Thus Λ is a maximal order with $S \otimes_{R_0} (\Lambda / rad(\Lambda)) \simeq (S/rad(S))_t$, and we conclude, that Λ decomposes into t isomorphic left modules.

Moreover, R_0 lies in the center of Λ , and $dim_{R_0}(\Lambda) = dim_{R_0}(S^{(t)}) = t^2$. Hence the only possibility for Λ is $\Lambda \simeq (R_0)_t$. Since $\Lambda = End_{RG}(M \uparrow_I^G)$, this shows that as RG-module, $M \uparrow_I^G$ decomposes into t RG-modules, say $X_i, 1 \le i \le t$; however, as RI-module, $M \uparrow_I^G \simeq M^{(t)}$, and we conclude that $M \uparrow_I^G$ is the direct sum of t isomorphic RG-modules, which when restricted to I are isomorphic to M .

Thus M extends to an RG-module. It is also an R_0G-module. This completes the proof of Lemma 1.11. q.e.d.

Remark 1.13. We shall now treat the situations "splitting field" or "non splitting field" separately, and we shall first deal with the split situation, which is the classical set up for Clifford theory [C-R1; 82, (11.1)] :

M is as above an irreducible SN-lattice with $End_{SN}(M) = S$ and inertia group $I_S(M)$. Moreover, $(|I_S(M) : N|, |N|) = 1$, and so M extends as S-module to its inertia group $I_S(M)$ (see Proposition 1.7).

[9] Every Λ-lattice is projective [Ro; 96,I] 3.1.

Theorem 1.14. [Mo; 58, Ga; 79] *With the above assumptions, the group-ring SG has a ring direct summand of the form*

$$B_M = Mat_{|G:I_S(M)|}(SI_S(M)/N \otimes_S \Lambda_M) \simeq Mat_{|G:I_S(M)|\cdot n}(SI_S(M)/N) ,$$

where $\Lambda_M \simeq End_S(M) = Mat_n(S)$ is the two sided ideal of SN corresponding to M.
In particular,

$$SG = \prod Mat_{|G:I_S(M)|}(S(I_S(M)/N) \otimes_S \Lambda_M) ,$$

where the product is taken over representatives of indecomposable SN-lattices, which are not G-conjugate. Thus SG is Morita equivalent to $\prod S(I_S(M)/N)$.

Proof: CASE 1: First let $G = I_S(M)$.

Let e_M be the central primitive idempotent of SN, which corresponds to the indecomposable SN-lattice M; i. e. $SN \cdot e_M = End_S(M) = (S)_n$. According to the definition of the inertia group, e_M is also a central idempotent in SG, not necessarily primitive. In fact, the conjugate idempotents ${}^g e_M$ are central idempotents in RN, which are all equivalent to e_M, since G is the inertia group of M.

We shall now focus our attention first on the ring direct summand $SG \cdot e_M$ of SG.

We claim that the module – the ordinary tensor product of SG-modules –

$$P_M = SG/N \otimes_S M \text{ is projective for } SG .$$

For this it suffices to show that the restriction to a Sylow p-subgroup P is projective 4.4; however, $(|G/N|, |N|) = 1$ and $|N| \cdot S = S$. Hence G is a semidirect product of G/N and N, and G/N contains a Sylow p-subgroup of G. Consequently, SG/N is SG-projective. But then the tensor product $SG/N \otimes_S M$ is projective (cf. [Ro; 96,I], Note 1.3) as SG-module, it is even a cyclic SG-module, generated by $1 \otimes m_0$, $0 \neq m_0 \in M$.

We now compute the ring of SG-endomorphisms of P_M:

$$\begin{aligned} End_{SG}(SG/N \otimes_S M) &= (End_{SN}(SG/N \otimes_S M))^{G/N} \\ &= (End_{SN}(SG/N) \otimes_S End_{SN}(M))^{G/N} \\ &= (SG/N \otimes_S S), \end{aligned}$$

(N acts trivially on SG/N) where $X^{G/N}$ are the G/N fixed points on X. G/N acts on SG/N from the left and on S it acts trivially. Hence

$$End_{SG}(SG/N \otimes_S M) \simeq SG/N .$$

We now have to compute the multiplicity of P_M in SG. The group-ring SG is a twisted tensor product 3.16, $SG \simeq SG/N \otimes_S SN$, and since e_M is a central idempotent of SG, the two sided

$$SG \simeq SG/N \otimes_S SN - \text{module } SG/N \otimes_S \Lambda_M$$

is a ring direct summand of SG, which is as left module isomorphic to $P_M^{(n)}$ – recall $\Lambda_M = End_S(M)$. Thus the group-ring SG has a ring direct summand isomorphic to $(SG/N)_n$, which is the statement of Theorem 1.14 in case $I_S(M) = G$.

CASE 2: $N \le I_S(M) < G$.

Let

$$G = \bigcup_{1 \le i \le n} g_i \cdot I_S(M)$$

be a system of left coset representatives of $I_S(M)$ in G. e_M is the central primitive idempotent in SN corresponding to M. Since we are outside of the inertia group of M, the idempotents $\{{}^{g_i}e_M\}_{1 \le i \le n}$ are different primitive orthogonal central idempotents of SN – note that N is normal.

We put $e_i = {}^{g_i}e_M$, $1 \le i \le s$. Then $e = \sum_{1 \le i \le n} e_i$ is a central idempotent in SG.

We recall from above that e_M is also a central idempotent in $SI_S(M)$. The group G acts on the idempotents e_i as follows:

If $g_j g_i = g_k x$, $x \in I_S(M)$, then $g_j e_i = e_k x$ and hence if L is an $SI_S(M)$-module on which e_M acts as identity, then $\bigoplus e_i L$ is the induced module $L \uparrow_{I_S(M)}^G$.

Claim 1.15. *SG contains a ring direct summand, which is Morita equivalent to*

$$\Gamma := SI_S(M)/N \otimes_S End_S(M),$$

which in turn is Morita equivalent to $SI_S(M)/N$.

Proof: The idempotents $\{e_i\}_{1 \le i \le n}$ are surely orthogonal and their sum $e := \sum_{1 \le i \le s} e_i$ is a central idempotent in SG. Thus we have a Pierce decomposition

$$B := \begin{pmatrix} e_1 \cdot SG \cdot e_1 & \dots & e_1 \cdot SG \cdot e_j & \dots & e_1 \cdot SG \cdot e_s \\ \dots & \dots & \dots & \dots & \dots \\ e_i \cdot SG \cdot e_1 & \dots & e_i \cdot SG \cdot e_j & \dots & e_i \cdot SG \cdot e_s \\ \dots & \dots & \dots & \dots & \dots \\ e_s \cdot SG \cdot e_1 & \dots & e_s \cdot SG \cdot e_j & \dots & e_s \cdot SG \cdot e_s \end{pmatrix}.$$

On the other hand, the module induced to G from the projective $I_S(M)$-module $P_M = SI_S(M)/N \otimes_S End_S(M)$ is

$$\begin{aligned} P_0 &:= (SI_S(M)/N \otimes_S End_S(M)) \uparrow_{I_S(M)}^G \\ &\simeq SG \otimes_{SI_S(M)} (SI_S(M)/N \otimes_S End_S(M)) \\ &\simeq \bigoplus_{1\leq i\leq n} e_i \cdot SI_S(M)/N \otimes_S End_S(M). \end{aligned}$$

This latter isomorphism is a consequence of the discussion preceding Claim 1.15.

Moreover, e acts as identity. Surely the modules $P_0 \cdot e_j$ are isomorphic to P_0, $1 \leq j \leq s$, and hence their direct sum is a two sided direct summand of SG, and it is $e \cdot SG$. So the claim is proved. q.e.d.

THIS FINISHES THE ARGUMENT IN THE SPLITTING SITUATION. WE NOW TURN TO THE GENERAL SET UP:

1. M is an irreducible RN-lattice, with $End_{RN}(M) = S$, and hence can be viewed as an SN-module.
2. $I(M)$ is the inertia group of M as RN-module, $I_S(M)$ is the inertia group of M as SN-module. $I_S(M)$ is a normal subgroup in $I(M)$ with cyclic quotient $T(M) = I(M)/I_S(M)$, which injects into $Gal(S/R)$ (see Lemma 1.5). $T(M)$ has in S the fixed ring R_0.
3. We assume as above that $(|I_S(M) : N|, |N|) = 1$. By Theorem 1.14 the module M extends to an $SI(M)$-module, with $End_{SI(M)}(M) = S$, it extends to an $RI(M)$-module with $End_{RI(M)}(M) = R_0$. Moreover, the induced $SI(M)$-module $M \uparrow_{I_S(M)}^{I(M)}$ is irreducible; as $SI(M)$-module $M \uparrow_{I_S(M)}^{I(M)}$ is the direct sum of the Galois conjugates (cf. Claim (1.10)), whereas the induced $RI(M)$ (even $R_0I(M)$)-module $M \uparrow_{I_S(M)}^{I(M)}$ decomposes in $t = |T(M)|$ copies of M (see Lemma 1.11).
4. By Theorem 1.14 the group-ring $SI(M)$ contains a ring direct summand
$$\Delta_S(M) := (SI_S(M)/N \otimes_S (S)_n)_t \simeq (SI_S(M)/N)_{n\cdot t}. \qquad (2)$$
5. Moreover, we have seen in Theorem 1.14 that the $SI(M)$-module $(SI_S(M)/N \otimes_S M)\uparrow_{I_S(M)}^{I(M)}$ and hence also $(SI_S(M)/N \otimes End_R(M))\uparrow_{I_S(M)}^{I(M)}$ are projective over $SI(M)$. But since
$$S \otimes_R ((RI_S(M)/N \otimes_R M) \uparrow_{I_S(M)}^{I(M)}) \simeq (SI_S(M)/N \otimes_S M) \uparrow_{I_S(M)}^{I(M)},$$
we conclude that $(RI_S(M)/N \otimes_R M) \uparrow_{I_S(M)}^{I(M)}$ is a projective $RI(M)$-module.

Remark 1.16. In general I can not say any more but that RG contains a ring direct summand, which is a full matrix ring over

$$End_{RI(M)}((RI_S(M)/N \otimes_R M) \uparrow_{I_S(M)}^{I(M)}).$$

However, the situation is much more transparent, if we ASSUME THAT THE GROUP EXTENSION $I(M)/N$ OVER $I_S(M)/N$ IS SPLIT.

That this need not always be so shows an example in Section 1.1, Example 2. In the non split situation, the 2-cocycle $\psi \in H^2(T(M), S^*)$ will play an important role.

Assumption 1.17. The exact sequence of groups

$$1 \to I_S(M) \to I(M) \to I(M)/I_S(M) \to 1$$

is split. Let $T_0(M)$ be a subgroup of $I(M)$, which is mapped isomorphically onto $T(M) := I(M)/I_S(M)$.

Recall that the $R_0T_0(M)$-module S° is S with $T_0(M)$ acting as Galois automorphism.

Lemma 1.18. *S° is as left module $R_0T_0(M)$-isomorphic to $R_0T_0(M)$. Let $R_0(I_S(M)^\circ)$ is the free left $R_0I_S(M)$-module of rank one with $T_0(M)$ acting by conjugation. Then the $R_0I(M)$-module*

$$SI_S(M)^\circ := S^\circ \otimes_{R_0} R_0(I_S(M)^\circ)$$

is as left $R_0I(M)$-module isomorphic to $R_0I(M)$.

Proof: The fact that S° is free as $R_0Gal(S/R_0) \simeq R_0T_0(M)$-module is a result of David Hilbert [Que; 80, p.219] [Hil; 1897], since S is unramified over R_0. The remaining statement is the general fact about semidirect products [Ro; 96,I] 3.16: Let $A \rtimes B$ be a semidirect product of groups, with B acting on A by conjugation, then $R(A^\circ) \otimes_R RB$ with B acting on $R(A^\circ)$ by conjugation is isomorphic to RG.

q.e.d.

N° is the group N but $T(M)$ acting via conjugation.

Lemma 1.19. *The $RI(M)$-module $RI_S(M)/N^\circ \otimes_R M$, where $I(M)$ is the semidirect product $I(M) = I_S(M) \rtimes T_0(M)$ and $T_0(M)$ acts on $RI_S(M)/N^\circ$ by conjugation, is a projective $RI(M)$-module.*

Proof: Since the Sylow p-subgroup of $I(M)$ injects into the Sylow p-subgroup of $I(M)/N$, the module $SI_S(M)/N^\circ$ is a projective $R_0I(M)$-module by Lemma

1.18, and hence the tensor product $SI_S(M)/N^\circ \otimes_{R_0} M$ is a projective $R_0I(M)$-module, and hence also a projective $RI(M)$-module. However, we have the following chain of isomorphisms

$$\begin{aligned} SI_S(M)/N^\circ \otimes_R M &\simeq_{R_0I(M)} SI_S(M)/N^\circ \otimes_R R_0 \otimes_{R_0} M \\ &\simeq_{R_0I(M)} (R_0 \otimes_R S)I_S(M)/N^\circ \otimes_{R_0} M \\ &\simeq_{R_0I(M)} S^{|R_0:R|} I_S(M)/N^\circ \otimes_{R_0} M \\ &\simeq_{RI(M)} \oplus RI(M)/N^\circ \otimes_{R_0} M\,. \end{aligned}$$

The third isomorphism holds since S is unramified over R. Thus the statement of the lemma follows. q.e.d.

The tensor product $X := R(I_S(M)/N^\circ) \otimes_R S$ is then a $T_0(M)$-module, where $T_0(M)$ acts as Galois automorphisms via $T(M)$ on S and by conjugation on $I_S(M)$. Then $End_{T_0(M)}(X) \simeq H^0(T_0(M), R(I_S(M)/N^\circ) \otimes_R S)$ is the ring of fixed points in $R(I_S(M)/N^\circ) \otimes_R S$ under the diagonal action of $T_0(M)$.

Lemma 1.20. *The $RI(M)$-module $Y := RI_S(M)/N^\circ \otimes_R M$ has*

$$End_{RI(M)}(Y) = H^0(T_0(M), R(I_S(M)/N^\circ) \otimes_R S)\,.$$

Proof: We compute the endomorphism ring of Y as

$$\begin{aligned} End_{RI(M)}(Y) &= End_{RI(M)}(RI_S(M)/N^\circ \otimes_R M) \\ &= (End_{RI_S(M)}(RI_S(M)/N^\circ \otimes_R M))^{T_0(M)} \\ &= (RI_S(M)/N^\circ \otimes_R S^\circ)^{T_0(M)} \\ &= H^0(T_0(M), RI_S(M)/N^\circ \otimes_R S^\circ). \end{aligned}$$

q.e.d.

We now can state the main theorem in this section:

Theorem 1.21. *Assume that N is a normal subgroup of the finite group G, with $|N| \cdot R = R$ and $(|N|, |G : N|) = 1$. Let M be an irreducible RN-lattice with $End_{RN}(M) = S$ a finite unramified extension of R; let $n = dim_S(M)$, and denote by $I(M)$ and $I_S(M)$ the inertia groups of M as RN-module and SN-module resp. Then $I_S(M) \trianglelefteq I(M)$ with quotient $T(M)$, which acts as group of R-Galois automorphisms on S. Assume furthermore that $I_S(M)$ has a complement $T_0(M)$ in $I(M)$. Denote by $SI_S(M)/N^\circ$ the $RI(M)$-module, where $I_S(M)$ acts by left multiplication and $T_0(M)$ acts by conjugation. S° is the $RT_0(M)$-module, where $T_0(M)$ acts via $T(M)$ as Galois automorphism.*

Then the group-ring RG contains a ring direct summand of the form

$$B := Mat_{|G:I_S(M)|\cdot n}(H^0(T_0(M), RI_S(M)/N^\circ \otimes_R S^\circ))\,.$$

Proof: Putting together the statements from the Lemmata 1.18, 1.19, 1.20 we conclude that the group-ring $RI(M)$ contains a ring direct summand of the form

$$B_{I(M)} := Mat_{|I(M):I_S(M)|\cdot n}(H^0(T_0(M), RI_S(M)/N^\circ \otimes_R S^\circ)) .$$

Passing from $I(M)$ to G is done exactly as in the proof of Theorem 1.14, where one passed from the inertia group of M, $I_S(M)$, to G. This process gives a full $|G : I(M)|$- matrix ring over $B_{I(M)}$. q.e.d.

1.1. Examples for Clifford theory.

EXAMPLE 1: The structure of the inertia groups and also the structure of blocks is quite different in case $R = \hat{\mathbb{Z}}_p$ and in case R has a field of fractions which is a splitting field for the underlying group and all of its subgroups, as shows the following EXAMPLE, which arose in a discussion with Gerhard Hiss:

Let

$$G = < a, b, c \mid a^4, b^3, c^2, [a, b], {}^c a = a^{-1}, {}^c b = b^{-1} > ,$$

and put $R = \hat{\mathbb{Z}}_2$ and $S = \hat{\mathbb{Z}}_2[\zeta]$, where ζ is a primitive 3-rd root of unity.

We observe that $O_{2'}(G)$, the largest normal subgroup of order prime to 2, is cyclic of order 3, generated by b. Now the group-ring SG or RG has two blocks, the principal block B_0 generated by $\{a , c\}$ and a block B, where we can apply Clifford theory, since b does not act trivially.

Let $M_i = S$ be the non-trivial $R(< b >)$ module with b acting via ζ^i for $i = 1, -1$. As $R(< b >)$-modules $M_1 \simeq M_{-1}$ but as $S(< b >)$-modules we have M_1 not isomorphic to M_{-1}. Since the Galois automorphism of S over R is moving M_1 to M_{-1}, and since this action is also achieved by c, we conclude:

1. The inertia group $I(M_1)$ over R is G.
2. The inertia group $I_S(M_1)$ is just $< a, b >$.
3. The group $T := I(M)/I_S(M) =< c >$ is just isomorphic to the Galois-group of S over R.

So we apply the theory: We have to compute $H^0(< c >, SI_S(M)/ < b >^0)$; but c is the Galois action. Hence it acts just on S, and so the fixed points are

$$H^0(< c >, SI_S(M)/ < b >) = RI_S(M)/ < b > \simeq R(< a >) .$$

We now have to tensor over R with S under the Galois-action and what we get is $S(< a >)^0$. Hence the group ring RG has a direct summand of the form:

$$B_R = (S(< a >)^0)_2 .$$

Similar calculation show that SG – now we are doing Clifford theory for SG – has a ring direct summand of the form $B_S = (S(< a >))_2$.

Note that on B_S the group $< c >$ acts trivially but on B_R it does not; as a matter of fact not even the rational rings are isomorphic, since B_S has an epimorphic image $\hat{Z}_2[\zeta]$ and B_R has an epimorphic image $\hat{Z}_2[\zeta, i]^\Delta$, where Δ is the diagonal involution in $Gal(hat\mathbb{Z}_2[\zeta, i])$ and i is a primitive 4-th root of unity.

This example shows at the same time that the THEOREM OF PUIG for nilpotent blocks [Pu; 81], which in our case states that in the splitting case B is Morita equivalent to the group-ring of the defect group $< a >$ DOES NOT HOLD IN THE NON SPLITTING SITUATION.

EXAMPLE 2: We next shall give an example that the splitting of $I(M)$ over $I_S(M)$ is not automatic:

Let $H = \mathbb{F}_7 \rtimes \mathbb{F}_7^*$ be the affine group of the line over $\mathbb{F}_7$. Then $C_3 \leq \mathbb{F}_7^*$ leads to the semidirect product $K := \mathbb{F}_7 \rtimes C_3$. We form the pull-back along $\alpha : C_9 \to C_3$ to get the central extension of K, the group $G = \mathbb{F}_7 \rtimes C_9$.

Let M be a faithful irreducible $\hat{\mathbb{Z}}_3C_7$ lattice. Since its dimension over $\hat{\mathbb{Z}}_3$ is 6, it is unique. We choose for Clifford's Theorem $N = \mathbb{F}_7$ and $R = \hat{\mathbb{Z}}_3$. Then $S = End_{\hat{\mathbb{Z}}_3C_7}(M) = \hat{\mathbb{Z}}_3[\zeta_7]$ with ζ_7 a primitive 7^{th} root of unity. The exact sequences

$$0 \longrightarrow I(N)\uparrow_N^G \longrightarrow RG \longrightarrow RC_9 \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow I(N)\uparrow_N^{G/C_3} \longrightarrow RG/C_3 \longrightarrow RC_3 \longrightarrow 0$$

are two sided split. Since $I(N)\uparrow^{G/C_3}$ is a block of defect 0, which contains M, the block $I(N)\uparrow_N^G$ is a block of defect at most 1. Hence $I_S(M) = \mathbb{F}_7 \times C_3$, however $I(M) = G$, since M has no conjugates over R. Therefore $I_S(M)$ has no complement in $I(M)$.

2. AN OUTER AUTOMORPHISM OF A GROUP BECOMES INNER IN THE INTEGRAL GROUP-RING AND A COUNTEREXAMPLE TO THE ISOMORPHISM PROBLEM FOR POLY-CYCLIC GROUPS

2.1. **Introduction.** The results in this section were obtained in joint work with Alexander Zimmermann [RoZi1; 95], [RoZi2; 95].

In this section we shall construct a finite group G which has an AUTOMORPHISM α OF G, WHICH IS NOT INNER; however, the INDUCED AUTOMORPHISM ON SG IS INNER, where S is the ring of algebraic integers in a suitably chosen algebraic number field. A consequence of our arguments is that α is inner in KG for every field K.

We shall apply this then to CONSTRUCT TWO POLY-CYCLIC (INFINITE) GROUPS H_1 AND H_2, WHICH ARE NOT ISOMORPHIC, BUT THE GROUP-RING SH_1 AND SH_2 ARE ISOMORPHIC for S as above.

It is a RESULT OF COLEMAN [Co; 64] that the natural map

$$\Phi : Out(G) \longrightarrow Out(SG)$$

is injective for p–groups[10]. Here S is the ring of integers in a global or a local number field K, in which p is not invertible.

Remark 2.1. J. Krempa proved that there can at most be an elementary abelian two group in the kernel of the group homomorphism from $Outcent(G)$ to $Outcent(\mathbb{Z}G)$ (cf. [JaMa;87]).

It is a question of Jackowski and Marciniak ([JaMa;87]) whether it happens that Φ is injective for all finite groups G and coefficient ring $\mathbb{Z} = S$. Our example is not directly a counterexample, since S is a finite extension of $\mathbb{Z}$. Using algebraic K-theory as in [RoSc; 87] one surely can also construct a counterexample for the coefficient ring $\mathbb{Z}$. We did though not elaborate on this.

If one wants to construct an automorphism $\alpha \in Aut(G)$ such that $\Phi(\alpha)$ becomes inner on SG, then

1. α must be the identity on the conjugacy classes of G,
2. α must be inner in G on the Sylow p–subgroups, combining Sylow Theorems and the result of Coleman (cf. [Co; 64]).

These are PURELY GROUP THEORETICAL PROPERTIES, and it is relatively easy to construct such an α. However, in order to show that α becomes inner on SG we have to use the following INGREDIENTS FROM INTEGRAL REPRESENTATION THEORY:

1. We show that α is inner on $\mathbb{Z}G$ semi-locally. From this, one can not automatically conclude that α is inner on $\mathbb{Z}G$. The OBSTRUCTION is an element in the CLASS GROUP of $\mathbb{Z}G$. We use CLASS FIELD THEORY – this is where the field K and its ring of integers enter – to kill this obstruction [Ro; 96,I], Section 2.
2. The passage from the LOCAL TO THE SEMI-LOCAL situation becomes possible by INTERPRETING AUTOMORPHISMS AS INVERTIBLE BIMODULES and using Fröhlich's exact sequence of Picard groups [Fr; 73].

[10] $Outcent(X)$ denotes for a group the conjugacy preserving automorphisms modulo the inner automorphisms, for a ring the automorphisms leaving the center element-wise fixed modulo conjugation by units.

3. Finally we are in the local respectively complete situation[11]. Here – in the complete situation – we use Clifford theory (cf. Section 2.2) to show, that α acts as inner automorphism on the inertia groups, after having applied the theorem of Noether-Deuring (cf. Lemma [Ro; 96,I] 2.2), to pass to a SPLITTING FIELD. The key point in our construction is to involve quaternion groups in order to keep the inertia groups small.
4. Let now K be any field, then there exists a rational prime p, such that $KG \simeq K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p G$. Thus α induces an inner automorphism on KG.

2.2. **Clifford theory for automorphisms.** Clifford theory will be essential for our arguments later on. We recall the setup (cf. Section 1). The HYPOTHESES ARE THE FOLLOWING.

Assumption 2.2. 1. N is a normal complemented subgroup of a finite group G with $(|G/N|, |N|) = 1$.
2. R is a complete local Dedekind domain of characteristic 0 with residue field $\mathfrak{k}$ of characteristic $p > 0$ such that the quotient field K of R is a SPLITTING FIELD for G and all of its subgroups.
3. p does not divide the order of N.
4. $I(M)$ is the inertia group of an irreducible RN-lattice M in G (cf. Section 1 1.1).
5. $I(M)$ is normal in G.
6. e_M is the central primitive idempotent of RN which is the identity on M. For each t in $I(M)$, the conjugate $t \cdot e_M \cdot t^{-1}$ is again a central primitive idempotent of RN which acts trivially on M, hence $t \cdot e_M \cdot t^{-1} = e_M$. Thus e_M is centralized by $I(M)$.
7. $\{g_1, ..., g_s\}$ is a set of left coset representatives of $I(M)$ in G. Then
$$e_i := g_i \cdot e_M \cdot g_i^{-1}; i = 1, ..., s$$
are the DIFFERENT conjugates of e_M in RG. Since the $\{g_i\}$ form a transversal with respect to the inertia group of M, the conjugates e_i are pairwise orthogonal idempotents in RG, and $\sum_{i=1}^{s} e_i =: e$ is a CENTRAL IDEMPOTENT IN RG. Thus $RG \cdot e$ is isomorphic to an $(s \times s)$-matrix ring, whose (i, j)-entry equals
$$B_{i,j} := e_i \cdot RG \cdot e_j \text{ for } 1 \leq i, j \leq s,$$
the Pierce decomposition.
8. α is a central group automorphism, which thus induces an inner automorphism of KG.
9. α acts as the identity on N and so $\alpha(e_M) = e_M$.

[11] Recall that lattices are locally isomorphic if and only if they are isomorphic over the completion.

10. $x^{-1} \cdot \alpha(x) \in I(M)$ for all $x \in G$. This implies

$$\alpha(e_i) = \alpha(g_i \cdot e_M \cdot g_i^{-1}) = \alpha(g_i) \cdot \alpha(e_M) \cdot \alpha(g_i)^{-1} = e_i .$$

WE LOOK AT $B_{i,j}$ MORE CLOSELY:

$g \in G$ can be written as $n \cdot h$ with $n \in N$ and $h \in G/N$. Then $h = g_k \cdot t$ with $t \in I(M)$.

We denote by χ_i THE CHARACTER OF N AFFORDED BY e_i.

Using the normality of $I(M)$ we calculate

$$\begin{aligned} e_i \cdot n \cdot h \cdot e_j &= \chi_i(n) \cdot e_i \cdot g_k \cdot t \cdot e_j \\ &= \chi_i(n) \cdot g_i \cdot e_M \cdot g_i^{-1} \cdot g_k \cdot g_j \cdot e_M \cdot g_j^{-1} \cdot t . \end{aligned}$$

If $g_k \cdot g_j \notin g_i \cdot I(M)$, which is equivalent to $g_k \notin g_i \cdot g_j^{-1} \cdot I(M)$, then – again by the normality of $I(M)$ – the above equals 0. Hence,

$$B_{i,i} = g_i \cdot e_M RI(M) \cdot e_M \cdot g_i^{-1}.$$

More precisely, the above calculations show

$$e_i \cdot RG \cdot e_j = g_i \cdot e_M \cdot RI(M) \cdot e_M \cdot g_j^{-1} .$$

THIS ALLOWS US TO DESCRIBE EXPLICITLY THE ACTION OF α ON $B_{i,j}$:

$$\alpha : g_i \cdot e_M \cdot x \cdot e_M \cdot g_j^{-1} \longrightarrow \alpha(g_i) \cdot e_M \cdot \alpha(x) \cdot e_M \cdot \alpha(g_j^{-1}) \qquad (3)$$

for $x \in RI(M)$. A central automorphism α of RG fixes $B_{i,j}$ as set for all $1 \leq i, j \leq s$ if and only if it is given by conjugation with an element of the form

$$\begin{pmatrix} u_1 & 0 & & & 0 \\ 0 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ 0 & & & 0 & u_s \end{pmatrix},$$

with elements $\{u_i \in (RI(M) \cdot e_M) \cap U(KI(M) \cdot e_M) \mid i = 1, .., s\}$ [12].

IF α ACTS AS THE IDENTITY ON $I(M)$ AND IF MOREOVER $\alpha(g_i) \in g_i \cdot Z(I(M))$, then it acts as identity on $g_i \cdot e_M \cdot RI(M) \cdot e_M \cdot g_i^{-1}$, and then $u_i \in Z(RI(M) \cdot e_M)$ for all $i = 1, \ldots, s$. Since R is local,

$$Outcent(RI(M) \cdot e_M) \simeq Outcent(Mat_n(RI(M) \cdot e_M)) .$$

Consequently, there is an inner automorphism γ of $Mat_n(RI(M) \cdot e_M) = RG \cdot e$ such that

$$\alpha = \gamma \circ \delta$$

[12] Note that this can always be achieved by multiplication with a central element.

with a central automorphism δ acting via the identification

$$B_{i,j} = e_i \cdot RG \cdot e_j = g_i \cdot e_M \cdot RI(M) \cdot e_M \cdot g_j^{-1}$$

on each of the entries $B_{i,j}$. Since δ is then conjugation by an element

$$v \in (RI(M) \cdot e_M) \cap U(KI(M) \cdot e_M),$$

the unit γ has to have a diagonal form analogous to that of α. Thus there are elements $\gamma_i \in U(RI(M) \cdot e_M)$ such that

$$u_i = \gamma_i \cdot v \in Z(RI(M) \cdot e_M).$$

Consequently,

$$u_i \cdot u_j^{-1} \in U(RI(M) \cdot e_M) \text{ for all } 1 \leq i, j \leq s.$$

We summarize these observations, which we shall apply in the next section, as

Proposition 2.3. *With the above notation, assume that one of the elements u_i can be chosen to be a unit in $RI(M) \cdot e_M$, then the automorphism α is inner.*

2.3. **The construction of the group and the automorphism.** We construct our group G step by step. The construction is done very carefully, having the goal described in Section 2.1 in mind:

Let $\overline{H}$ be the semidirect product of the cyclic group of order 8, generated by a, with its automorphism group $C_2 \times C_2$, generated by $\overline{b}$ and $\overline{c}$, where $\overline{b}$ inverts a and $\overline{c}$ raises a to the third power.

$\overline{H}$ has a central automorphism σ sending $\overline{c}$ to $a^4\overline{c}$ and fixing $\overline{b}$ and a. This automorphism is NOT INNER.

Let now

$$H_0 := (C_8 \rtimes (C_2 \times C_2)) \rtimes < \sigma > .$$

We note that $< a , \sigma >$ is a normal subgroup of H_0.

The next step is to involve quaternion groups. Let $Q_8 :=< b , c >$ be the quaternion group of order 8. Modulo its center, Q_8 maps onto $C_2 \times C_2$, say via π_0. We can thus identify the image of b with $\overline{b}$ and that of c with $\overline{c}$. We thus can form the pull-back

$$\begin{array}{ccccccccc} 1 & \to & C_8 & \longrightarrow & H_0 & \longrightarrow & C_2 \times C_2 \times < \sigma > & \to & 1 \\ & & \| & & \uparrow & & \uparrow \pi_Q \times id_{<\sigma>} & & \\ 1 & \to & C_8 & \longrightarrow & H & \longrightarrow & Q_8 \times C_2 & \to & 1 . \end{array}$$

Up to now we have constructed a two group. Next we form the semidirect product with two finite irreducible modules of odd order: Both modules are fixed by $< a >$. We note that $H / < a > \simeq Q_8 \times < \sigma >$. We choose odd primes

p_1 , p_2 and let M_i be the irreducible 2-dimensional $\mathbb{F}_{p_i}Q_8$-module[13]. σ fixes M_1 and inverts M_2 .

Finally G is the semidirect product of M with H :

$$G := (M_1 \times M_2) \rtimes H .$$

We shall construct – again very carefully – our automorphism α of G.

α will be conjugation with σ on H , i. e. it is inner on H . Since σ is the identity modulo $< a >$, and since a acts trivially on M_i , we may extend the conjugation with σ to an automorphism α on G , by letting it act trivially on M_i .

Lemma 2.4. *α is a central automorphism of G , which is not inner.*

In fact, on H the elements h and $\alpha(h)$, which are conjugate by σ are also conjugate by either a or a^2 or 1 . All these elements centralize $M_1 \times M_2$, and therefore, α as automorphism of G is central. In fact, for every integers i and j the elements $a^i \cdot b \cdot c \cdot \sigma^j$ and $a^i \cdot b^3 \cdot c \cdot \sigma^j$ are centralized by $a \cdot \sigma$, the elements $a^i \cdot c \cdot \sigma^j$ and $a^i \cdot b^2 \cdot c \cdot \sigma^j$ are centralized by $a^2 \cdot \sigma$ and the rest of H is centralized by σ itself.

From the above considerations it also follows that α can not be inner. However, this can also be checked with the group theory computer-system GAP, which has been introduced at this conference.

2.4. **The automorphism is semi-locally inner.** We shall show that the automorphism α of G is inner on RG for a suitable semi-local ring of integers R , where neither 2 nor p_1 nor p_2 are invertible. Since we interpret the question of whether α is inner as a question on invertible bimodules, by the validity of the Noether-Deuring Theorem (cf. [Ro; 96,I] Lemma 2.2), it is NO LOSS OF GENERALITY IF WE ASSUME THAT THE QUOTIENT FIELD OF R IS A SPLITTING FIELD FOR G AND ALL SUBGROUPS.

Using Fröhlich's result ([Fr; 73]) for semi-local domains R

$$Outcent(RG) \simeq \prod_{\wp \in Spec(R)} Outcent(R_\wp G)$$

and interpreting α as an invertible bimodule IT IS ENOUGH TO SHOW THAT α IS INNER FOR ALL COMPLETIONS OF R AT FINITE PRIMES $\wp$.

For a global Dedekind domain R the OBSTRUCTION to globalize local automorphisms is a certain subgroup of the CLASS GROUP of the center of the group-ring.

[13] Recall that $\mathbb{F}_{p_1}Q_8 \simeq (\mathbb{F}_{p_1})_2 \prod_{i=1}^{4} \mathbb{F}_{p_1}$.

In fact, denoting by $Cl(\Lambda)$ the locally free class group of the order Λ, the sequence

$$1 \to Cl(centre(RG)) \longrightarrow Picent(RG) \longrightarrow \Pi_{\wp \in Spec(R)} Picent(R_\wp G) \to 1$$

is exact (cf. [Fr; 73]).

Using class field theory (cf. [Ja; 68a]) this implies that there exists a ring of algebraic integers S in an algebraic number field L, L being finite over the quotient field of R, such that α becomes inner as an automorphism of SG.

2.5. **The crucial calculation.** We now come to the central point in the proof: We shall indicate in the next subsection that the following three groups will occur as inertia groups at various primes $\wp$:

1. $I_1 := < M, a, \sigma >$ is normal in G.
2. $I_2 := < M, a, \sigma \cdot b^2 >$ is normal in G.
3. $I_3 := < M, a > = I_1 \cap I_2$.

We take this for granted for the moment.

At the prime $\wp$ the group-ring has a ring direct factor of the form

$$(g_i \cdot e_M \cdot R_\wp I \cdot e_M \cdot g_j^{-1})_{1 \le i,j \le s}$$

with $I \in \{I_1, I_2, I_3\}$ (cf. 1 1.14). Recall that $e_i = {}^{g_i}e_M$ for $g_i \in H_k$ and H_k suitably chosen according to the inertia groups I_k.

We now consider the three cases I_1, I_2 and I_3 separately.

In case $I = I_1$ we can choose the transversal

$$H_1 := \{1, b, b^2, b^3, c, cb, cb^2, cb^3\} = H_1^0 \cup c \cdot H_1^0 .$$

In case $I = I_2$ we can choose the same transversal

$$H_2 := \{1, b, b^2, b^3, c, cb, cb^2, cb^3\} = H_2^0 \cup c \cdot H_2^0 .$$

In case $I = I_3$ we can choose the transversal

$$H_3 := \{1, b, b^2, b^3, \sigma, b\sigma, b^2\sigma, b^3\sigma, c, cb, cb^2, cb^3, c\sigma, cb\sigma, cb^2\sigma, cb^3\sigma\} = H_3^0 \cup c \cdot H_3^0 .$$

We have seen that

$$e_i \cdot R_\wp G \cdot e_j = g_i \cdot e_M \cdot R_\wp I \cdot e_M \cdot g_j^{-1} .$$

Thus, by the above formula, in case $I = I_1$ and in case $I = I_2$ the action of α on $(e_i \cdot R_\wp G \cdot e_j)_{1 \le i,j \le s}$ is multiplication by a^4 if $4 \le |i - j|$ and the identity otherwise, hence, conjugation by the matrix

$$\begin{pmatrix} 1_{4\times 4} & 0 \\ 0 & a^4 \cdot 1_{4\times 4} \end{pmatrix} .$$

Here we denote by $1_{m\times m}$ the $m \times m$ unit matrix. Therefore, α acts as inner automorphism on the ring direct factor corresponding to e_M.

In case $I = I_3$ a similar observation yields that α acts as conjugation by

$$\begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & a^4 \cdot 1_{8\times 8} \end{pmatrix}.$$

Hence, by Proposition 2.3, α acts as inner automorphism on the ring direct factor corresponding to e_M as well.[14]

2.6. **Determination of the inertia groups.** We choose a prime $\wp \in Spec(R)$, then there are various possibilities for the inertia groups. We shall only discuss one case in detail:

$$2 \in \wp .$$

Then the group-ring decomposes as

$$R_\wp G = I_{R_\wp}(M)G \times R_\wp H .$$

On $R_\wp H$ the automorphism α is conjugation by σ. For $I_{R_\wp}(M)G$ we apply Clifford theory (Section 1). Let χ be an irreducible nontrivial character of M as abelian groups (the quotient field of R is a splitting field for M) with kernel $K \subseteq M_1 \times M_2$.

For χ there are three cases which have to be considered separately:

1. If $M_2 \subset K$, then the inertia group I_χ contains M, σ and a. If $h \in H$ lies in I_χ, we observe that the sequence

$$1 \longrightarrow \; < a, \sigma > \; \longrightarrow H \longrightarrow Q_8 \longrightarrow 1$$

   splits. Hence, we may assume that $h \in Q_8$, but there are no fixed points on either M_1 or M_2, a contradiction. Thus

$$I_\chi =< M, a, \sigma >= I_1 ,$$

   and α is inner.
2. If $M_1 \subset K$, then obviously $I_\chi \supseteq < M, a, b^2 \cdot \sigma >$. Similar arguments as above show that

$$I_\chi =< M, a, b^2 \cdot \sigma >= I_2 .$$

   Hence α is inner.
3. If neither M_1 nor M_2 is contained in K, then the inertia group

$$I_\chi =< M, a >= I_3$$

[14] Note how sensitive this construction is with respect to the group G.

is the intersection of the two inertia groups of the previous cases, since $M = M_1 \times M_2$ and $(\wp, |M|) = 1$. Again we see that α acts as inner automorphism on the factor corresponding to that character χ of M.

Our automorphism α from Section 2.3 now acts as inner automorphism on each of the factors and $\alpha_\wp$ is inner in $R_\wp G$ provided $2 \in \wp$.

The cases $p_1 \in \wp$, $p_2 \in \wp$ and $|G|R_\wp = R_\wp$ are treated similarly.

This shows that α is an inner automorphism of RG.

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ON THE ZASSENHAUS CONJECTURE AND ČECH COHOMOLOGY

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ABSTRACT. We introduce some kind of Čech-cohomology which is very well suited to treat the Zassenhaus Conjecture, which states that in $\mathbb{Z}G$ all group bases are rationally conjugate. For solvable groups we then discuss several applications to the Isomorphism problem and the Zassenhaus Conjecture. In particular, we can give a necessary and sufficient condition, purely in terms of the group G, for when the Isomorphism problem is true for a large quotient of the integral group ring.

1. Zassenhaus Conjecture

The origins of these results go back to joint work with L. L. Scott and are obtained in collaboration with W. Kimmerle [KiRo: 93].

1.1. Introduction.

Problem 1.1. [IP:] The ISOMORPHISM PROBLEM asks: Does

$$\mathbb{Z}G = \mathbb{Z}H$$

imply that there is a group isomorphism

$$\rho : G \longrightarrow H\ ?$$

We note that every isomorphism between integral group-rings can be modified to an augmented automorphism, i. e. it commutes with the augmentations of G and H resp.

In the sequel we shall always ASSUME THAT HOMOMORPHISMS BETWEEN GROUP RINGS ARE AUGMENTED.

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Remark 1.2. The CLASS SUM CORRESPONDENCE implies, that there exists a BIJECTION – not necessarily an isomorphism

$$\beta : G \longrightarrow H \text{ with } \underline{K_g} = \underline{K_{\beta(g)}} .$$

where $\underline{K_g} := \sum_{x \in G/C_G(g)} {}^x g$ is the CLASS SUM OF g IN $\mathbb{Z}G$.

Problem 1.3. [ZC:]

1. The ZASSENHAUS CONJECTURE asks whether the above map

   β can be chosen to be an isomorphism of groups.

   This is equivalent to the existence of

   $$a \in \mathbb{Q}G^\times \text{ with } a \cdot G \cdot a^{-1} = H ,$$

   since on the center β is the identity. Here $\mathbb{Q}G^\times$ denotes the units in $\mathbb{Q}G$. This is a very strong positive answer to the Isomorphism problem. As a matter of fact, it is too strong, as was shown in collaboration with L. L. Scott, by constructing a COUNTEREXAMPLE ([RoSc; 87]; see also [Ro; 89] [Ro; 92dmv], Chapter IX and [Kl; 91]). We should point out here, that our COUNTEREXAMPLE IS A METABELIAN GROUP .
   Using a trick of Kimmerle, one can show that the Zassenhaus Conjecture is more or less equivalent to:
   (a) The Isomorphism problem has a positive answer.
   (b) Every augmented automorphism α of $\mathbb{Z}G$ has the form $\alpha = \gamma \cdot \rho$, where γ is a central automorphism of $\mathbb{Z}G$ and ρ is an automorphism of G.
2. The p-VERSION OF THE ZASSENHAUS CONJECTURE , asks whether there exists an isomorphism (in particular the Isomorphism problem has a positive answer)

   $$\rho = \rho_p : G \longrightarrow H ,$$

   such that on a Sylow p-subgroup $P \subset \mathbb{Z}G = \mathbb{Z}H$ of G the isomorphism ρ_p is given by conjugation with an element $a_P \in \mathbb{Q}G^\times$; i. e. it induces β on the class sums of p-power elements.

The most far reaching result thus far, which was obtained by L. L. Scott when collaborating with the author in 1987, is the following, which I shall only formulate for SOLVABLE groups:

Theorem 1.4 (Scott [Sc; 87], [Sc; 90]). *Let G be a solvable group with $O_{p'}(G) = 1$, then the Zassenhaus Conjecture is true. (Note that since G is solvable, $O_{p'}(G) = 1$ is equivalent to the fact that $\mathbb{Z}_pG$ consists only of the principal block.)*

Remark 1.5. 1. In order to deal with the Zassenhaus Conjecture for solvable groups in general, we may thus assume that $O_{p'}(G) \neq 1$ for every prime divisor p of the order $|G|$ of G.
For a solvable group we are thus in the following situation:
For every prime divisor p of G, we have an epimorphism

$$\phi_p : G \longrightarrow G/O_{p'}(G).$$

On each of the images the Zassenhaus Conjecture is true. We are going to develop an OBSTRUCTION THEORY for the Zassenhaus Conjecture resp. the Isomorphism problem to hold for G.

2. We shall deal here with the ISOMORPHISM PROBLEM , the ZASSENHAUS CONJECTURE and certain p-VERSION OF THE ZASSENHAUS CONJECTURE from a conceptional point of view by defining a Čech style cohomology set – this idea goes back to the collaboration of L. L. Scott with the author – which yields obstructions for these conjectures to be true.

Using this we can give a criterion on the structure of H purely in terms of the group theory of G, provided $\mathbb{Z}G = \mathbb{Z}H$ (cf. Theorem 2.4).

Moreover, we have the following new class of groups, for which the Isomorphism Problem has a positive answer – here we do not necessarily require that G is solvable. For the TERMINOLOGY, we shall write

$$Aut_p(G)$$

for those automorphisms ρ of G such that for every p and every p-power element $x \in G$ the elements x and $x\rho$ are conjugate.

$$Aut_c(G)$$

stands for the group of conjugacy class preserving automorphisms of G. Then surely $Aut_c(G) \subset Aut_p(G) \subset Aut(G)$.

Theorem 1.6 ([KiRo; 93]). *Assume that*

1. *G has a nilpotent normal subgroup N such that G/N is nilpotent,*
2. *for each quotient X of G/N, the group $Aut_p(X)$ consists of inner automorphisms only. (Note that this is a group theoretical condition. Moreover, since G/N is nilpotent, we have $Aut_p(G) = Aut_c(G)$.)*

Then the p-version of the Zassenhaus Conjecture holds for G and all p. In particular, the Isomorphism problem has a positive answer.

Corollary 1.7. *Assume that $G/F(G)$ is abelian, where the Fitting subgroup $F(G)$ is the largest nilpotent normal subgroup of G. Then the p-version of the Zassenhaus Conjecture holds for G and all p. (We point out that a corresponding result for the Zassenhaus Conjecture in case of an abelian normal* *subgroup with nilpotent quotient was proved by L. L. Scott* [Sc; 85] *and the author* [RoSc; 85], *and was extended by A. Zimmermann* [Zi; 90].*)*

Remark 1.8. We point out that the counterexample to the Zassenhaus Conjecture is metabelian, and hence for this example the p-version of the Zassenhaus Conjecture is true.

Theorem 1.9 ([KiRo; 93]). *If G is a Frobenius group or a 2-Frobenius group, then the p-version of the Zassenhaus Conjecture holds for G and all p.*

Recall that a Frobenius group is a group G with a non trivial subgroup H such that $H \cap {}^xH = 1$ for every $x \in G \setminus H$. Then there exists a normal subgroup N of G, the Frobenius kernel of G with $G/N \simeq H$. A 2-Frobenius group has a chain of normal subgroups

$$1 < N < T < G,$$

such that T is a Frobenius group with kernel N, and G/N is a Frobenius group with kernel T/N.

Let me point out that the class of solvable groups, for which the integral augmentation ideal decomposes, consists exactly of the Frobenius and the 2-Frobenius groups (cf. [GR1; 75], [GR2; 76], [GR3; 75]) – an interesting coincidence?

2. Čech cohomology

2.1. **Projective limits.** We shall assume from now on that G is solvable.
In order to set up our obstruction theory, we TRY to describe a finite group G as PROJECTIVE LIMIT with respect to certain families of normal subgroups $\{N_i\}_{1\leq i\leq n}$. We put $G_i := G/N_i$. The projective limit is then defined as

$$lim.proj.(G_i) = \{(g_1, \ldots, g_n) \mid g_i \in G_i \,:\, g_i \equiv g_j \text{ in } G/(N_i \cdot N_j)\}. \qquad (1)$$

We then have an induced natural map $G \longrightarrow lim.proj.(G_i)$.

We do not have a general criterion for when this map is an isomorphism; however, we have

Lemma 2.1 ([KiRo; 93]). *The map in Equation 1 is an isomorphism, provided $\bigcap_i N_i = 1$ and for every $p \in \pi(G)$ [1] there is an index i with $(p, |N_i|) = 1$. This applies in particular, if*

$$N_i = O_{p_i'}(G), \text{ where } \{p_i\} = \pi(G).$$

[1] For the finite group G we denote by $\pi(G)$ the set of rational prime divisors of $|G|$.

The basic questions around the Zassenhaus Conjecture, mentioned above, should also be seen as a COMPARISON BETWEEN THE CATEGORY OF GROUPS AND THE CATEGORY OF GROUP-RINGS.

THE BEHAVIOR OF PROJECTIVE LIMITS IS QUITE DIFFERENT IN BOTH CASES. If G is the projective limit of the quotients $G_i := G/N_i$, then the projective limit

$$\Gamma_G := \Gamma_G(\{G_i\}) := lim.proj.(\mathbb{Z}G_i) \tag{2}$$

of the group-rings $\mathbb{Z}G_i$ does not coincide with $\mathbb{Z}G$. As a matter of fact,

$$\Gamma_G(\{G_i\}) := \{(\gamma_1, \ldots, \gamma_n) \mid \gamma_i \in \mathbb{Z}G_i : \gamma_i \equiv \gamma_j \text{ in } \mathbb{Z}G/(N_i \cdot N_j)\},$$

and over the rationals it is in general a relatively small proper quotient of $\mathbb{Z}G$. The map σ induced from the universal property of projective limits, $\sigma : \mathbb{Z}G \longrightarrow \Gamma(G)$ has kernel

$$Ker(\sigma) = \bigcap\nolimits_{1 \leq i \leq n} I(G, N_i), \tag{3}$$

where $I(G, N_i)$ is the kernel of the natural map $\mathbb{Z}G \longrightarrow \mathbb{Z}G_i$.

We should point out that G EMBEDS INTO $\Gamma(G)$, and in many cases $\Gamma(G)$ determines G. For example we have

Lemma 2.2. *If $G = \prod_{i=1}^{s} P_i$ is a finite nilpotent group with Sylow p_i-subgroup P_i for $1 \leq i \leq s$, then the projective limit $\Gamma_G(\{P_i\})$ determines G uniquely, up to isomorphism. It should be noted that $P_i = G/O_{p_i'}$.*

Proof: In fact, G is uniquely determined by P_i and $\Gamma_G(P_i)$ projects onto $\mathbb{Z}P_i$. q.e.d.

Example 2.3. Let $G := \prod_{1 \leq i \leq n}(P_i)$ be a nilpotent group with P_i a Sylow p_i-subgroup. Since direct products are special cases of projective limits,

$$G = lim.proj._{1 \leq i \leq n} P_i$$

is a projective limit, and $\Gamma(G)$ is the product of $\{\mathbb{Z}P_i\}_{1 \leq i \leq n}$ in the category of $\mathbb{Z}$-augmented algebras. This shows that $\Gamma(G)$ is a VERY NATURAL CONSTRUCTION. The corresponding group-ring is the tensor product, which is the PRODUCT IN THE CATEGORY OF $\mathbb{Z}$-ALGEBRAS. More precisely, if $\epsilon_G : \mathbb{Z}G \longrightarrow \mathbb{Z}$ is the augmentation, then

$$\Gamma(G) = \{(x_i)_{1 \leq i \leq n} : x_i \in \mathbb{Z}P_i \mid \epsilon_{P_i}(x_i) = \epsilon_{P_j}(x_j)\};$$

i. e., rationally, $\Gamma(G)$ consists only of those irreducible representations, where at most one Sylow p-subgroup acts non trivially.

The MAIN RESULT GIVES AN EXPLICIT DESCRIPTION OF H IN TERMS OF G PROVIDED $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras, in case G is solvable. Before we can state it, we have to introduce some more notation:

We know that G is the projective limit of the groups $G_i = G/O_{p_i'}(G)$, and we put

$$G_{ij} = G/(O_{p_i'}(G) \cdot O_{p_j'}(G)).$$

WE DENOTE THIS PROJECTIVE LIMIT OF THE GROUP-RINGS $\mathbb{Z}G_i$ BY $\Gamma_G(\mathcal{O})$. Assume that we are given a 'central cocycle'

$$\rho = (\rho_{ij})\,; \text{ i. e. } \rho_{ii} = 1 \text{ and } \rho_{ij} = \rho_{ji}^{-1},$$

where ρ_{ij} are conjugacy class preserving automorphisms of G_{ij}; i. e. $\rho_{ij} \in Aut_c(G_{ij})$, the group of conjugacy class preserving automorphisms of G. Then

$$G(\rho) = \{(g_i) \in \prod_{1 \leq i \leq n} G_i \mid (g_i \cdot G_{ij})\rho_{ij} = (g_j \cdot G_{ij})\,\}$$

is a group.

One of the main results is then as follows. The crucial application is given in Proposition 3.3, where this is used to give a necessary and sufficient condition for the Zassenhaus Conjecture to hold for Γ.

Theorem 2.4 ([KiRo; 93]). 1. *Assume that $\Gamma_G(\mathcal{O}) = \Gamma_H(\mathcal{O})$ as augmented algebras and that G is solvable. Then there exists a central cocycle ρ such that $H \simeq G(\rho)$.*

2. *Moreover, $G(\rho) \simeq G$ if, and only if there are automorphisms $\rho_i \in Aut(G_i)$ - not necessarily in $Aut_c(G_i)$ - with*

$$\rho_{ij} = \rho_i \cdot \rho_j^{-1}.$$

Hence the obstruction is a question of lifting automorphisms.

LET US POINT OUT THAT THE ABOVE IS A PURELY GROUP THEORETICAL DESCRIPTION.

Note that the hypothesis of Theorem 2.4 is in particular satisfied, if $\mathbb{Z}G = \mathbb{Z}H$. So the result shows that then

$$H \simeq G(\rho),$$

which shows in particular that G AND H SHARE MANY PROPERTIES.

We point out that the ρ_{ij} are CENTRAL automorphisms.

This theorem can be used to construct examples:

Example 2.5. There are TWO NON ISOMORPHIC GROUPS G AND H written as projective limits, such that semi-locally – $\mathbb{Z}_{\pi(G)}$ is the semi-localization of $\mathbb{Z}$ at the primes dividing $|G|$ –

$$\mathbb{Z}_{\pi(G)} \otimes_{\mathbb{Z}} \Gamma_G(\mathcal{O}) \simeq \mathbb{Z}_{\pi(G)} \otimes_{\mathbb{Z}} \Gamma_H(\mathcal{O}),$$

showing that the Isomorphism problem has a negative answer for projective limits of these group rings. We shall describe this example in subsection 3.3.

Using class field theory, we then can find a ring of algebraic integers R such that

$$lim.proj.(RG_i) \simeq R \otimes_{\mathbb{Z}} \Gamma_G(\mathcal{O}) \simeq R \otimes_{\mathbb{Z}} \Gamma_H(\mathcal{O}) \simeq lim.proj.(RH_i).$$

We do not know, whether or not for these groups $\mathbb{Z}_{\pi(G)}G \simeq \mathbb{Z}_{\pi(G)}H$; the groups are just too big.

Remark 2.6. The counterexample to the Zassenhaus Conjecture constructed by L. L. Scott and the author shows that the Zassenhaus Conjecture may fail for $\mathbb{Z}G$ but hold for $\Gamma_G(\mathcal{O})$.

The importance of the p-version of the Zassenhaus Conjecture lies in the following property:

Proposition 2.7 ([KiRo; 93]). *Let $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras. Then the p-version of the Zassenhaus Conjecture holds for the pair of groups G and H, if and only if it holds for $\Gamma_G(\mathcal{O}) = \Gamma_H(\mathcal{O})$.*

In the above proposition we have to start with $\mathbb{Z}G = \mathbb{Z}H$ though.

Unfortunately we do not have a similar result for the Isomorphism problem.

2.2. **Čech Cohomology.** We shall set up the Čech cohomology in a general setting: Let $\mathcal{K}$ be either the category of groups, or of rings or of modules.

Definition 2.8. *Assume that for each $X \in ob(\mathcal{K})$ we are given a 'natural' subgroup $Aut_*(X) \leq Aut(X)$ – e. g. inner automorphism or central automorphisms etc. Let us be given objects and surjective epimorphisms*

$$\{X_i, X_{ij} = X_{ji}, X_{ii} = X_i \mid \phi_{ij} : X_i \longrightarrow X_{ij}, 1 \leq i,j \leq n\}.$$

Then we can form the projective limit

$$\underline{X} = \{(x_i) : x_i \in X_i \mid x_i\phi_{ij} = x_j\phi_{ji}\}.$$

In addition, we assume that the kernels of the maps ϕ_{ij} are '$*$-characteristic'; i. e. they are preserved under the automorphisms in $Aut_*(X)$.

We shall use $\underline{X}$ to stress that we are working with a projective limit.

1. *The* COCYCLES *are defined as:*

$$Z(\underline{X}, Aut_*(\underline{X})) = \{(\rho) = (\rho_{ij}) : \rho_{ij} \in Aut_*(X_{ij}) \mid \rho_{ij} = \rho_{ji}^{-1}\,,\ \rho_{ii} = id_{X_i}\}\,.$$

2. *The* COBOUNDARIES *are defined as*

$$B(\underline{X}, Aut_*(\underline{X})) = \{(\rho_{ij}) \in Z(\underline{X}, Aut_*(\underline{X})) \mid \rho_{ij} = \rho_i \cdot \rho_j^{-1} \text{ for } \rho_i \in Aut_*(X_i)\}\,.$$

3. *Two cocycles are said to be* EQUIVALENT,

$$(\rho_{ij}) \sim (\rho'_{ij}) \Longleftrightarrow \rho_i \cdot \rho_{ij} \cdot \rho_j^{-1} = \rho'_{ij} \text{ for } \rho_i \in Aut_*(X_i)\,.$$

(Here we mean the maps induced on the quotients.)

4. *The equivalence classes of the cocycles are denoted by*

$$\check{H}(\underline{X}, Aut_*(\underline{X}))\,.$$

This is a pointed set, with point the class of the coboundaries: it is called ČECH COHOMOLOGY CLASS OF $\underline{X}$ WITH RESPECT TO $Aut_*(X)$.

5. *For* $\rho = (\rho_{ij}) \in Z(\underline{X}, Aut_*(\underline{X}))$ *we define*

$$X(\rho) := \{(x_i) : x_i \in X_i \mid x_i\phi_{ij} \cdot \rho_{ij} = x_j\phi_{ji}\}\,.$$

This definition should be compared with the Čech cohomology of a topological space with respect to an open covering.

We point out that these definitions depend strongly on the specially chosen subgroup $Aut_*(-) \leq Aut(-)$!

Direct calculations show:

Lemma 2.9 ([KiRo; 93]). *There is an isomorphism of projective limits*

$$\sigma = (\sigma_i) : X(\rho) \longrightarrow X(\rho') \text{ with } \sigma_i \in Aut_*(X_i) \Longleftrightarrow \rho \sim \rho' \text{ in } Z(\underline{X}, Aut_*(\underline{X}))\,.$$

3. Isomorphism problem

3.1. **The Isomorphism problem and projective limits.** Assume that $G \doteq lim.proj._{1 \leq i \leq n}(G_i)$ with common quotients G_{ij}, where

$$N_i = ker(G \longrightarrow G_i)$$

are characteristic subgroups, and assume furthermore that the relative augmentation ideals $I(G, N_i)$ are characteristic in $\mathbb{Z}G$.

We assume that the Isomorphism problem holds for the groups G_i. We denote by Γ_G the corresponding projective limit of the group-rings $\mathbb{Z}G_i$. We note that Γ_G is an augmented algebra.

Assume that $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras. Then the hypothesis implies that we obtain ISOMORPHISMS

$$\beta_i : G_i \longrightarrow H_i \text{ and hence AUTOMORPHISMS } \beta_i \cdot \beta_j^{-1} =: \rho_{ij} \in Aut(G_{ij}),$$

which gives rise to a cocycle $\rho \in Z(\underline{G}, Aut(\underline{G}))$. The above Lemma 2.9 now translates to:

Proposition 3.1 ([KiRo; 93]). 1. $H \simeq G(\rho)$.
2. $H \simeq G \Longleftrightarrow \rho \in B(\underline{G}, Aut(\underline{G}))$.
3. $\Gamma_G \simeq \Gamma_H \Longleftrightarrow \rho \in B(\underline{\Gamma_G}, Aut_{augm}(\underline{\Gamma_G}))$.

We can formulate this more conceptually for the projective limit Γ_G as follows: We have a natural map

$$\Phi : Z(\underline{G}, Aut(\underline{G})) \longrightarrow Z(\underline{\Gamma_G}, Aut(\underline{\Gamma_G})).$$

Lemma 3.2 ([KiRo; 93]). *Given* $\rho \in B(\underline{\Gamma_G}, Aut_{augm}(\underline{\Gamma_G})) \cap Im(\Phi)$. *Then*

$$\Gamma_G \simeq \Gamma_{G(\rho)} \text{ and } G \simeq G(\rho) \Longleftrightarrow \rho \in \Phi(B(\underline{G}, Aut(\underline{G}))).$$

3.2. **Zassenhaus Conjecture.** Let G be solvable and put $N_i := G/O_{p_i'}(G)$. Then G is the projective limit $G = lim.proj._{1\leq i\leq n}(G_i)$ with $G_i = G/N_i$ and common quotients G_{ij}. Then the groups N_i are characteristic subgroups; and also the ideals $I(G, O_{p_i'})$ are characteristic.
Thus the Zassenhaus Conjecture is true for the groups G_i (cf. [Sc; 87], [Sc; 90]). We denote by $\Gamma_G(\mathcal{O})$ the corresponding projective limit of the group rings $\mathbb{Z}G_i$.

Let $\alpha : \mathbb{Z}G \longrightarrow \mathbb{Z}G$ be an augmented automorphism. Then we obtain from the validity of the Zassenhaus Conjecture a family of ISOMORPHISMS

$$\alpha_i = \gamma_i \cdot \rho_i : \mathbb{Z}G_i \longrightarrow \mathbb{Z}G_i,$$

where the maps γ_i are central automorphisms of $\mathbb{Z}G_i$ and $\rho_i \in Aut(G_i)$. Since α_i and α_j are induced from the automorphism α on $\mathbb{Z}G$, the maps α_i and α_j must coincide on $\mathbb{Z}G_{ij}$:

$$\gamma_i \cdot \rho_i = \gamma_j \cdot \rho_j \text{ on } \mathbb{Z}G_{ij};$$

i. e. the map $\rho_{ij} := \rho_i \cdot \rho_j^{-1} = \gamma_i^{-1} \cdot \gamma_j$ is a CENTRAL AUTOMORPHISM of G_{ij}. Hence we have associated to an automorphism α of $\mathbb{Z}G$ an element

$$\rho := (\rho_{ij}) \in \check{H}(\underline{G}, Aut_c(\underline{G})). \tag{4}$$

Here $Aut_c(-)$ stands for the group of conjugacy class preserving automorphisms; note that this is a very small subgroup of $Aut(-)$.

It should be noted that

$$\check{H}(\underline{G}, Aut_c(\underline{G}))$$

is a purely GROUP THEORETICAL INVARIANT.

Similar arguments as above hold, if we start with an automorphism of the projective limits $\alpha : \Gamma_G(\mathcal{O}) \longrightarrow \Gamma_G(\mathcal{O})$. The above Lemma 2.9 here translates to:

Proposition 3.3 ([KiRo; 93]). *The Zassenhaus Conjecture holds for our automorphism* $\alpha : \Gamma_G(\mathcal{O}) = \Gamma_G(\mathcal{O})$ *if and only if* ρ *is a coboundary. (I. e. there are* $\rho_i \in Aut_c(G_i)$ *with* $\rho_{ij} = \rho_i \cdot \rho_j^{-1}$ *.)*

Note again that this is a purely GROUP THEORETICAL QUESTION.

This can now be used to construct a semi-local counterexample for the Zassenhaus Conjecture for $\Gamma_G(\mathcal{O})$.

It is more complicated to get similar results for the p-version of the Zassenhaus Conjecture.

Also in this situation we have similar results as in Proposition 3.1 and Lemma 3.2:

We have a natural map

$$\Phi : Z(\underline{G}, Aut(\underline{G})) \longrightarrow Z(\underline{\Gamma_G\mathcal{O}}, Aut(\underline{\Gamma_G\mathcal{O}}))\,.$$

We also have a natural map

$$\Xi : Z(\underline{G}, Aut_c(\underline{G})) \longrightarrow Z(\underline{G}, Aut(\underline{G}))\,.$$

Lemma 3.4 ([KiRo; 93]). *Given* $\rho \in Z(\underline{G}, Aut_c(\underline{G}))$ *(cf. Equation 4).*

1. *Then* $\Gamma_G(\mathcal{O}) \simeq \Gamma_{G(\rho)}(\mathcal{O}) \Longleftrightarrow \rho\Xi \cdot \Phi \in B(\underline{\Gamma_G(\mathcal{O})}, Aut_{augm}(\underline{\Gamma_G(\mathcal{O})}))$.
2. $G \simeq G(\rho) \Longleftrightarrow \rho\Xi \in B(\underline{G}, Aut(\underline{G}))$.
3. *The Zassenhaus Conjecture holds for*

$$\Gamma_G(\mathcal{O}) \Longleftrightarrow \rho \in Z(\underline{G}, Aut_c(\underline{G}))\,.$$

3.3. A counterexample to the Isomorphism problem for $\Gamma_0(G)$.

Theorem 3.5. *There are two non isomorphic solvable groups* G *and* H *such that*

$$\mathbb{Z}_\pi \otimes_{\mathbb{Z}} \Gamma_0(G) \simeq \mathbb{Z}_\pi \otimes_{\mathbb{Z}} \Gamma_0(H)\,.$$

Remark 3.6. This is a slight modification of the construction which was used to find a counterexample to the Zassenhaus Conjecture for the integral grouprings by L. L. Scott and the author [RoSc; 87].

We define the group

$$H_0 := \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

as a subgroup of $GL(3,4)$. Three elements will play an important rôle in our construction:

$$s := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } t := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } c := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

these matrices all lie in $GL(3,2)$.

If $\mathbb{F}_4$ is the field with four elements with a $\mathbb{Z}/2\cdot\mathbb{Z}$-basis $\{1, r\}$, then the group

$$K := \begin{pmatrix} 1 & \delta \cdot r & * \\ 0 & 1 & \delta \cdot r \\ 0 & 0 & 1 \end{pmatrix},$$

where δ is either 1 or 0, is a normal subgroup of H_0 with quotient generated by s and t.

The group H_0 has an automorphism σ_0, defined via the map

$$\phi : H_0 \longrightarrow H_0/K \longrightarrow H_0 \, : \, s \longrightarrow c \text{ and } t \longrightarrow c \text{ as } \sigma_0(h) = h \cdot \phi(h).$$

Then $\sigma_0 \in Aut_c(H_0)$; i. e. for every $h \in H_0$ the elements h and $\sigma_0(h)$ are conjugate in H_0.

We now define three modules for H_0:

1. On $< m \,|\, m^4 >$ the normal subgroup K centralizes m, but ${}^s m = m^{-1}$ and ${}^t m = m^{-1}$. We denote by H the semidirect product $H := < m > \rtimes H_0$.
2. On $D := < d \,|\, d^3 >$ the normal subgroup K centralizes f, but ${}^s f = d$ and ${}^t d = d^{-1}$. We denote by H_3 the semidirect product $D \rtimes H$.
3. On $F := < f \,|\, f^5 >$ the normal subgroup K centralizes f, but ${}^s f = f^{-1}$ and ${}^t f = f$. We denote by H_5 the semidirect product $F \rtimes H$.

The following result was shown in [RoSc; 87] cf. also [Ro; 89].

Lemma 3.7. 1. *σ_0 can be extended to $\sigma \in Aut_c(H)$, by letting σ centralize m.*

2. *σ can be extended to $\sigma_i \in Aut(H_i)$, by letting σ_i centralize d and f resp. $i = 1, 2$.*
3. *σ does not extend to an automorphism in $Aut_c(H_i)$. Even more is true:*
4. *There do not exist $\gamma_i \in Aut_c(H_i)$, $i = 3, 5$ such that*

$$\sigma = \gamma_5 \cdot \gamma_3$$

in the common quotient H.

We now look at all possibilities of pairs

$$(\rho_3^k, \rho_5^k) \in Aut(H_3) \times Aut(H_5) \text{ such that } \sigma = \rho_3^k \cdot \rho_5^k, \; k = 1, \ldots, n.$$

By the above lemma, not both ρ_3 and ρ_5 can lie in $Aut_c(H_3)$ and $Aut_c(H_5)$ resp. After renumbering we may assume that

$$\{\rho_3^k : 1 \leq k \leq n_1\} \subset Aut(H_3) \setminus Aut_c(H_3) \text{ and}$$
$$\{\rho_5^k : n_1 + 1 \leq k \leq n\} \subset Aut(H_5) \setminus Aut_c(H_5)\,.$$

We now let p and q be primes such that $\hat{\mathbb{Q}}_p$ is a splitting field for H_3 and $\hat{\mathbb{Q}}_q$ is a splitting field for H_5. For each ρ_3^k, $1 \leq k \leq n_1$ we pick an irreducible $\hat{\mathbb{Z}}_3 H_3$-lattice M_p^k such that M_p^k and ${}^{\rho_3^k}M_p^k$ are non isomorphic. This can be done, since these ρ_3^k move conjugacy classes of H_3. We now choose a large enough power of p, say p^ν such that these modules stay non isomorphic after reduction modulo p^ν. Let

$$\overline{M_p^k} := M_p^k/(p^\nu \cdot M_p^k)\,.$$

The point now is that by a result of [Zi; 90] ρ_3^k does not extend to an AUTOMORPHISM of $\overline{M_p^k} \rtimes H_3$. We now put

$$M_p := (\bigoplus_{1\leq k\leq n_1} \overline{M_p^k}) + M_0\,,$$

where the only purpose of M_0 is to make M_p a faithful H_3-module.

In a similar fashion we construct the $\hat{\mathbb{Z}}_q G_5$-module M_q.

Let us recall, where we stand:

Definition 3.8. *We put $N_p := M_p \rtimes D$ and $N_q := M_q \rtimes F$, and then $G_p := N_p \rtimes H$ and $G_q := N_q \rtimes H$. We now define G as the pull-back*

$$\begin{array}{ccc} G_q & \longrightarrow & H \\ \uparrow & & \uparrow \\ G & \longrightarrow & G_p \end{array}\,.$$

Then it is clear that $O_{p'}(G) = N_q$ and $O_{q'}(G) = N_p$, $O_{3'}(G) = N_q$ and $O_{5'}(G) = N_p$. Moreover $O_{2'}(G) = N_p \times N_q$. Since $G/(N_p \times N_p) \simeq H$ the above diagram is the projective limit of the various $G/O_{p'_i}(G)$.

Claim 3.9. *The groups G and $G(\sigma)$ are not isomorphic. But the pull-backs of $\Gamma_0(G)$ and $\Gamma_0(G(\sigma))$ are isomorphic.*

Proof: Assume there are automorphisms $\tau_p \in Aut(G_p)$ and $\tau_q \in Aut(G_q)$ such that $\tau_q \cdot \tau_p = \sigma$ on the common quotient H. Then $\rho_3 := \tau_p\, mod(M_p)$ and $\rho_5 := \tau_q\, mod(M_q)$ lie in $\rho_3 \in Aut(H_3)$ and $\rho_5 \in Aut(H_5)$ and they form a pair with $\rho_3 \cdot \rho_5 = \sigma$ on H.
However, our construction of M_p and M_q shows that ρ_3 or ρ_5 does not extend to G_p or G_q, a contradiction. As for $\Gamma_0(G)$, our construction is such that the

central automorphism σ does extend to $\mathbb{Z}_3G_p$ by Proposition 3.1, and so $\Gamma_0(g)$ and $\Gamma_0(G(\sigma))$ are semi-locally isomorphic at $\mathbb{Z}_{\{2,3,5,p,q\}}$. q.e.d.

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AUSLANDER–REITEN SEQUENCES AND TILTING THEORY

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ABSTRACT. Given a ring, say R, two of its most important invariants are its module category $R - Mod$ and, coarser, its derived module category $D^b(R - Mod)$. This is an introduction into various techniques which have been developed to study these categories, in particular for finite dimensional algebras.

This is a survey paper (written by a non-expert), thus there will be no proofs, and only a few recent results will be mentioned. The emphasis is on explaining important notions by means of examples. Thus we will not try to present statements in their most general forms. Instead we will restrict to finite dimensional algebras over algebraically closed fields, where the theorems have their easiest form.

There are four sections: First we define path algebras, which are both important examples and important tools for the development of the whole theory. In section two we discuss the approach of Auslander and Reiten to the first main question; how to view (or even compute) a module category or a combinatorial picture of it, or how to organize the information contained in a module category. The second question is, how to compare module categories; in section three we discuss the approach via tilting theory. This leads us naturally to the final topic, derived equivalences, which is discussed in section four.

There are no dramatic assumptions on the knowledge of the reader; a little elementary ring theory and elementary homological algebra clearly will be enough to understand all definitions and assertions (but, of course, more work has to be done in order to understand the proofs which will be omitted here).

These lectures being given in Constantza — the former Tomis, where Ovid had to spent his last years — it seems appropriate to see the ideas which are recorded here as a collection of metamorphoses.

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A consequence of this result is the fact that the representation theory of $k\Gamma$ does not really depend on the orientation of Γ, although for two different orientations one does not get equivalent representation theories. This surprising fact has been studied by Bernstein, Gelfand, and Ponomarev [9], and their results can be seen as the starting point both of Auslander–Reiten theory and of tilting theory. We will come back to this point later on.

Quivers which do not have Dynkin diagrams as underlying graphs, are of infinite representation type. That is, however, not the end of the story. In fact, infinite type decomposes into two rather different situations, called tame and wild representation type. **Tame** representation type is somehow similar to Jordan normal form (which is the representation theory of a polynomial ring, that is, a quiver with one vertex, one loop, and no relation): By definition, for any fixed dimension n, each indecomposable n–dimensional representation M (up to finitely many exceptional representations) of a tame algebra A can be written as $M_i \otimes_{k[T]} V$ where V runs through the indecomposable representations of $k[T]$ (which is the prototype of a tame algebra) and M_i runs through finitely many $A-k[T]$–bimodules, which are finitely generated and free over $k[T]$. This says, in a fixed dimension the representation theory of a tame algebra looks like finitely many Jordan normal forms (but, of course, when changing the fixed dimension n, the number of needed Jordan normal forms may change as well). In contrast, the representation theory of a **wild** algebra contains as a subproblem the representation theory of a free algebra in two variables, hence (by a result of S. Brenner [11]) the representation theory of any finite dimensional algebra. An algebra cannot be both tame and wild (but tame as defined here of course includes finite type). A deep result of Drozd says that there are no other algebras.

Theorem 1.4. *(Drozd [21]) A finite dimensional algebra is either tame or wild.*

The rather complicated proof by Drozd uses a metamorphosis of algebras, the so called bocses (or boxes, as they tend to be called now). A variation on Drozd's proof has been given by Crawley–Boevey [16]. A rather different and more explicit proof (using another generalization of algebras, the so called subspace categories of vector space categories) recently has been given by Gabriel, Nazarova, Roiter, Sergeijchuk, and Vossieck [26].

There are (rather involved) algorithms to decide whether a finite dimensional algebra is of finite or infinite type (we will see one in the next subsection), but there are no algorithms known to decide (in general) between tame and wild.

For group algebras of finite groups, the situation is much easier:

Theorem 1.5. *(Higman [36], Bondarenko and Drozd [10]) Let G be a finite group and k a field whose characteristic p divides the order of G. Then the* *group algebra kG is of finite representation type if and only if the p–Sylow subgroups of G are cyclic. It is of tame representation type if and only p equals 2 and in addition the p–Sylow subgroups are elements of a small and special list of groups (dihedral, semidihedral, Klein's four group, and generalized quaternion groups). In all other cases, the group algebra kG is wild.*

The tame group algebras have been studied and their basic algebras have been more or less classified (in terms of quivers and relations) by K. Erdmann, and their representation theory has been determined by various authors [43, 17, 22]. Recall that any of the groups occuring in the theorem has precisely two generators, say a and b, and the relations are as follows: for dihedral groups, $a^n = b^2 = (ab)^2 = 1$, for semidihedral groups, $a^{2^{n+1}} = b^2 = 1$ and $bab^{-1} = a^{2^n - i}$, and, finally, for generalized quaternion groups, $a^{2m} = 1$, $b^2 = a^m$, and $bab^{-1} = a^{-1}$ (where n, m, i are parameters).

1.6. To have an example for later use, we compute the dimension vectors of the indecomposable representations of any quiver with underlying graph the Dynkin diagram A_3. By Gabriel's Theorem, these dimension vectors are precisely the positive roots of the corresponding semisimple complex Lie algebra.

A convenient way to compute positive roots, is to use, that a positive root is a vector with non–negative integral entries which has value one under the **quadratic form** associated with the Dynkin diagram. Thus we first have to compute the quadratic form. It is $q(x, y, z) = x^2 - xy + y^2 - yz + z^2$ (since we have vertices at x, y, and z, and edges between x and y and between y and z). We can rewrite q as $(x - y/2)^2 + y^2/2 + (y/2 - z)^2$ (which implies that the form is positive definite). It is now easy to check that the following is a complete list of all positive roots of this quadratic form:

$$100, 110, 111, 011, 010, 001$$

In the next section we will continue this example by writing down the corresponding indecomposable modules for some fixed orientation.

1.7. A classical example of the representation theory of an Euclidean diagram is the **Kronecker quiver** $\bullet \rightrightarrows \bullet$ (which is the smallest quiver in the series $\tilde{A}_n$). Its representations $V \overset{(\varphi,\psi)}{\rightrightarrows} W$ are determined by pairs of matrices (φ, ψ), and isomorphism of representations means simultaneous conjugation of both matrices (that is, changes of the bases of V and W). That is, classifying indecomposable representations is equivalent to classifying pairs of matrices modulo simultaneous conjugation. Thus one can see this problem either as a problem of linear algebra or as a problem in representation theory. In fact, linear algebra is enough to solve it; the first solution has been achieved by Kronecker [38] (for a short elegant version of his proof, see [8], section 4.3). Using the representation theoretic tools we will discuss in the next chapter, one can save a lot of computational effort, and one can generalize this solution to a solution of the whole class of problems given by Euclidean diagrams (see [19] for a complete solution).

Now we list all the indecomposable representations of the Kronecker quiver: There are three different cases. If we fix a natural number n, and assume $dim(V) = n$ then $dim(W)$ is either $n+1$ or n or $n-1$. In the first case, there is up to isomorphism precisely one indecomposable representation; the two maps $V = k^n \to W = k^{n+1}$ are as follows: $\varphi(v_i) = w_i$ and $\psi(v_i) = w_{i+1}$ where $\{v_i\}$ and $\{w_i\}$ are bases of V and W. The third case of $dim(W) = dim(V) - 1$ is dual to the first one; thus again, there is precisely one indecomposable representation for each dimension vector. A much larger set of indecomposable representations arises for the dimension vectors (n, n). Here, we can identify the vector spaces V and W. If φ is an isomorphism we can choose a basis, such that φ is the identity map; in that case, ψ can be represented by a Jordan block

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & \dots \\ 0 & \lambda & 1 & 0 & \dots \\ \dots & 0 & \dots & \dots & 0 \\ \dots & \dots & 0 & \lambda & 1 \\ \dots & \dots & \dots & 0 & \lambda \end{pmatrix}$$

The only indecomposable representation with φ not invertible is the one having (up to isomorphism) $\psi = id$ and φ represented by the Jordan block

$$\begin{pmatrix} 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & \dots & \dots & 0 \\ \dots & \dots & 0 & 0 & 1 \\ \dots & \dots & \dots & 0 & 0 \end{pmatrix}$$

2. Almost split sequences

In this section we define almost split sequences (which sometimes are called Auslander–Reiten sequences) and explain their importance. They tell us how to **view** (and in many cases, how to compute) a module category, or how to **organize** information we have on a module category.

Throughout we will work (for simplicity) with a finite dimensional algebra A over an algebraically closed field k. Modules always are finitely generated (= finite dimensional), sometimes left, sometimes right modules. (The k–duality $Hom_k(-, k)$ is an antiequivalence between the categories of left and of right modules over a k–algebra, thus it does not matter which kind of modules we study.)

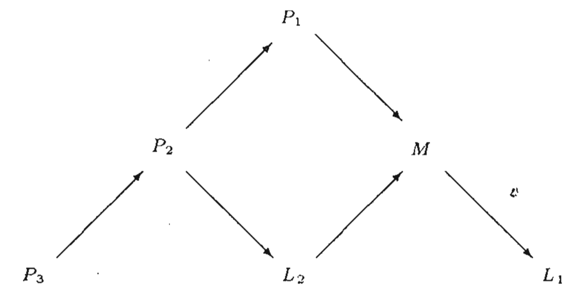
2.1. In order to have an example at hand, we first compute the modules over the upper 3×3–triangular matrices, that is, the representations of the quiver $\bullet \to \bullet \to \bullet$ with underlying graph A_3. By Gabriel's Theorem 1.3 we know the dimension vectors of the indecomposable modules. Hence it is enough to find one indecomposable module for each of the dimension vectors in the list given above. If we deal with right modules, then the first projective module (the first row in the 3×3–matrices) has dimension three and dimension vector 111. The second projective module has dimension vector 011, the third one has 001. Of course, the first and the second projective have unique simple quotients $L_1 = 100 = P_1/P_2$ and $L_2 = 010 = P_2/P_3$. There remains just one dimension vector 110 in the list, which belongs to the indecomposable module $M = P_1/P_3$.

Now we know all the indecomposable modules, thus by Krull-Remak-Schmidt all the finitely generated modules (which are in a unique way direct sums of indecomposable ones). But this of course does not mean that we know now the whole module category. The homomorphisms are missing! Going through the list of indecomposable modules we can check that they all have endomorphism ring k. For example, the endomorphisms of the projective representation $P_1 = k \overset{id}{\to} k \overset{id}{\to} k$ are triples (a, b, c) of field elements, that is 1×1–matrices which make the following diagram commutative:

$$\begin{array}{ccccc} k & \overset{id}{\to} & k & \overset{id}{\to} & k \\ \downarrow a & & \downarrow b & & \downarrow c \\ k & \overset{id}{\to} & k & \overset{id}{\to} & k \end{array}$$

And we also can find some other homomorphisms between these modules, as is shown in the following picture:

Γ_A :



Here, the maps indicated by arrows are the obvious inclusions or projections. There are more than these maps, for example one can compose some of them to get other non–zero homomorphisms. But trying to find more maps one gets the impression, that any homomorphism between these modules is a composition of isomorphisms with some of the maps in the picture. Thus the maps in the picture deserve special attention. They will be called irreducible maps, and will be seen to be the maps in the almost split sequences which will be defined later on. What properties do the maps in the picture have? They are not isomorphisms, but they generate (multiplicatively) all the non-isomorphisms (in this special example as well as in the case of finite representation type in general).

Definition 2.1. (Auslander–Reiten [4, 5]) Let X and Y be A–modules and $f : X \to Y$ an A–homomorphism. Then f is called **irreducible** if it is neither a split monomorphism nor a split epimorphism, but in any factorization $f = gh : X \xrightarrow{g} Z \xrightarrow{h} Y$ (for Z any finitely generated A–module) the map g is split mono or the map h is split epi.

In other words, an irreducible map is a non–isomorphism which cannot be factored in any non–trivial way. Clearly, an irreducible map must be either injective or surjective (factor the map over its image).

Now let us look at the example again in order to see how the irreducible maps are organized. Surprisingly, it turns out that they form exact sequences. More precisely, there are exact sequences

$$0 \to P_3 \to P_2 \to L_2 \to 0$$

$$0 \to P_2 \to P_1 \oplus L_2 \to M \to 0$$

$$0 \to L_2 \to M \to L_1 \to 0$$

which contain all irreducible maps occuring in the picture, and all maps in these sequences (except the zero maps) are irreducible.

This gives us a first definition of almost split sequences. It can be shown to be equivalent to another definition (the original one) which will be given in the next subsection and which provides a way to prove the existence of such sequences and hence the existence of enough irreducible maps.

Definition 2.2. An exact sequence $0 \to M \to E \to N \to 0$ of A–modules, with indecomposable starting and ending terms, is called an **almost split sequence** (short: ass) if and only if all non–zero maps in the sequence are irreducible .

Sometimes, almost split sequences are called Auslander–Reiten sequences.

Later on, we will see that almost all irreducible maps occur in such sequences. In particular, for any module M there are finitely many irreducible maps starting in M and finitely many irreducible maps ending in M. This tells us how to view the module category as a quiver:

Definition 2.3. The **Auslander–Reiten quiver** of A is an oriented graph having as vertices the isomorphism classes of indecomposable A–modules, and containing n arrows from M to N if and only the space of irreducible maps from M to N is n–dimensional.

The picture given above is the Auslander–Reiten quiver of the path algebra of the quiver $\bullet \to \bullet \to \bullet$.

If the module M is projective, then it is easy to find all irreducible maps ending in M; they are just the direct summands of the inclusion of $rad(M)$ into M. Dually, if M is injective, the irreducible maps starting at M are the summands of the projection $M \to M/soc(M)$. All other irreducible maps can be found via almost split sequences.

Note that we encounter here two more metamorphoses; irreducible maps are turned into almost split sequences, and the module category is turned into the Auslander–Reiten quiver (the latter change may cause some loss of information, since we cannot recover maps from the AR quiver which can be factored into an infinitely long chain of irreducible maps; such maps exist in case of infinite representation type).

2.2.

Definition 2.4. (Auslander–Reiten [4, 5]) Let $0 \to M \xrightarrow{f} E \xrightarrow{g} N \to 0$ be an exact sequence with indecomposable starting and ending terms. It is called an **almost split sequence** if and only if f is **left almost split**, that is, f is not split mono, and any homomorphism $M \to X$ which is not split monomorphism, factors via E, and g is **right almost split**, that is, g is not split epi, and any homomorphism $Y \to N$ which is not split epimorphism, factors via E.

Note that it is necessary to require that the maps to be factored are not split, since otherwise the sequence itself would split, which we do not want.

Theorem 2.5. *(Auslander–Reiten) If N is indecomposable and not projective, then there is an almost split sequence $0 \to M \to E \to N \to 0$ ending in N. If M is indecomposable and not injective, then there is an almost split sequence $0 \to M \to E \to N \to 0$ starting in M.*

For **lattices over orders** (over a discrete valuation domain) there is a completely analogous theorem, independently due to Auslander and to Roggenkamp [46]; the construction of almost split sequences differs, however, in a rather subtle way from that for modules over finite dimensional algebras. For other modules over orders, or for other classes of modules over more general rings, almost split sequences usually do not exist.

Together with the previous remarks on irreducible maps ending in projective or starting in injective modules, this shows that for any indecomposable module in a later subsection implies that these modules are all the indecomposable B-modules (since B is indecomposable as a ring).

Note that there is one almost split sequence (and also one indecomposable module) less than in the previous example. However, the two module categories of A and B look rather similar. This similarity will be explained in the section on tilting theory.

2.4. In order to have an example of an infinite Auslander–Reiten quiver, we look at the **Kronecker quiver**. Recall that its representations are of the form $V \overset{(\varphi,\psi)}{\rightrightarrows} W$, that is, they are pairs of matrices. The list of indecomposable representations we gave before, distinguishes three kinds of indecomposable representations, which have dimension vectors $(n, n+1)$ or (n, n) or $(n, n-1)$, respectively. The first kind of representations includes the projective ones, which have dimension vectors $(0,1)$ and $(1,2)$; these representations are called **preprojective**. For each n, there is precisely one indecomposable preprojective representation of dimension vector $(n, n+1)$. How do their almost split sequences look like? Let us start with the projective representation $(0,1)$; it has two irreducible maps into $(1,2)$, the cokernel is $(2,3)$ which is the Auslander–Reiten translate of $(0,1)$. Hence the almost split sequence is $0 \to (0,1) \to (1,2) \oplus (1,2) \to (2,3) \to 0$. Continuing in that way, one can produce an infinite component of the Auslander–Reiten quiver, which contains precisely all the preprojective representations:

$$(0,1) \rightrightarrows (1,2) \rightrightarrows (2,3) \rightrightarrows (3,4) \rightrightarrows \cdots \rightrightarrows (n, n+1) \rightrightarrows (n+1, n+2) \ldots$$

The representation $(n, n+1)$ has two linear independent irreducible maps into $(n+1, n+2)$ (which send v_i to v_i or to v_{i+1} respectively), and the almost split sequence looks like

$$0 \to (n, n+1) \to (n+1, n+2) \oplus (n+1, n+2) \to (n+2, n+3) \to 0$$

Of course, the representations $(n+1, n)$ form another Auslander–Reiten component; they are called **preinjective** (and include the injective representations).

How many components do we need for the remaining representations (which are called **regular**)? They are not uniquely determined by their dimension vector, thus to fix them we have to add a parameter λ (which fixes the Jordan block defining φ, and where we follow the convention that $\lambda = \infty$ gives the representation with $\psi = id$). Computing Auslander–Reiten translates one sees, that each (n, n, λ) is sent to itself. Thus there are modules, which are starting and ending term of the same almost split sequence. There are two irreducible maps $(n, n, \lambda) \to (n+1, n+1, \lambda)$ and $(n, n, \lambda) \to (n-1, n-1, \lambda)$ (the latter in case $n > 1$ only), one of them injective, the other surjective. They fit into an exact sequence, which thus is the almost split sequence

$$0 \to (n, n, \lambda) \to (n+1, n+1, \lambda) \oplus (n-1, n-1, \lambda) \to (n, n, \lambda) \to 0$$

So for each λ we get an Auslander–Reiten component, which contains precisely the representations (n, n, λ) (where n runs and λ is fixed). The shape of these Auslander–Reiten components is as follows:

$$(1, 1, \lambda) \rightleftarrows (2, 2, \lambda) \rightleftarrows \cdots \rightleftarrows (n, n, \lambda) \rightleftarrows (n+1, n+1, \lambda) \rightleftarrows \ldots$$

A good way to imagine these components is as **tubes** with $(1, 1, \lambda)$ at the mouth of the tube.

In general, many other shapes of Auslander–Reiten components can occur; the shapes occuring for a given algebra can be used to get information on this algebra (for group algebras this method has been used by Erdmann [22]).

The other Euclidean diagrams have similar representation theories; always there are two exceptional components containing preprojective and preinjective modules and all the other modules lie in tubes (up to three of these tubes may look different from the above ones by having more than one indecomposable module at the mouth). This is the typical behaviour of a tame algebra (by a result of Crawley–Boevey [17]). For wild algebras completely different shapes of Auslander Reiten components do occur (but there also may be many tubes).

Once one has such a precise knowledge on the representation theory of a certain algebra, one can use it to study other algebras which are somehow related. In the section on tilting theory we will review a method for doing so. However, using ad hoc methods one can for example relate the Klein four group to the Kronecker quiver and arrive (by some clever computations, see [8], 4.3) at a complete classification of the representations of this group (which in characteristic two has a tame representation theory).

2.5. An important application of the existence of almost split sequences is to provide an easy proof of **Brauer Thrall I**. Before we give that proof we have to recall the contents of the Brauer Thrall Conjectures (which have not been stated as conjectures by Brauer or Thrall). They have been central problems in the area for a long time.

An algebra can be either of finite or of infinite representation type. There are two possible reasons, why it could be of infinite representation type: Either there are indecomposable modules of arbitrary large dimension, or there are infinitely many non–isomorphic indecomposable modules having the same dimension. Brauer Thrall I says that the first possibility always must occur. Brauer Thrall II says that in fact a combination of both possibilities must occur: there are infinitely many natural numbers such that for each of them there exist infinitely many pairwise non–isomorphic indecomposable modules having this number as their dimension. Recall the example of the Kronecker quiver:

Here (over an algebraically closed field!), for each even dimension $2n$ there is an infinite series of pairwise non-isomorphic indecomposable representations given by $V = k^n = W$ and φ represented by a Jordan block

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & \dots \\ 0 & \lambda & 1 & 0 & \dots \\ \dots & 0 & \dots & \dots & 0 \\ \dots & \dots & 0 & \lambda & 1 \\ \dots & \dots & \dots & 0 & \lambda \end{pmatrix}$$

(there are infinitely many λ, since k is algebraically closed).

Brauer Thrall I was proved by Roiter [47] using a very tricky argument. Brauer Thrall II follows from another big theorem (due to Bautista, Gabriel, Roiter, and Salmeron [7]), the multiplicative basis theorem, saying that a large class of algebras — including algebras of finite type and algebras of minimal infinite type (the latter being the ones important for Brauer Thrall II) — have bases which are multiplicative in the sense that the product of two basis elements is either zero or a basis element again. Brauer Thrall II has been proved [6, 23, 14] only for finite dimensional algebras over algebraically closed fields, whereas Brauer Thrall I has been shown for arbitrary artin algebras by an argument of Auslander which we are going to discuss now.

Now we sketch a version of Auslander's proof of Roiter's result. It makes use of the Harada–Sai Lemma:

Lemma 2.6. *(Harada-Sai, [35]) Let n be a natural number and $M_0, \dots, M_{2^n-1}$ be indecomposable A-modules (not necessarily pairwise different), each of dimension n, and pick non-isomorphisms $f_i : M_{i-1} \to M_i$. Then the composition $f_1 \cdot \dots \cdot f_{2^n-1}$ equals zero.*

The proof is clever, but elementary.

Now we prove Roiter's Theorem. Assume that all indecomposable A-modules have dimension less than a fixed natural number n. We have to show that A has finite representation type.

Let M be any indecomposable module and $f : N \to M$ a non-zero homomorphism which is not a split epimorphism. Clearly, M is not simple projective. If f is not irreducible, then f factors over an irreducible map f_1 ending in M: $f = gf_1$ (where f_1 either occurs in the almost split sequence ending in M, or if M is projective, f_1 is the inclusion $rad(M) \to M$). If f_1 starts in a module which is not semisimple projective and g is not a split homomorphism (that is, neither a split monomorphism nor a split epimorphism), then g factors, too. Continuing by induction we arrive at one of three possibilities: Either at some step, g becomes a split homomorphism — then we have connected N and M by a finite chain of irreducible maps having non-zero composition. Or we can continue factoring for ever — this contradicts the Harada–Sai Lemma. Or at some step we arrive at a simple projective module Q, which then must be a direct summand of N (which maps non-trivially to Q). Hence again, N and M are connected by a finite chain of irreducible homomorphisms having non-zero composition. Since for any M there exists an indecomposable projective module $P = N$ mapping non-trivially to M, we see that any indecomposable module is connected by a finite chain of irreducible maps with an indecomposable projective module. And this chain can be chosen in such a way that its composition is not zero. By Harada–Sai it follows that the chain has length bounded by $2^n - 1$. By the theorem of Auslander and Reiten, the Auslander–Reiten quiver is a locally bounded graph. Hence we can reach only finitely many indecomposable modules by chains of irreducible maps having non-zero composition, which start in one of the finitely many indecomposable projective modules. Thus we have finite representation type. This finishes the proof of Roiter's Theorem.

If the algebra A is **connected**, that is indecomposable as a ring, then its projective modules can be connected by chains of non-zero maps. Then repeating the above argument gives a stronger statement:

Theorem 2.7. *(Auslander) Let A be connected. Assume there is a component in the Auslander–Reiten quiver of A such that the length of the modules in this component is bounded. Then A is of finite representation type, and the Auslander–Reiten quiver has precisely one connected component.*

Thus the best way to prove Brauer Thrall I is to prove its metamorphosis in Auslander's Theorem.

So, if one is given an algebra, one may try to compute the Auslander Reiten components containing indecomposable projective modules. If these components turn out to be finite (since the process of computing almost split sequences stops at a certain time) one has shown that the algebra has finite representation type, and at the same time one knows already all the indecomposable modules. This is precisely the argument we need for showing in our example that we have found all indecomposable B-modules.

By a result of Bongartz, in case of finite representation type, the number of indecomposable modules is bounded by a number which only depends on the dimension of A. Hence the above test for finite representation type really is an algorithm. There are however much better (and much more involved) techniques to decide whether or not an algebra has finite representation type, using certain lists compiled by Bongartz and by Happel and Vossieck.

2.6. In the case of group algebras the computation of DTr is very canonical:

Proposition 2.8. *Let kG be the group algebra of a finite group. Then for M indecomposable and not projective, there is an isomorphism $DTr(M) \simeq \Omega^2(M)$, where the latter module is the second syzygy, that is the second kernel in the minimal projective resolution of M.*

A connection to defect groups (see Zimmermann's lectures) is given by:

Proposition 2.9. *Let $A = kG$ be the group algebra of a finite group G. Then an almost split sequence $0 \to M \to E \to N \to 0$ splits after restriction to a subgroup H of G if and only if the subgroup H does not contain a vertex of M (or equivalently of N).*

3. Tilting theory

The aim of this section is to discuss how to **compare** the module categories of two algebras. The starting point of course is the classical result of Morita which settles the case of equivalence of two module categories (see for example the book [1]). The development of tilting theory can be seen as a sequence of metamorphoses of this result.

3.1.

Theorem 3.1. *(Morita [39]) Let R and S be two rings (with unit) and assume there are additive functors $F : R-Mod \to S-Mod$ and $G : R-Mod \to S-Mod$ which are mutually inverse equivalences. Set $P = F(R)$, which is an $S-R$-bimodule, and $Q = G(S)$, which is an $R-S$-bimodule. Then the following statements are true:*

(a) The natural ring homomorphisms $R \to End_S(P)$ and $S \to End_R(Q)$ are isomorphisms.

(b) The modules P_R, ${}_SP$, ${}_RQ$ and Q_S are finitely generated projective generators.

(c) There are isomorphisms of bimodules ${}_SP_R \simeq Hom_S({}_RQ_S, S) \simeq Hom_R({}_RQ_S, R)$ and ${}_RQ_S \simeq Hom_R({}_SP_R, R) \simeq Hom_S({}_SP_R, S)$.

(d) There are natural isomorphisms $F \simeq Hom_R({}_RQ, -) \simeq P_R \otimes_R -$ and $G \simeq Hom_S({}_SP, -) \simeq Q_S \otimes_S -$.

Conversely, given bimodules satisfying the conditions in (a) and (b), the functors in (d) define equivalences between $R - Mod$ and $S - Mod$.

This result gives a complete characterization of when two rings have equivalent module categories. Starting with one of the rings, it is by condition (a) enough to know one of the modules P or Q in order to find the other ring.

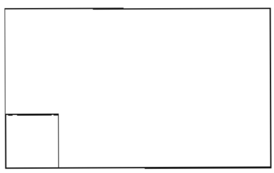
There are three conditions on the module ${}_SP$ in order to define an equivalence between $S - Mod$ and $End_S(P) - Mod$: P must be finitely generated, it must be projective, and it must be a generator.

If there is an equivalence, then it has a very special form: it is given by a *Hom* or a $\otimes$ functor with the module P.

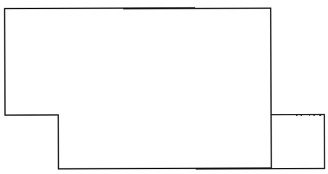
A typical example of a Morita equivalence is between a ring R and the matrix ring $M_n(R)$ of $n \times n$–matrices with entries in R (a special case of that is Wedderburn theory which has been known long before Morita's result).

3.2. Morita's result looks like finishing the subject, so it is surprising enough that this is actually the starting point of a new development. The next step which is important for us was not done before 1973, when Bernstein, Gelfand and Ponomarev [9] gave a new proof of Gabriel's Theorem 1.3. When discussing Gabriel's Theorem we observed already that it implies that for all orientations of a fixed Dynkin diagram one gets similar representation theories; the dimension vectors of the indecomposable modules over any of these algebras are independent of the orientation (although the modules themselves are not). So there is a need to explain this similarity. In their proof, Bernstein, Gelfand, and Ponomarev introduced what has been called **reflection functors** and what is just a metamorphosis of the classical Weyl group. They are functors between $A - mod$ and $A' - mod$, where A and A' are path algebras of the same quiver, but with different orientation such that the change of orientation is done by a reflection at a sink (where no arrow starts) or at a source (where no arrow ends). It turns out that via this correspondence almost all indecomposable A–modules 'are' indecomposable A'–modules and vice versa. So the reflection functors are almost equivalences. Pictorially, the situation is as follows:

$A - Mod$:



$A' - Mod$:



Let us look at our above example. We compare the two path algebras $A = k(\bullet \to \bullet \to \bullet)$ and $A' = k(\bullet \to \bullet \leftarrow \bullet)$. A representation of A is a triple of vector spaces with a pair of linear maps:

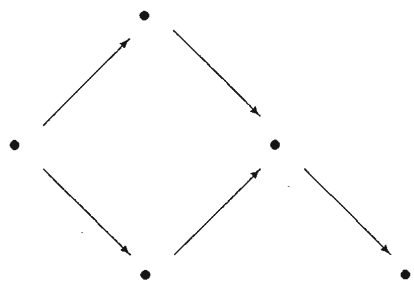
$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3.$$

What we have to do is to reverse the arrow $2 \to 3$. The functor doing that on the level of representations is the reflection functor S^+; it takes the above representation and sends it to a representation of A' which is defined as follows:

$$V_1 \xrightarrow{f} V_2 \xleftarrow{incl} kernel(g).$$

On morphisms, S^+ does the obvious thing. To go back from A' to A we have a reflection functor S^- which uses a cokernel to define the vector space at vertex 3. Check now what these functors do on representations; they send an indecomposable representation to another indecomposable representation with only one exception: S^+ sends the simple module at vertex 3 (where the arrow is reversed) to zero, and S^- does the same with the simple representation at 3 (where again the arrow is reversed). So if we remove these simples from the module categories, the reflection functors give an equivalence of what remains.

$\Gamma_A \cap \Gamma_{A'}$:

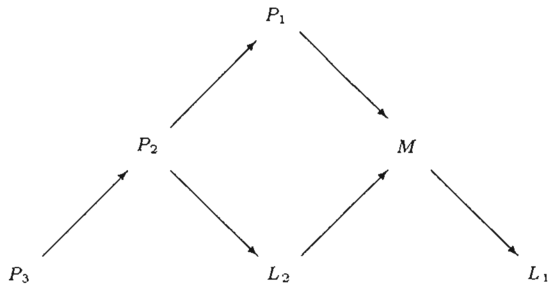


For defining reflection functors one just needs a sink or a source (the vertex, where all arrows are reversed). If the graph underlying Γ is a tree, then there must be a sink or a source. Reflection functors can be composed in order to give a functor from $A - mod$ into itself. It turns out that the resulting combinatorial operation (called **Coxeter transformation**) on dimension vectors coincides with the action of the Auslander–Reiten translation DTr on dimension vectors (for indecomposable non–projective representations of path algebras). This purely combinatorial way to do Auslander Reiten theory is an important tool of representation theory of path algebras of quivers (see the book [45]).

The idea of Bernstein, Gelfand, and Ponomarev has been to prove Gabriel's Theorem by looking at the group generated by reflections. In the case of a Dynkin diagram, this is just the Weyl group of the corresponding semisimple complex Lie algebra. Now the positive roots can be produced from the simple roots by applying a sequence of reflections, and conversely, a positive root can be transformed into a simple one by applying the inverse sequence of reflections. Reflection functors do precisely the same on dimension vectors of indecomposable modules. Hence one can prove the theorem by induction, the main point being to have full control about when a positive root or an indecomposable module are being sent to a negative root or the zero module by a reflection or a reflection functor respectively. An exposition of the Bernstein–Gelfand–Ponomarev proof of Gabriel's Theorem can be found in chapter 8 of [41].

3.3. The next step in the development of tilting theory is due to Auslander, Platzeck, and Reiten [3] who interpreted reflection functors in a module theoretical way. We describe this metamorphosis (which is a special case of tilting) and check it in our example. Recall the Auslander–Reiten quiver of the path algebra A:

Γ_A :



There are three indecomposable projective modules, the third of them, P_3, being simple. Morita's Theorem tells us that we get algebras with equivalent module categories by taking endomorphism rings of copies of these projective modules. Now to get 'almost' an equivalence we replace the projective module P_3 (the one at the vertex, where the arrows are reversed) by its TrD which is the representation $L_2 = 0k0$. The endomorphism algebra of the module $T := P_1 \oplus P_2 \oplus L_2 = kkk \oplus 0kk \oplus 0k0$ is the algebra

$$\begin{pmatrix} k & k & 0 \\ 0 & k & 0 \\ 0 & k & k \end{pmatrix}$$

which is the path algebra A' of the quiver $\bullet \rightarrow \bullet \leftarrow \bullet$. Now there is an obvious functor from $A - mod$ to $A' - mod$, namely $Hom_A(T, -)$. Let us make a table with its values:

A–mod	A'–mod
00k	000
0kk	0k0
kkk	kk0
0k0	0kk
kk0	kkk
k00	k00

Looking at the Auslander–Reiten quiver, one observes that all indecomposables except the simple at vertex 3 are mapped 'in an equivalent way' (that is, preserving homomorphisms) to the other Auslander–Reiten quiver (where the simple at 3 is not in the image).

We can even improve the situation a little bit, if we use another natural functor, which also sends A–modules to A'–modules: the functor $Ext^1_A(T,-)$ sends the simple at vertex 3 to the simple at vertex 3 and everything else to zero. Thus this second functor corrects the 'mistake' made by $Hom(T,-)$! Observe that it 'moves' the simple module at vertex 3 from the left hand side of the Auslander–Reiten quiver to the right hand side. This is why this process is called **tilting**.

3.4. Reflection functors and APR–tilting functors provide a link between two module categories which are not equivalent but 'very similar'. This is a very special case of a more general behaviour which has been discovered by Brenner and Butler [13] and extended by Happel and Ringel [33]. They generalize Morita's Theorem by replacing the progenerator P from Morita's Theorem with a tilting module T (generalizing the APR tilting we saw before). Thus a tilting module is just a metamorphosis of a progenerator.

Definition 3.2. Let A be a finite dimensional algebra. An A–module T is called a **tilting module** if and only if it satisfies the following conditions:

(a) T is finitely generated,

(b) its projective dimension is 0 or 1 (that is, there is a short exact sequence $0 \to A_1 \to A_2 \to T \to 0$ with A_1 and A_2 being projective),

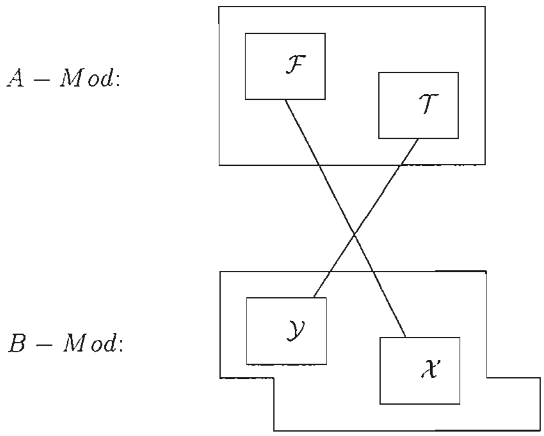
(c) it has no self–extensions (that is $Ext^1_A(T,T) = 0$),

(d) there is an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with T_1 and T_2 being direct summands of finite direct sums of copies of T.

Comparing these conditions with those of Morita's Theorem, we have again a condition saying that T is finitely generated, as well as homological conditions and a condition which says that T is large enough.

With a projective generator comes an equivalence between module categories, with a tilting module come equivalences between subcategories:

Theorem 3.3. *(Brenner–Butler) Let A be a finite dimensional algebra, T a tilting module and B its endomorphism ring $End_A(T)$. Then the functors $Hom_A(T,-)$ and $T \otimes_B -$ provide mutually inverse equivalences between the subcategory $\mathcal{T}(_AT) = \{_AM \in A-mod \mid Ext^1_A(T,M) = 0\}$ of $A-mod$ and the subcategory $\mathcal{Y}(T_B) = \{N_B \in B-mod \mid Tor^B_1(N,T) = 0\}$ of $B-mod$. And the functors $Ext^1_A(T,-)$ and $Tor^B_1(-,T)$ provide mutually inverse equivalences between the subcategories $\mathcal{F}(_AT) = \{_AM \in A-mod \mid Hom_A(T,M) = 0\}$ and $\mathcal{X}(T_B) = \{_BN \in B-mod \mid T \otimes_B N = 0\}$.*



If $T = P$ is a projective generator, then the second pair of subcategories is trivial and the statement on the first pair just repeats the statement of Morita's Theorem.

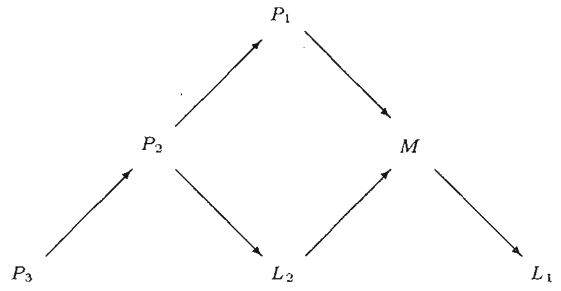
In general, the subcategories do not exhaust neither $A-mod$ nor $B-mod$. But if A is hereditary, the two subcategories $\mathcal{X}$ and $\mathcal{Y}$ together give all of $B-$mod; so the knowledge of the representation theory of A can be used for giving a complete picture of B–mod.

The tilting module T itself lies in $\mathcal{T}(T)$ and is mapped to $Hom_A(T,T) = B$, thus its image is a projective generator.

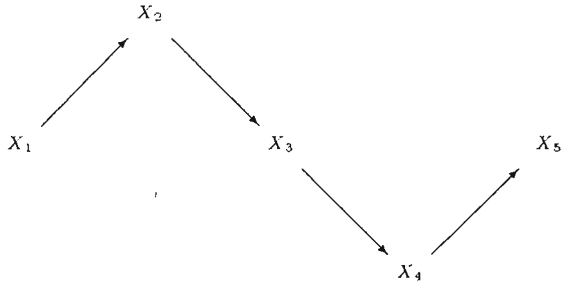
An algebra A and the endomorphism algebra B of an A–tilting module may look quite different. For example, they may have different global dimension (as the next example shows) or different representation types, and none of them need to be a subalgebra or a quotient of the other (as APR tilting shows).

Using this general version of tilting theory we can now include our second example, that is, we can apply tilting in order to come from the path algebra A to its quotient $B = A/rad^2(A)$. Recall the two Auslander–Reiten quivers:

Γ_A :



Γ_B :



A tilting module which brings us from A to B is $T = P_3 \oplus P_1 \oplus L_1$ (which has endomorphism ring B). The subcategory $\mathcal{T}$ contains the indecomposable modules P_3, P_1, M, and L_1 (which are mapped to X_1, X_2, X_3, and X_4). The subcategory $\mathcal{F}$ (which is tilted from left to right) has just one indecomposable object, L_2, which is sent to X_5.

The theorem of Brenner and Butler can be further generalized to tilting modules which have finite global dimension (possibly bigger than one); then one has to use all the functors $Ext^i_A(T, -), \ldots$ and a whole bunch of subcategories. Tilting modules of global dimension one thus may be called classical tilting modules.

3.5. One of the reasons to be interested in tilting theory is that starting with some algebras one has a good knowledge of the representation theory of, one can construct new algebras and study their module categories. The most important class of examples is obtained by looking at tilting modules over hereditary algebras. The algebras obtained in this way are called **tilted algebras**. A more general class, that of **quasitilted algebras**, has been studied by Happel, Reiten, and Smalø [31]. A quasitilted algebra is the endomorphism algebra of a tilting object in a hereditary abelian k-category. This includes tilted algebras, but also canonical algebras and other algebras obtained by tilting in certain categories of sheaves. So the module category of a quasitilted algebra is a metamorphosis of a hereditary abelian category.

These algebras have a nice homological characterization:

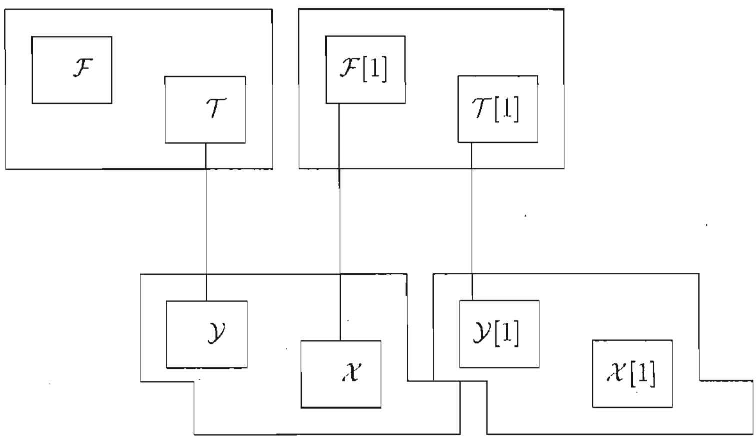
Theorem 3.4. *(Happel, Reiten, Smalø [31]) A finite dimensional algebra A is quasitilted if and only if its global dimension is less than or equal to two and in addition each indecomposable A–module has projective dimension or (not: and!) injective dimension less than or equal to one.*

4. Derived categories

During the last decade, tilting theory has become a very intensively studied area, which is the source of many examples. A new metamorphosis of tilting theory has been found by Happel [28, 29] who brought derived categories into the game:

4.1.

Theorem 4.1. *(Happel) Let A be a finite dimensional algebra and T a tilting module over A with endomorphism ring B. Then the derived module categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories. Equivalences are given by the functors derived from $Hom_A(T,-)$ and $T \otimes_B -$.*

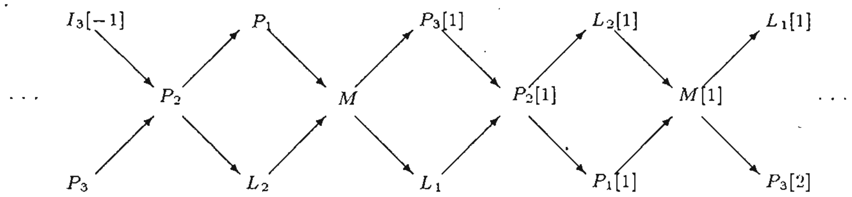


The equivalence between the derived categories restricts to equivalences between the categories $\mathcal{T}$ and $\mathcal{Y}$ and between the categories $\mathcal{F}[1]$ and $\mathcal{X}$.

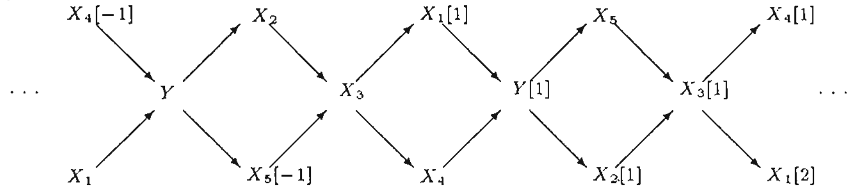
Thus equivalence of derived categories is a handy and elegant formulation for the 'similarity' of module categories.

The two derived categories of the path algebra A of $\bullet \to \bullet \to \bullet$ and of its quotient $B = A/rad^2(A)$ look as follows:

$D^b(A)$:



$D^b(B)$:



It is clear from the picture that they are equivalent!

In $D^b(A)$ all indecomposable objects are — up to shift — modules. In contrast, in $D^b(B)$ there is a new indecomposable object, called Y, which is not a module. It is the complex $Y = (P_2 \to P_1)$, which makes the sequence $Y \to X_2 \oplus X_5[-1] \to X_3$ into a triangle. One may check the equivalence by first applying the derived functor of $Hom(T, -)$ to the A-modules inside the two subcategories appearing in the Brenner Butler Theorem (since for these modules, the derived functor coincides with the functors $Hom_A(T, -)$ and $Ext^1_A(T, -)$ in the Brenner Butler Theorem). The only A-module which is left is P_2. Since the equivalence is one of triangulated categories, P_2 must be mapped to Y (which can be checked by chosing a triangle containing P_2 and two objects with known images, and then looking at the image of the triangle).

4.2. The derived category of a hereditary algebra is especially easy; its indecomposable objects are just shifted indecomposable modules, that is, an indecomposable complex is isomorphic to a stalk complex with only one non-zero entry. Thus it is not surprising, that there is an Auslander Reiten structure on the indecomposable complexes. Of course, this derived Auslander-Reiten quiver contains as subquivers shifted versions of the Auslander-Reiten quiver of the path algebra itself. But there are additional Auslander-Reiten triangles relating shifted projective modules $P[n]$ and shifted injective modules $I[n-1]$, as one can see in the above example (note that inserting these additional irreducible maps 'twists' the shifted copies of the Auslander-Reiten quiver). On dimension vectors, this additional Auslander–Reiten translation is still given by the Coxeter transformation (except that one has to change a sign). For example, we have a new Auslander Reiten triangle

$$(0 \to P_1) \overset{(0,1)}{\to} (P_3 \to P_1) = M \overset{(1,0)}{\to} (P_3 \to 0) = P_3[1]$$

which uses the obvious maps. All of the new Auslander–Reiten triangles are of that easy shape.

In general, the construction of Auslander–Reiten sequences can be modified to a construction of Auslander–Reiten triangles of finite complexes, either by working directly in the derived category (Happel [30]) or by working in the category of complexes (Roggenkamp) and then passing to the quotient. However, this works only for finite complexes. In fact, Happel [30] has shown that Auslander–Reiten triangles $X \to Y \to Z \to X[1]$ exist precisely for Z and X being bounded complexes of finitely generated projective modules. All other complexes are neither starting nor ending terms of Auslander–Reiten triangles. (Note that the almost split sequences for modules need not be Auslander–Reiten triangles, as one can see in the example of $D^b(B)$.) A consequence is:

Theorem 4.2. *(Happel [29]) The bounded derived category $D^b(A - mod)$ of a finite dimensional algebra A has Auslander-Reiten triangles if and only if the global dimension of A is finite.*

This applies in particular to tilted algebras. More general, applying a derived equivalence to a hereditary algebra one gets what is called a **piecewise hereditary algebra**. Happel, Rickard, and Schofield [32] have shown that the class of piecewise hereditary algebras coincides with the class of iterated tilted algebras which are obtained by starting with a hereditary algebra and then apply classical tilting finitely many times. Brenner [12] found an algorithm which allows us to find the (unoriented) graph of the original hereditary algebra (which is uniquely determined) within the derived category of a piecewise hereditary algebra.

An important technique used in the proof of the result of Happel, Rickard, and Schofield — and in many other papers — is to consider **perpendicular categories**. Let us restrict to the case of a hereditary algebra A and a finite dimensional A–module X: The right perpendicular category $X^\perp$ is the full subcategory of $A - mod$ having objects M which satisfy $Hom_A(X, M) = 0 = Ext^1_A(X, M)$. If X is indecomposable without selfextensions, then its perpendicular category $X^\perp$ is the module category of a hereditary algebra B which has one simple module less than A. (This is the special case of a much more general result.) And the embedding $B - mod \to A - mod$ is fully faithful and exact. This allows us to proceed by induction.

4.3. The existence of a tilting module T over A with $End_A(T) = B$ is sufficient but not necessary for the existence of a derived equivalence $D^b(A - mod) \simeq D^b(B - mod)$.

So the next step in tilting theory was to develop a Morita theory for derived categories. This has been done by Rickard [42] (and also by Keller). Here, the final metamorphosis of progenerators, the tilting complexes, appear:

Definition 4.3. Let R be a ring (with unit) and T a complex over R, that is $T \in K(R)$, the category of complexes over R with homomorphisms modulo homotopy. Then T is called a **tilting complex** if and only if it satisfies the following conditions:

(a) T is a bounded complex of finitely generated projective R-modules,

(b) for all $i \neq 0$, the homomorphism set $Hom_{K(R)}(T, T[i])$ is 0, that is, all homomorphisms from T to its i-th translate are homotopic to zero,

(c) $add(T)$ generates $K^b(proj - R)$, the category of bounded complexes of finitely generated projective R-modules, as a triangulated category.

Theorem 4.4. *(Rickard [42]) Let R and S be two rings. Then there is an equivalence between $D^b(R - Mod)$ and $D^b(S - Mod)$ if and only if there is a tilting complex T over R which has (in the derived category) endomorphism ring S.*

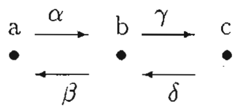
In contrast to the previous results, there is no statement about the functors giving the derived equivalence. The obvious choices of functors do not work in general (since S acts on T only up to homotopy). Hence in order to prove this theorem one has to find a new construction of functors, which involves solving hard technical problems.

In special cases, e.g. for finite dimensional algebras over fields, it has been shown by Rickard that in case there is a derived equivalence, there is another one which is a derived Hom or $\otimes$ functor.

Rickard's fundamental theorem is not only appealing from an aesthetic point of view, it also opens a way to completely new applications. The reason is that there are lots of rings, even finite dimensional algebras, which are derived equivalent, but not iterated tilting equivalent. Selfinjective algebras provide the most prominent examples: Over a selfinjective algebra, each module of finite projective dimension, in particular each tilting module, is projective. Thus tilting theory in the sense of Brenner and Butler reduces to classical Morita theory. However, there are many examples of selfinjective algebras A and B which are derived equivalent, but not Morita equivalent. Prominent examples are blocks of cyclic defect, that is, Brauer tree algebras.

From the existence of a derived equivalence between two rings, one gets back various kind of information on these rings. Among derived invariants are the Grothendieck groups, the center, and several homologies, in particular Hochschild cohomology and cyclic homology.

4.4. Let us finish by an example of a tilting complex. Let A be an algebra we met before, given by quiver and relations as follows:



modulo the relations: $\alpha \cdot \gamma = 0$, $\delta \cdot \beta = 0$, $\beta \cdot \alpha = \gamma \cdot \delta$.

This is a Brauer tree algebra with Brauer tree being a stem consisting of three edges. We will define a tilting complex which has endomorphism ring being a Brauer tree algebra associated with a star with three edges.

By $P(a), P(b), P(c)$ we denote the indecomposable projective A-modules. Let us consider the following complex $P(a) \oplus P(b) \oplus P(b) \stackrel{(0,0,f)}{\longrightarrow} P(c)$ where $f : P(b) \to P(c)$ is a non-zero homomorphism. This is a bounded complex of finitely generated projective A-modules. We check that it does not have selfextensions (which are just shifted endomorphisms): The only possibility to get selfextensions would be in degree one; but there is no homomorphism $P(a) \to P(c)$, and any homomorphism $P(b) \to P(c)$ factors via f. Moreover, T has $P(a)$ and $P(b)$ as direct summands. And there is a triangle $(P(b) \to 0) \stackrel{id}{\to} (P(b) \to P(c))$ which has suspension $P(c)$, hence the direct summands of T generate the same triangulated subcategory of $D^b(A)$ as the indecomposable projective modules. Thus T is a tilting complex. Up to homotopy, the set of homomorphisms between any two different direct summands of T is precisely k, whereas the endomorphisms up to homotopy in all cases are two-dimensional. Thus we get a Brauer tree algebra associated with a star (with three edges).

This example illustrates a result of Rickard, saying that each Brauer tree algebra is derived equivalent to another one, which has Brauer tree a star (with the same number of edges as the original Brauer tree).

4.5. **Acknowledgment.** These are the slightly extended notes of lectures which have been given at a workshop in Constantza (Romania) in September 1995. I would like to thank the organizers, K. W. Roggenkamp and M. Stefanescu, for inviting me to this unusually pleasant and well-organized conference. I am grateful to the Romanian colleagues for their hospitality and their interest, and to Henning Krause for allowing me to borrow some TEXfiles of Auslander-Reiten quivers from the notes of his lecture given at a workshop in Pappenheim in 1994.

4.6. Representation theory of finite dimensional algebras is too huge a subject to make a complete list of references possible. Thus before starting the ordinary references we give a list of books which are related to these lectures.

The best and most extensive introduction into almost split sequences and related topics is the recent book of M.Auslander, I.Reiten, and S.Smalø (Cambridge University Press). More of a survey nature and oriented towards applications to finite groups are the very readable two volumes of D.Benson (Cambridge University Press). Different in style and in terminology is the book of P.Gabriel and A.Roiter (Springer, Algebra VIII in the Encyclopedia).

More specialized are the following books: C.M.Ringel (Springer Lecture Notes) discusses tame algebras. K.Erdmann (Springer Lecture Notes) concentrates on tame group algebras (giving a more or less complete classification). Both books contain lots of examples. D.Happel (Cambridge University Press) surveys tilting theory and introduces into derived categories. D.Simson (Gordon and Breach) is interested in representations of partially ordered sets, that is, in a very special but important class of algebras (the incidence algebras of partially ordered sets). U.Jensen and H.Lenzing (Gordon and Breach) and M.Prest (Cambridge University Press) discuss connections to model theory which are important for studying infinite dimensional modules.

A good source for information on research topics are various conference proceedings, usually entitled Representations of algebras or Representation theory of algebras (older ones: Springer Lecture Notes, most recent ones: AMS and Kluwer, two new ones to be published in 1996 by the AMS).

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COMPUTER ALGEBRA AND REPRESENTATION THEORY

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ABSTRACT. This is an extended version of four lectures given at the workshop on representation theory of groups, orders and algebras at Constanta in September 1995.

INTRODUCTION

Computer algebra is an expanding subject in many areas of mathematics. Even, if one restricts oneself to the interaction of computer algebra and representation theory, it is impossible to describe the present research and activities in that area completely in four lectures. Thus the object of these lectures is the attempt to give a flavour of this topic to an audience which is not specialized in it, which wishes however to be presented with typical parts of the topic starting from the foundations and ending by questions which are in the middle of present research.

In section 1 we shall discuss some statements about sense and nonsense of computers in mathematics. We try to find some answers to such questions by looking at specific projects and problems which at the same time give a rough oversight about the actual research going on in computer algebra.

Section 2 deals with the classical question how ordinary character tables of finite groups may be calculated. The development of character theory dates back to the end of the 19th century and is due to Frobenius, Burnside and Schur. Already Burnside's results allow in principle the calculation of ordinary character tables. Starting with these origins we describe the Dixon - Schneider algorithm which is nowadays implemented in computer algebra systems like GAP for the calculation with characters. At the end of the section we sketch very briefly other methods like Clifford matrices and methods for modular representations.

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In section 3 we describe the Gröbner basis algorithm, often also called Buchberger algorithm, in its rudimentary basic form. This algorithm situated inside the commutative algebra and the algebraic geometry should be seen as a typical general tool of computer algebra which always enters the picture when systems of algebraic equations have to be solved. Many questions in the representation theory of group rings lead directly to such algebraic questions. A typical example for this has been given by M. Wursthorn in his MAPLE demonstration, cf. [54].

Finally section 4 is devoted to an application of computer algebra methods to a conjecture of Hans Zassenhaus concerning the unit group of integral group rings. The method developed in [7] represents an interplay between modular and ordinary representation theory. As starting point it uses the information available from the ordinary character table. In this sense it continues section 2 but under a different aspect. It will become clear that computer algebra is not only a topic the task of which is the calculation of mathematical objects or the computation of complicated examples. Section 4 demonstrates in which way computers may be used in order to get and to prove theorems. The point is that generic methods allow calculations for infinitely many objects and not only for one specific example. Finally we survey results on the isomorphism problem of integral group rings which have been obtained very recently with support of computer algebra.

1. About Sense and Nonsense of Computers

1.1. The central questions are as follows.

- Can you produce mathematics with a computer? If yes, what kind of mathematics can you produce with a computer?
- What is the part of computer algebra within mathematics?

1.2. Possible answers and provocative statements.

(a) A computer is mainly an algebraic machine. Thus it is primarily made for algebra.
(b) With a computer you can only produce and calculate examples or say better counterexamples.
(c) Computer algebra is a constructive part of mathematics. This means it is not sufficient to know there is a solution. You want to be able to compute this solution.
(d) Computer algebra should be applied also to problems where an algorithmic solution is unknown.

(e) It is impossible to prove theorems (solely) with computers or, if you really need a computer for a proof, you would at least have doubts about the correctness of the result.

(f) The task of computer algebra splits into two parts.
- The quick computation of an object.
- The development and the finding of algorithms for a given problem. This means in particular that as a first step for the solution of the problem a **constructive proof** has to be given.

1.3. Examples. The following examples illustrate the statements above.

ad (e). Probably most mathematicians are unhappy with the "solution" of the 4-colour problem. On the other hand this example is one of the striking ones which demonstrate the power of computers. Another example is the classification of the finite simple groups. This classification does not only cause problems by the number and the lengths of the papers which altogether yield finally the list of the finite simple groups. The existence of some of the sporadic groups still depends on computers and therefore also the list. The most prominent example perhaps is the construction of the 112-dimensional representation of the sporadic simple group J_4 over the field of order 2 [33]. For a description how such matrix representations are achieved see [40].

ad (d). Typical examples for problems where no deterministic solution is known are:

- Given finite groups G and H, decide whether their integral group rings $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic.
- Given a finite group G with a cyclic Sylow p–subgroup. Compute the Brauer tree of the principal block of kG, where k denotes an algebraic closed field of characteristic p. For groups where the precise shape is unknown see [20].

ad (c). It is not sufficient to know that $f \in \mathbb{Q}[x]$ has a Galois group G. You want to be able to compute G whenever f is given.

ad (f)

i) In practice often a quick algorithm is obtainable when the problem is reduced to

"Linear Algebra over $\mathbb{F}_p$".

ii) There might exist a priori an algorithm but it is not practicable. For example the modular isomorphism problem. This poses the question whether for a p-group P a ring isomorphism between group algebras $\mathbb{F}_p P$ and $\mathbb{F}_p Q$ implies that P and Q are isomorphic as groups.

This is clearly a finite problem. But note that in the case of $|P| = 2^7$ the unit group $U(\mathbb{F}_p P)$ has order 2^{127}. A computer algebra package called Sisyphos which handles automorphisms and isomorphisms of such huge groups has been developed by M. Wursthorn, cf. [55].

ad (b). Clearly counterexamples to conjectures are always of value. But it is also possible to calculate first with a computer some examples which are of suitable size for a computer but which are not computable by hand. It is a special aspect of the use of computer algebra then to discover a general recipe and finally to prove this recipe completely theoretically.

ad (a). This is the battle for ever young between

Numerical Mathematics and Algebra

Note that algebraic numbers may be handled as roots of polynomials with rational coefficients in an exact manner because calculations with such polynomials may be exactly done. However transcendental numbers always must be controlled by bounds in order to calculate with them in an accurate way.

1.4. The classification of the finite simple groups ($\sim$ 1980) has had a big influence on the development of computer algebra. This is clearly demonstrated by

- 1985 The atlas of the finite simple groups [9].
- 1995 The atlas of the Brauer characters of the finite simple groups [22].

May be around 2005 an atlas of generic character tables completes this picture. One should however expect that such an atlas is available in the form of a computer algebra system which does not simply store the tables but which calculates and displays on demand a specific part of such a table.

To archive character tables is justified because simple groups are determined up to isomorphism by their character table. More generally the following holds.

Theorem. [26] The chief factors of a finite group G are determined up to isomorphism by its character table $\mathrm{CT}(G)$
In particular semi-simple groups and their automorphism groups are determined by $\mathrm{CT}(G)$.
Note that the proof uses in a strong sense the classification.

The atlanta mentioned contain with the character table of a simple group G also the tables of the quasi-simple groups (this means central extensions G which are perfect as group) and of the almost simple groups (this means subgroups of the automorphism group of G which contain $G = \mathrm{Inn}G$) which belong to G. It seems to be unknown whether the ordinary character table CT(G) determines such groups up to isomorphism.

If one tries to prove a result for every finite group one always have to prove it for the simple ones and so in particular for the 26 sporadic groups. Most of the information known about these 26 groups is contained in [9] and in [22]. Thus the use of this part of computer algebra often enters the picture of a proof of a result on a general finite group. It is desirable that also the simple groups of Lie type may be handled similarly. These simple groups occur in series, mainly parameterized by their dimension and the characteristic of the underlying field. This shows clearly that generic character tables of groups of Lie type, i. e. parameterized character tables representing such series of simple groups, are of great interest.

1.5. The inverse problem of Galois Theory. The so-called constructive Galois-Theory is a typical example how computer algebra may be used for the solution of a classical problem.

- Realize (using the classification of the finite simple groups) any finite simple group as Galois group over $\mathbb{Q}$.
- Find a procedure for composite groups G to realize G as Galois group provided its composition factors are well-behaved.

There are criterions for (series of) simple groups to be Galois groups over $\mathbb{Q}$ depending only on their (generic) ordinary character table. Some composite groups may be realized as Galois groups if their composition factors are well behaved, cf. [30]. Thus simple groups enter here the picture precisely in the way as described in section 1.4.

1.6. Units of integral group rings.

It is difficult to compute non-trivial units for $\mathbb{Z}G$, by hand, even for G of small order. With a computer it is not too difficult to produce such units. Usually two such units generate an infinite subgroup of the unit group of $U(\mathbb{Z}G)$. Thus it is a priori not clear how finite subgroups of $U(\mathbb{Z}G)$ may be handled with computational methods. Already the determination of the finite subgroups of GL($n, \mathbb{Z}$) for small numbers n is a formidable task, cf. [32]. From this point of view it appears hopeless to prove with a computer that for $\mathbb{Z}B$, where B denotes the Baby Monster, the Zassenhaus Conjecture (ZC 2), cf. section 4.2., is valid. In section 4 we shall see how such a problem may be attacked by character tables.

2. Calculation of Character Tables

Notations. Throughout G denotes a finite group.

$$\begin{aligned} k &= \#\text{ of conjugacy classes of } G\,, \\ C_1,\dots,C_k &= \text{conjugacy classes of } G\,, \\ h_1,\dots,h_k &= \text{length of } C_1,\dots,C_k\,, \\ \chi_1,\dots,\chi_k &= \text{irreducible } \mathbb{C}\text{-characters of } G\,, \\ \chi_i(C_j) &= \text{character value of a representative of } C_j\,, \\ d_1,\dots,d_k &= \text{degrees of } \chi_1,\dots,\chi_k\,, \\ \overline{C_j} &= \text{class sum of } C_j = \sum_{x\in C_j} x\,, \\ c_{rst} &= \#\text{ of pairs } (x,y)\in C_r\times C_s \text{ with } xy=z, \\ & \quad\ \text{where } z \text{ is a fixed element of } C_t. \end{aligned}$$

2.1. Proposition.

$$\overline{C_r}\cdot\overline{C_s} = \sum_{t=1}^{k} c_{rst}\overline{C_t}\,.$$

Proof. The equation follows immediately from the definition of the numbers c_{rst} and the multiplication of the class sums in $\mathbb{Z}G$.

2.2. Proposition.

$$\frac{h_r\cdot\chi_1(C_r)}{d_i}\cdot\frac{h_s\cdot\chi_i(C_s)}{d_i} = \sum_{t=1}^{k} c_{rst}\cdot\frac{h_t\cdot\chi_i(C_t)}{d_i}\,.$$

Proof. By Maschke's Theorem $\mathbb{C}G$ is semisimple. It follows from Wedderburn's structure Theorem for semisimple algebras that $\mathbb{C}G$ is a direct product of matrix rings of the form

$$\mathbb{C}G \cong M(d_1,\mathbb{C})\times\ldots\times M(d_k,\mathbb{C}).$$

Class sums are central. Thus

$$\overline{C_r} = A_1\times\ldots\times A_n \text{ with } A_i = \begin{pmatrix} \lambda_{i,r} & & 0 \\ & \ddots & \\ 0 & & \lambda_{i,r} \end{pmatrix}.$$

Now

$$\chi_i(\overline{C_r}) = tr(A_i) = d_i\cdot\lambda_{i,r} = |C_r|\cdot\chi_i(C_r) = h_r\cdot\chi_i(C_r),$$

where tr denotes the trace.

Hence $\lambda_{i,r} = \frac{h_r \cdot \chi_i(C_r)}{d_i}$. Analogously $\lambda_{i,s}$ of $\overline{C_s}$ and $\lambda_{i,t}$ of $\overline{C_t}$ may be calculated. Consider now equation (1.1) with respect to $M(d_i, \mathbb{C})$. Then

$$\lambda_{i,r} \cdot \lambda_{i,s} = \sum_{t=1}^{k} c_{rst} \lambda_{i,t} .$$

This completes the proof.

2.3. Rewrite (1.2) as follows. Fix the indices r and i . Put

$$x_i = \begin{pmatrix} \lambda_{i,1} \\ \vdots \\ \lambda_{i,k} \end{pmatrix} \text{ and } M_r = \begin{pmatrix} c_{r11} & \cdots & c_{r1k} \\ \vdots & & \vdots \\ c_{rk1} & \cdots & c_{rkk} \end{pmatrix} .$$

Then Proposition 2.2. has the form

$$\lambda_{i,r} \cdot \chi_i = M_r \cdot x_i .$$

In other words $\lambda_{i,r}$ is an eigenvalue of M_r with respect to the eigenvector χ_i. The desired character values $\chi_i(C_j)$ are factors of $\lambda_{i,j}$ and are determined provided h_j and d_i are known. Thus the original problem is reduced to a problem in linear algebra, compare 1.3 ad f).

2.4. Burnside's algorithm.
The finite group G is given. The goal of the algorithm is the calculation of the character table

$$CT(G) = \begin{pmatrix} \chi_1(C_1) & \cdots & \chi_1(C_k) \\ \vdots & & \vdots \\ \chi_h(C_1) & \cdots & \chi_h(C_k) \end{pmatrix} .$$

The algorithm consists of six steps.

(i) Calculate $C_1, \ldots, C_k$ and $h_1, \ldots, h_k$. Choose $C_1 = \{1_G\}$.
(ii) Calculate $M_1, \ldots, M_k$ (these matrices are usually called the class matrices of G).
(iii) Find a set of k linearly independent vectors $x_1, \ldots, x_k$ such that each x_i is an eigenvector for each M_j.
(iv) Normalize $x_1^t = (x_{11}, \ldots, x_{1k}), \ldots, x_k^t = (x_{k1}, \ldots, x_{kk})$ by $x_{11} = \ldots = x_{k1} = 1$.
(v) Calculate the degrees d_i via

$$d_i^2 = \frac{|G|}{s} \text{ with } s = \sum_{j=1}^{k} x_{ij} x_{ij'} \cdot \frac{1}{h_j} \text{ and } d_i > 0.$$

The index j' is defined by $g \in C_j \Leftrightarrow g^{-1} \in C'_j$.

(vi) Calculate $\chi_i(C_j)$ by

$$\chi_i(C_j) = \frac{x_{ij} \cdot d_i}{h_j}.$$

Comments. Steps (i) and (ii) consist of calculations purely inside the group G. The normalization in step (iv) is possible because x_{j1} are non–zero multiples of the character degrees and therefore non–zero.

The calculation of s in step (v) may easily be derived from the orthogonality relations

$$\sum_{i=1}^{k} \chi_i(C_j)\chi_i(C_j') \cdot h_j = |G|.$$

The main problem are the calculations necessary for step (iii). A look at small groups shows that the determination of the eigenvectors is a non-trivial problem, if the underlying field has infinitely many elements.

2.5. Example. Let G be the symmetric group of degree 3. Denote by C_1 the conjugacy class of the identity, let C_2 be the class of a transposition and let C_3 that one of a 3 - cycle. We have

$\bar{C_2} \cdot \bar{C_2} = 3\bar{C_1} + 3\bar{C_3}, \bar{C_2} \cdot \bar{C_3} = 2\bar{C_2}, \bar{C_2} \cdot \bar{C_1} = \bar{C_2}, \bar{C_3} \cdot \bar{C_1} = \bar{C_3}, \bar{C_3} \cdot \bar{C_2} = 2\bar{C_2}, \bar{C_3} \cdot \bar{C_3} = 2\bar{C_1} + \bar{C_3}$.

This gives the class matrices

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Note that M_2 has three pairwise different eigenvalues. Thus we find the common eigenvectors of all class matrices already by calculating the eigenvectors of M_2. M_3 has only two different eigenvalues. Thus the use of this class matrix does not lead directly to the eigenspaces of all class matrices, cf. also section 2.7.

The eigenspaces of M_2 are generated by $(1,3,2), (1,-3,2), (1,0,-1)$. E.g. from $(1,3,2)$ we get the trivial character χ_1 and $(1,0,-1)$ gives rise to the irreducible character of degree 2 with values $(2,0,-1)$.

2.6. Dixon's improvement. [12] The idea is to take a prime p such that $p \equiv 1 \mod e$, where e denotes the exponent of G. Find $z \in \mathbf{N}$ such that $z^e \equiv 1 \mod p$ and $z^f \not\equiv \mod p \; \forall f$ with $0 < f < e$.

Consider now the homomorphism $\Theta : \mathbb{Z}[\zeta] \longrightarrow \mathbb{F}_p$ defined by sending ζ to z. Then Θ transfers the problem of finding the common eigenvectors of the class matrices into a finite problem or more precisely into a question on

"Linear Algebra over $\mathbb{F}_p$".

Applying Θ component-wise the vectors $x_1, \dots x_n$ are mapped into a set of linearly independent vectors over an $\mathbb{F}_p$ - vector space which are eigenvectors for the class matrices $M_1, \dots, M_k$ mod p. Thus step (iii) of Burnside's algorithm 1.4 is replaced by

(iii)$_p$ Find a set of k linearly independent vectors $\bar{x_1}, \dots, \bar{x_k}$ such that each $\bar{x_i}$ is an eigenvector for each M_j mod p.

Clearly the eigenvalues may be found just by checking the elements of $\mathbb{F}_p$ and then the determination of their eigenspaces is a homogeneous system of linear equations over $\mathbb{F}_p$. This big advantage however causes on the other hand some new problems. At the end of the algorithm 1.4. one gets a new step.

(vii) Translate $\Theta(\chi_i(C_j))$ back to $\chi_i(C_j)$.

Also in step (v) of 1.4. one has to recognize the degrees d_i in a unique way mod p. For this the prime p has to be chosen large enough. It suffices to choose p such that no squares of numbers between 1 and d_i are equal mod p. For a given t the equation $x^2 \equiv t$ mod p has two solutions. But $(p+a)^2 \equiv (-p+a)^2$ mod p. Hence $p > 2 \cdot d_i$ will work and because $d_i^2 \leq |G|$ it follows that $p > 2 \cdot \sqrt{|G|}$ suffices.

Note that this choice is possible is guaranteed by Dirichlet's Prime Number Theorem which in particular says that there are infinitely many primes $\equiv 1$ mod ϵ.

Thus we are left with **step (vii)**. This means that we know

$$\Theta(\chi(C) = \zeta^{s_1} + \dots \zeta^{s_d},$$

where ζ denotes a primitive e - th root of unity. The goal is to find $\chi(C)$.

Now use that $\sum_{i=0}^{e-1} \zeta^{it} = e$, if e divides t, and that the sum is zero, if e does not divide t.

Claim. From

$$\chi(C) = \sum_{s=0}^{e-1} m(s)\zeta^s$$

we get that

$$m(s) = \frac{1}{e} \cdot \sum_{n=0}^{e-1} \chi(C^n)\zeta^{-sn}.$$

Proof. $x \in C$ implies that $x^n \in C^n$.. Thus

$$\chi(C^n) = \zeta^{s_1 n} + \dots + \zeta^{s_d n}.$$

It follows that

$$\sum_{n=0}^{e-1} \chi(C^n)\zeta^{-sn} = \sum_{n=0}^{e-1} \zeta^{(s_1-s)n} + \ldots + \zeta^{(s_d-s)n}.$$

On the right hand side we get an e as summand, if and only if e divides $(s_i - s)$. This happens only if $s_i - s = 0$ because all the s_i are $\leq e-1$. The claim follows immediately. q.e.d.

We get finally that $\chi_i(C_j) = \sum_{s=0}^{e-1} m_{ij}(s)\zeta^s$ with

$$m_{ij}(s) = \frac{1}{e} \cdot \sum_{n=0}^{e-1} \Theta(\chi_i(C_{j(n)})z^{-sn} \bmod p,$$

where $j(n)$ is defined by $x \in C_j$ if and only if $x^n \in C_{j(n)}$. Since $m_{ij}(s) \leq d_i < p$ the multiplicity $m_{ij}(s)$ is uniquely determined mod p.

2.7 Schneider's refinement. [53]

The Dixon algorithm still may be improved. Denote by $\bar{M}_i$ the class matrices mod p. Dixon's algorithm calculates the eigenspaces of these $\bar{M}_i$ in a direct manner.

(i) Assume that the class matrices $\bar{M}_1, \ldots, \bar{M}_{r-1}$ have been handled and comon eigenspaces $E_1, \ldots, E_k$ are determined.
(ii) For $0 \leq \lambda \leq p-1$ consider for each eigenspace E_i of dimension ≥ 2 the action of $\bar{M}_r - \lambda I)$. Split E_i into a direct sum of eigenspaces E_{ij} of $\bar{M}_r$.
(iii) If each eigenspace E_{ij} has now dimension 1 the process terminates.

Note that the ordering of the $\bar{M}_i$ is arbitrary. As already the trivial example 1.5. shows, a fast determination of the common eigenspaces depends on the ordering of the class matrices; as a matter of fact, it is not necessary to compute all class matrices in order to get the character table. The main parts of Schneider's refinement are

- to determine the linear characters first and then to compute their orthogonal complement in the underlying k - dimensional $\mathbb{F}_p$ - vector space,
- to organize a suitable ordering of the class matrices by a valuation process which prefers class matrices which split low dimensional (low means ≤ 6) eigenspaces,
- to avoid the full calculation of a new class matrix by choosing certain well organized bases of the eigenspaces E_i known before,
- the use of non-deterministic methods in order to split 2 - dimensional eigenspaces.

In many examples it is shown that these points reduce the costs for the calculation of the character table substantially. Also the non-deterministic methods need only few CPU-time. Thus in case of their failure their influence to the total costs is small. For more details we refer to [53].

2.8. Clifford theory. In the situation when G has a normal p - subgroup N and the character table of G/N is known there are other methods for the calculation of the character table of G. The method of Clifford matrices introduced by Fischer, cf. [15], has been successfully used in this situation. The theoretical background is the following fundamental result of the Clifford theory:

Let N be a normal subgroup of G. For $\zeta \in \mathrm{Irr}N$ denote by T its inertia group in G. This means that T is defined as

$$T = \{g \in G; \zeta^g = g\},$$

where the action of g on ζ is defined as $\zeta^g(n) = \zeta(g^{-1}ng), n \in N$. Denote by $<,>$ the scalar product of characters, i.e.

$$< \phi, \chi >= \frac{1}{|G|} \cdot \sum_{g \in G} \phi(g) \cdot \bar{\chi(g)}.$$

Let ψ^G be the character induced to G and let $\chi|_N$ be the character restricted to N. Put

$$A(\zeta) = \{\psi \in \mathrm{Irr}T; < \psi_N, \zeta > \neq 0\} \mathrm{and} B = \{\chi \in \mathrm{Irr}G; < \chi_N, \zeta > \neq 0\}.$$

Then the map $\psi \longrightarrow \psi^G$ is a bijection between A and B. Moreover, let $\zeta_1, \ldots, \zeta_m$ be the representatives of the orbits of the G - action on $\mathrm{Irr}N$, then

$$\mathrm{Irr}G = \bigcup_{i=1}^{m} A(\zeta_i).$$

If $\chi \in A(\zeta_i)$, then $\chi(g)$ is expressible as a sum of the form $\sum c_j \cdot \rho(y_j N)$, where ρ is a projective character of T_i/N, T_i denotes the inertia group of ζ_i). The coefficients c_j form the entries of the Clifford matrices.

The method requires a good knowledge of the characters of the factor group G/N and of its subgroups. Thus it works in particular good for the case when G/N is abelian and if N has a complement in G. Such a situation often occurs for almost simple groups.

2.9. Character Tables of finite groups of Lie type. The calculation of character tables of linear groups dates back to Frobenius [16] and Schur. Fundamental for the calculation of irreducible characters of arbitrary finite groups of Lie Type is the theory of Deligne and Lusztig [11]. A survey on the use of this theory in computer algebra is given in [31, S2]. The known generic character tables of finite groups of Lie type are collected in [10].

2.10. Modular Representations. The methods explained so far concern mainly ordinary representations of a finite group. The determination of the Brauer characters of a finite group is of course another fundamental part of computer algebra in the representation theory. More precisely let k be a finite field of characteristic p and let M be a finitely generated kG - module then a basic task in representation theory is the determination of the composition factors of M. Parkers Meat-Axe [39] was probably the first computer program which solved this problem in a satisfactory way (for matrix representations of dimensions up to around 1000). Combining the Meat-Axe with condensation methods one can nowadays handle representations of much larger dimension. The idea of condensation is as follows.

Let H be a p' - subgroup of G. Then

$$e = \frac{1}{|H|} \sum_{h \in H} h$$

is an idempotent of the group algebra kG and leads to the Hecke algebra $H = e(kG)e$. Now H acts on the right of Me. The H - module Me is called the condensed module to M. Information on the composition factors of M is now obtained by looking at the composition factors of Me, which is of course a module of much smaller k - dimension. For more details about condensation see [38] or [29].

In contrast to the Meat-Axe the MOC - system [19] deals with Brauer characters and not with representations. It relies on elementary methods computing decomposition numbers. In particular the restriction of ordinary irreducible characters of G to the p - regular conjugacy classes gives a good starting point. This leads to the study of basic sets of Brauer characters which are also of theoretical interest for finite groups of Lie type, cf. [17]. The advantage of MOC is that it applies to much larger degrees than Meat-Axe plus condensation. On the other hand system some problems really require the knowledge of the representation and not only of its character. Thus MOC, Meat-Axe and condensation should be seen together as a computational tool for the determination of the modular representations of a finite group. Most of the irreducible modular representations of the sporadic groups are nowadays known [22] and also most of their Brauer trees [20].

3. Gröbner bases

Notations. K=field, $K[X] = K[X_1, \ldots, X_\nu]$ polynomial ring in ν variables. $P = \{X_1^{\alpha_1} \cdot \ldots \cdot X_\nu^{\alpha_\nu}; \alpha_1, \ldots, \alpha_\nu \in \mathbb{N}_K\}$. Note that P is a K–basis.

3.1. Definition. A term order on P is a linear order $(P,<)$ which satisfies the following two conditions.

(i) $\forall_{p \in P\setminus\{1\}} : 1 < p$,
(ii) $\forall_{p,q,r \in P} : p < q \Longrightarrow pr < qr$.

3.2. Examples.

a) The lexicographical order.
$X_1^{\alpha_1} \cdot \ldots \cdot X_\nu^{\alpha_\nu} <_{lex} X_1^{\beta_1} \cdot \ldots \cdot X_\nu^{\beta_\nu}$ if and only if there exists $j \in \{1, \ldots, v\}$ such that $\alpha_j < \beta_j$ and $\alpha_k = \beta_k$ for all $k < j$.
b) The graded – lexicographical order.
$X_1^{\alpha_1} \cdot \ldots \cdot X_\nu^{\alpha_\nu} <_{glex} X_1^{\beta_1} \cdot \ldots \cdot X_\nu^{\beta_\nu}$ if and only if
$\sum\limits_{i=1}^{\nu} \alpha_1 < \sum\limits_{j=1}^{\nu} \beta_j$ or $\sum\limits_{i=1}^{\nu} \alpha_1 = \sum\limits_{j=1}^{\nu} \beta_j$ and $X_1^{\alpha_1} \cdot \ldots \cdot X_\nu^{\alpha_\nu} <_{lex} X_i^{\beta_1} \cdot \ldots \cdot X_\nu^{\beta_\nu}$.

3.3. Definitions. Assume that a term order $(P,<)$ is given.

a) Let $0 \neq f = \sum\limits_{p \in P} c_p P$ with $c_p \in K$. Then we put $L(f) = \max\{p; c_p \neq 0\}$ and $Lc(f) = c_{L(f)}$.
If $0 = f$, then define $L(f) = 0$.
We call $L(f)$ the leading monomial and $Lc(f)$ the leading coefficient.
b) Let $0 \neq F \subseteq K[x]$. Then we put
$L(F) = \{p \cdot L(f); p \in P, f \in F\{0\}\}$.
Note that $L(F)$ is a subset of P.
c) Let I be an ideal of $K[x]$. A finite subset G of I is called a Gröbner basis of I, if $L(G) = L(I)$.

3.4. Examples.

a) Let $m \in K[x]\setminus\{0\}, I =< m >$. Then $\{m\}$ is a Gröbner basis of I.
b) Any nonempty finite subset G of P is a Gröbner basis of $< G >$.
c) There exists $F = \{f_1, f_2\} \subseteq K[X]$ such that F is not a Gröbner basis of $I =< F >$, e.g.

$$f_1 = X_1^3 X_2 + X_1, \;\; f_2 = X_1^2 X_2^3.$$

Then with respect to "$<_{glex}$" we have $L(f_1) = X_1^3 X_2 L(f_2) = X_1^2 X_2^3$. Because $y = X_2 \cdot f_1 - f_2 = X_1 X_2$ is in I we get $L(y) = X_1 X_2 \in L(I)$. But $X_1 X_2 \notin L(F)$.
d) Each finite subset S of an ideal I of $K[X]$ which contains a Gröbner basis of I is itself a Gröbner basis of I.

3.5. Theorem. Each ideal I of $K[X]$ has a Gröbner basis.

Remarks on the proof of 3.5. A short proof may be given using Hilbert's Basis Theorem. But this proof is not constructive. Thus the goal is to find an algorithm which permits the calculation of a Gröbner basis.

3.6. The division by F. Let $F \subset K[X]$. Then $\sum_{p\in P, f\in F} c(p,f)\cdot pf$ with $c(p,f) \in K$ is called an admissible combination from F, if for all $p, p' \in P$ and for all $f, f' \in F$ with $c(p,f) \neq 0, c(p', f') \neq 0$ and $(p,f) \neq (p', f')$ the leading monomials $L(p \cdot f)$ and $L(p' \cdot f'))$ are different.

Proposition. Let $v \in K[X], F \subset K[X]$. Then exists an admissible combination w from F such that $L(v - w) \notin L(F)$.

$v - w$ may be explicitly calculated but is not unique. It may be regarded as a residue of v modulo F. We shall use the notation $res(v)$ mod F in the meaning that $res(v)$ is one residue of v modulo F. We describe the calculation of these residues.

Compute a sequence (v_n) as follows. Start with $v_0 = v$. Assume that v_1 is already computed. If $L(v_i) \in L(F)$, then there exists $f_i \in F$ and $p_i \in P$ with $L(v_i) = L(p_i f_i)$. Put

$$v_{i+1} = v_i - Lc(v_i)Lc(f_i)^{-1}p_i f_i.$$

If $L(v_i) \neq L(F)$, then

$$\omega = \sum_{j=0}^{i-1} Lc(v_j)Lc(f_j)^{-1}p_j f_j, \text{ if } i > 0$$

and $\omega = 0$ otherwise.

Proof. It is easy to see that $L(v_{i+1}) < L(v_i)$. Thus we get a descending sequence of monomials. But any subset of P has a minimal element. q.e.d.

Note that p_i and f_i are not unique. The knowledge of $L(v)$ gives a priori an estimate for the number of steps needed for the division by F.

3.7. The S - polynomial. Let $p = X_1^{a_1} \cdot \ldots \cdot X_\nu^{a_\nu}$ and $q = X_1^{b_1} \cdot \ldots \cdot X_\nu^{b_\nu}$, then the lowest common multiple is defined as

$$\operatorname{lcm}(p,q) = X_1^{c_1} \cdot \ldots \cdot X_\nu^{c_\nu} \text{ with } c_i = \max(a_i, b_i).$$

Moreover $\operatorname{lcm}(0,q) = \operatorname{lcm}(p,0) = 0$.

Define now for given $v, w \in K[X]$ the S-polynomial as

$$S(v,w) = Lc(w)\cdot p\cdot v - Lc(w)\cdot q\cdot w,$$

where $p, q \in P \cup \{0\}$ with $p \cdot L(v) = q \cdot L(w) = \operatorname{lcm}(L(v), L(w))$.

3.8. Algorithm for the construction of the Gröbner basis. Assume that $U =< F >$ with F finite is given.

(i) Put $F_0 = F$.
(ii) Suppose that F_i is constructed. Put

$$F_{i+1} = F_i \cup \{res(S(f,g)) \bmod F_i; f, g \in F_i \setminus \{0\}\}.$$

(iii) If $F_{i+1} \neq F_i$ then goto step (i) else F_{i+1} is a Gröbner basis of $< F >$.

3.9. The algorithm above may be improved substantially. For this we call $F \subset K[X]$ simplified, if the following holds.

$$\forall f \in F : L(f) \notin L(F \setminus \{f\}).$$

A Gröbner basis G of U is called reduced, if

1. G is simplified.
2. $\forall g \in G : Lc(g) = 1$
3. $\forall g \in G : g - L(g)$ is a K - linear combination of elements of $P \setminus L(G)$.

3.10. Basic Properties of Gröbner bases.

a) Simplified Gröbner bases are not unique. Reduced Gröbner bases are unique. Each ideal of $K[X]$ has a reduced Gröbner basis. Thus as a consequence one obtains that ideals I and J of $K[X]$ coincide if and only if their reduced Gröbner bases coincide.
b) If G is a Gröbner basis of I then $I =< G >$.
c) If G is a Gröbner basis of I and $v \in K[X]$ is given, then

$$v \in \Leftrightarrow 0 \text{ is a residue of } v \bmod G.$$

d) The Gröbner basis algorithm may be extended such that each element of the Gröbner basis is expressed as a sum of multiples of the input polynomials.

3.11. Gröbner bases and systems of algebraic equations. Let $F = \{p_1, \ldots, p_k\} \subset K[X]$ and consider the system of algebraic equations of the form

$$p_1 = 0, \ldots, p_k = 0.$$

Denote by $\bar{K}$ the algebraic closure of K and put $J =< p_1, \ldots, p_k >$. Let $N(F) = \{y \in \bar{K}^\nu; p_i(y) = 0 \ \forall i\}$ and let G be a Gröbner basis of J. Then

(i) $N(F) = \emptyset \Leftrightarrow G \cap K \neq \emptyset$.
(ii) $N(F)$ is finite $\Leftrightarrow P \setminus L(G)$ is finite.
(iii) Let $g \in K[X]$ and put $\tilde{J} =< F, g \cdot X_0 - 1 > \subset K[X_0, \ldots, X_n]$. Let $\tilde{G}$ be a Gröbner basis of $\tilde{J}$. Then $\exists n \in \mathbb{N}$ with

$$g^n \in J =< F > \Leftrightarrow \tilde{G} \cap K \neq \emptyset,$$

where K is identified with the constants in $K[X_0, \dots, X_n]$ and

$$\tilde{G} \cap K \neq \emptyset \Leftrightarrow N(F \cup \{g\}) = N(F).$$

(iv) There are procedures which produce solutions for certain systems of algebraic equations. For example in the case when $K = \mathbb{Q}$ and you know a priori that there are only finitely many solutions. Then one can calculate all solutions in $\mathbb{Q}^{\nu}$.

3.12. Remarks. The introduction of Gröbner bases given in this section follows [36]. For a detailed description of the development of Gröbner bases we refer to [2].
Gröbner bases exist in a more general context. The field K may be replaced by a principal ideal domain and instead of $K[X]$ one can construct analogously Gröbner bases for free modules over $K[X]$ of finite rank, cf. [2, S10]. For the polynomial ring $A = K < X_1, \dots, X_n >$ in non-commuting variables X_i there is no analogue to Gröbner bases. But for certain quotients of A the construction of Gröbner bases for arbitrary ideals is possible. This holds in particular for Weyl algebras, cf. [2, S11].
Other such quotients are basic algebras of the group algebra kG, where k is a field of characteristic $p > 0$ and G a finite group. Here the Gröbner bases are used to calculate projective resolutions of simple modules, cf. [13] and [14].

4. An Application to Integral Group Rings

Notations. Throughout R denotes an integral domain of characteristic zero and G is a finite group. We assume that no prime divisor of $|G|$ is invertible in R. K denotes a field containing R. The map $\varepsilon : RG \longrightarrow R$ defined by $\varepsilon(\sum r_g g) = \sum r_g$ is called the augmentation map. A unit u of RG is called normalized, if $\varepsilon(u) = 1$. The group of normalized units of RG is denoted by $V(RG)$. A subgroup H of $V(RG)$ is called a group basis, if H is an R-basis of RG. Let H be a group basis of RG and let C be a conjugacy class of H. Then the sum taken over all elements of C is called the class sum denoted by $\underline{C}$.
An R - algebra automorphism σ of RG is called normalized, if it preserves the augmentation, i.e. $\forall x \in RG : \varepsilon(\sigma(x)) = \varepsilon(x)$. The group of all normalized R - algebra automorphisms will be denoted by $\mathrm{Aut}_n(RG)$.

4.1. The class sum correspondence [47]. If H is a group basis of RG, then H is in class sum correspondence to G, i.e. there is a bijection $\sigma : G \to H$ such that the image of a conjugacy class C of G is a conjugacy class of H. Moreover their class sums $\underline{\sigma(C)}$ and $\underline{C}$ coincide. In particular the R-linear extension of σ fixes the centre of RG element-wise, since class sums form an R-basis of the centre of RG. Note however that σ is a priori not a group homomorphism. By

[35] σ is compatible with the **power map**, i.e. for each n the conjugacy class of $\sigma(g^n)$ coincides with that of $(\sigma(g))^n$.

Assume now for a moment that $R = \mathbb{Z}$. It is an open question whether the torsion subgroups of $U(\mathbb{Z}G)$ are determined by G. More precisely the following questions have been studied extensively in the last twenty years.

4.2. The Zassenhaus Conjectures

[51]

1. **(ZC 1)** Let u be a unit of finite order of $V(\mathbb{Z}G)$. Then u is conjugate within $\mathbb{Q}G$ to an element of G.
2. **(ZC 2)** Let H be a subgroup of $V(\mathbb{Z}G)$ with the same order as G. Then H is conjugate to G by a unit of $\mathbb{Q}G$.
3. **(ZC 3)** Let U be a finite subgroup of $V(\mathbb{Z}G)$. Then U is conjugate within $\mathbb{Q}G$ to a subgroup of G.

4.3. Remarks (ZC 2) is in general not true. Roggenkamp and Scott constructed metabelian counterexamples, cf. [28], [42]. Nevertheless for many classes of groups it is true and no counterexample to the following Sylow-version of (ZC 2) is known.

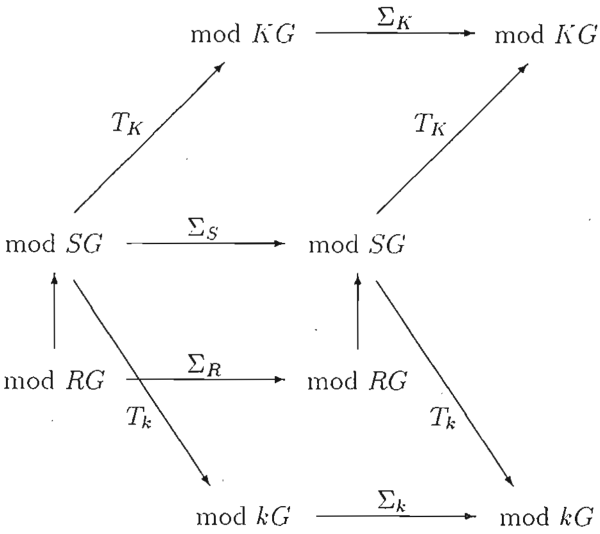
(ZC 2)$_p$ Let H be a group basis of RG and let p be a rational prime. Then there exists a ring automorphism σ_p of RG such that $\sigma_p(G) = H$ and σ_p restricted to the class sums of p-elements is the identity.

Note, if one knows that the isomorphism problem for G has a positive solution, i. e. $RG \cong RH \Rightarrow G \cong H$, then (ZC 2) is reduced to the study of normalized automorphisms of RG. It may be then phrased in that way that each $\sigma \in \mathrm{Aut}_n(RG)$ may be modified by a group automorphism τ of G such that the composition $\tau \cdot \sigma$ fixes all class sums. It is now clear that (ZC 2)$_p$ is a weaker statement than (ZC 2).

4.4. Character tables and blocks. It is a result of G. Glauberman that $\mathbb{Z}G \cong \mathbb{Z}H \longrightarrow \mathrm{CT}(G) = \mathrm{CT}(H)$, cf. [21, (3.20)]. The class sum correspondence, section 4.1., shows that $\sigma \in \mathrm{Aut}_n(RG)$ induces an automorphism of $\mathrm{CT}(G)$. Such a table automorphism consists of a pair of permutations (τ_1, τ_2) such that

$$\chi_i(C_j) = \chi_{\tau_1(i)}(C_{\tau_2(j)}), \forall i, j.$$

Now let (K, S, k) be a p-modular system which is sufficiently large for G. S is a discrete complete valuation ring containing R and $k = S/\mathrm{rad}S$ has characteristic p. Sufficiently large means that KG and kG are splitting fields for G. The inclusion map from S into K rsp. and the reduction from R onto k induce obvious functors T_K : mod $SG \longrightarrow$ mod KG and T_k : mod $SG \longrightarrow$ mod kG. We get the following picture:



Let $\sigma \in \mathrm{Aut}_n(RG)$. Twisting the action on a finitely generated RG - module M by σ we get an autoequivalence Σ_R of mod RG. More precisely $\Sigma_R(M) = M^\sigma$ is the RG - module which is as R - module identic with M. The G - action however is given by

$$g \star m = \sigma(g) \cdot m,$$

where $\cdot$ denotes the action of RG on M. Now E is a composition factor of M if and only if E^σ is a composition factor of M^σ. The automorphism σ extends linearly to group algebra automorphisms σ_S, σ_K and via reduction modulo the maximal ideal of S to $\sigma_k \in \mathrm{Aut}_n(kG)$. As in the case of Σ_R we get induced autoequivalences $\Sigma_S, \Sigma_K, \Sigma_k$ which make the above diagram obviously commutative. Looking at the composition factors we see that even more Σ_S and Σ_k commute with the decomposition map.
Finally let χ be the irreducible K - character afforded by the simple KG - module T. Let ξ be the irreducible K - character afforded by $T^\sigma = \Sigma_K(T)$. Then

$$\chi(C^\sigma) = \xi(C),$$

where C^σ denotes the conjugacy class of G whose class sum coincides with that one of the conjugacy class $\sigma_R(C)$ in $\sigma_R(G)$. Note that the class sum correspondence garantees that there is a unique conjugacy class of G with this property. Thus the action on $\mathrm{Irr}(G)$ induced from σ coincides with that one given by Σ_K.

The following result gives now the key information for establishing conjecture (ZC 2) via character tables. Note that the module theory of a block with cyclic defect is described more or less completely by a combinatorical object called Brauer tree. This is a tree whose vertices are labelled by the irreducible K - characters belonging to B. The edges are labelled by the irreducible k - characters of B. A leaf of a tree is an edge which is not adjacent to two other edges.

4.5. Theorem. [7]. Let Σ be an autoequivalence of a block B of kG with cyclic defect. Assume that Σ fixes the isomorphism class of a leaf of the Brauer tree of B. Then Σ fixes the isomorphism classes of all simple B - modules.

4.6. Example. We explain the idea of the proof of Theorem 4.5 with the following example. Let B be a cyclic p - block of kG with the following Brauer tree Γ.

$$\begin{array}{ccccc} & & o^6 & & \\ & & | & & \\ o^1 & - & o^8 & - & o^5 \\ & & | & & \\ & & o_7 & & \end{array}$$

The vertices of Γ are labelled by the indices i of the ordinary irreducible characters χ_i of G which belong to B. The Brauer tree provides an algorithm for calculating the composition series of the projective covers of the simple modules, cf. [1, Ch.V]. Denote by S_i the simple module which labels the edge between the vertices 8 and i in the tree. Then the projective covers P_{S_6} and P_{S_7} are uniserial and have the composition series

	S_6		S_7
	S_1		S_5
$P_{S_6} =$	S_7	$P_{S_7} =$	S_6
	S_5		S_1
	S_6		S_7

Note that the Brauer tree of our example corresponds to the principal 5 - block of M_{11}. But for the explanation of the proof of Theorem 4.5. this is of secondary interest.

Let now $\sigma \in \mathrm{Aut}_n(RG)$. σ induces on mod kG the autoequivalence Σ_k, compare the diagram in 4.4. Clearly Σ_k permutes the p - blocks of kG. We assume that Σ_k fixes the leaf with vertex 1 of Γ. Thus Σ_k fixes also B and induces finally an autoequivalence Σ_B of the block B. Observe that the last assumptions are automatically satisfied when B is the principal p - block and the leaf corresponds to the trivial simple module.

We explain first that Σ_B induces a graph automorphism of Γ. For the vertices

i, j rsp. of Γ let T_i, T_j rsp. be the irreducible KG - modules corresponding to χ_i, χ_j rsp. Let L_i, L_j be an S - form of T_i, T_j. Note that S - form means that L_i, L_j rsp. are SG - lattices such that $K \otimes L_i \cong T_i, K \otimes L_j \cong T_j$ rsp. Then i and j are joined by an edge if there is a simple kG - module occuring as composition factor of $k \otimes L_i$ and of $k \otimes L_j$. Because Σ_S and Σ_k commute with the decomposition map it follows that Σ_B acts as a graph automorphism on Γ. This still leaves the possibility that Σ_B interchanges χ_6 and χ_7 and therefore S_6 and S_7. But Σ_B also must map the projective cover of a simple module S_i into the projective cover of the simple module $\Sigma_B(S_i)$. Now looking at the composition series of P_{S_6}, P_{S_7} rsp. we see that this is impossible because then Σ_B has also to interchange S_1 and S_5.

4.7. Principal block algorithm. [7, section 3]

(i) Compute the ordinary character table $\mathrm{CT}(G)$.

(ii) Denote by $\mathrm{Aut}(\mathrm{CT}(G))$ the group of character table automorphisms. Compute the subgroup A of $\mathrm{Aut}(\mathrm{CT}(G))$ induced by $\mathrm{Aut}(G)$.

(iii) Compute $\mathrm{Aut}(\mathrm{CT}(G))$.

(iv) Put $M = \{\chi \in \mathrm{Irr}(G); \exists \sigma \in \mathrm{Aut}(\mathrm{CT}(G)) \text{ with } \sigma_1(\chi) \neq \chi\}$.

(v) Let $P = \{p;\ G$ has cyclic Sylow p-subgroups $\}$. Check for each $\chi \in M$ whether χ belongs to the principal p-block B_p for some prime p, where $p \in P$.

Note that $\chi \in \mathrm{Irr}(G)$ belongs to the principal p-block if, and only if, for all p-regular elements $g \in G$ the following holds.

$$\frac{\chi(g)}{\chi(1)}|\mathrm{Cl}(g)| \equiv |\mathrm{Cl}(g)| \mod p,$$

in the ring of algebraic integers of $\mathbb{Q}[\zeta]$, where ζ denotes a primitive $|G|$-th root of unity and $\mathrm{Cl}(g)$ the conjugacy class of g in G (see [18, (7.10)]).

(vi) Check whether χ is non-exceptional, i.e. check how many characters belonging to the principal p-block coincide with χ restricted to the p-regular classes.

(vii) Let T be the subset of M consisting of those characters which are for some prime $p \in P$ non-exceptional and belong to the principal p-block.

(viii) Determine the subgroup U of $\mathrm{Aut}(\mathrm{CT}(G))$ given by

$$U = \{\sigma \in \mathrm{Aut}(\mathrm{CT}(G); \sigma_1(\chi) = \chi \ \forall \ \chi \ \in T\}.$$

(ix) Note that the subgroup of $\mathrm{Aut}(\mathrm{CT}(G))$ induced by $\mathrm{Aut}_n(RG)$ is always a subgroup of U and contains A. Thus, if $A = U$, (ZC 2) holds.

4.8. Example We apply the principal block algorithm to show that (ZC 2) is valid for the groups $G = \mathrm{PSL}(2,p)$, where p denotes a rational odd prime, cf. [7]. Consider the following generic character tables of $\mathrm{PSL}(2,q)$, where

$q = p^f$. These tables are recorded in CHEVIE [10]. They were first implicitely calculated in [16]. For the first table we asumme that $q \equiv 1 \bmod 4$.

	C_1	C_2	C_3	$C_4(i)$	$C_5(i)$
χ_1	1	1	1	1	1
χ_2	q	0	0	1	-1
χ_3	$\frac{1}{2}(q+1)$	$\frac{1}{2} - \frac{1}{2}\sqrt{q}$	$\frac{1}{2} + \frac{1}{2}\sqrt{q}$	$(-1)^i$	0
χ_4	$\frac{1}{2}(q+1)$	$\frac{1}{2} + \frac{1}{2}\sqrt{q}$	$\frac{1}{2} - \frac{1}{2}\sqrt{q}$	$(-1)^i$	0
$\chi_5(k)$	$q+1$	1	1	$\zeta_1^{2ik} + \zeta_1^{-2ik}$	0
$\chi_6(k)$	$q-1$	-1	-1	0	$-\xi_1^{2ik} - \xi_1^{-2ik}$

$\zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1})$,
$\xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$.

One knows that $\mathrm{Out}(G)$ has order 2 and that G has an outer automorphism which changes the classes C_2 and C_3. Thus the group A according to step (ii) is given by the table automorphism $(\chi_3, \chi_4)(C_2, C_3)$. Instead of calculating $\mathrm{Aut}(\mathrm{CT}(G))$ and then M according to the steps (ii) and (iv) we consider the set

$$\tilde{M} = \{\chi \in \mathrm{Irr}(G); \exists \xi \in \mathrm{Irr}(G) \text{ with } \chi(1) = \xi(1) \text{ and } \xi \neq \chi\}.$$

Clearly $M \subseteq \tilde{M}$. From the table we see that $\tilde{M} = \{\chi_3, \chi_4, \chi_5(k), \chi_6(k)\}$.

The sizes of the centralizers, the class lengths and the ratios relevant to check step (v) of the algorithm are as follows. Note that g denotes in the table always a representative of the class.

class	$\lvert C_G(g)\rvert$	$\lvert Cl(g)\rvert$	$\chi_5(k)(g)\frac{\lvert Cl(g)\rvert}{\chi_5(k)(1)}$	$\chi_6(k)(g)\frac{\lvert Cl(g)\rvert}{\chi_6(k)(1)}$
1	$\frac{1}{2}q(q^2-1)$	1	1	1
2	q	$\frac{1}{2}(q^2-1)$	—	—
3	q	$\frac{1}{2}(q^2-1)$	—	—
4	$\frac{1}{4}(q-1)$	$2q(q+1)$	$2q(\zeta_1^{2ik} + \zeta_1^{-2ik})$	0
5	$\frac{1}{4}(q+1)$	$2q(q-1)$	0	$2q(-\xi_1^{2ik} - \xi_1^{-2ik})$

Looking at the centralizers one sees that for $p = q$ the classes 1, 4 and 5 are the p - regular classes. Using the criterion of step (v) we conclude that all characters of $\tilde{M}$ belong to the principal p - block. Note that we now have to assume that $f = 1$ and so $q = p$ because otherwise the principal p - block does not have cyclic defect.

χ_3 and χ_4 are exceptional but not the other ones. It follows that T consists of χ_3 and χ_4. Hence U coincides with A and (ZC 2) is established for $PSL(2,p)$.

For completeness we give also the generic character table in the case when $q \equiv 3 \bmod 4$.

	C_1	C_2	C_3	$C_4(i)$	$C_5(i)$
χ_1	1	1	1	1	1
χ_2	q	0	0	1	-1
χ_3	$\frac{1}{2}(q-1)$	$\frac{1}{2}(-1+\sqrt{q}\varepsilon)$	$-\frac{1}{2}(1+\sqrt{q}\varepsilon)$	0	$-(-1)^i$
χ_4	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(1+\sqrt{q}\varepsilon)$	$-\frac{1}{2}(1-\sqrt{q}\varepsilon)$	0	$-(-1)^i$
$\chi_5(k)$	$q+1$	1	1	$\zeta_1^{2ik}+\zeta_1^{-2ik}$	0
$\chi_6(k)$	$q-1$	-1	-1	0	$-\xi_1^{2ik}-\xi_1^{-2ik}$

$\varepsilon := \exp(\frac{2\pi\sqrt{-1}}{4})$,
$\zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1})$,
$\xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$.

Using this generic character table one gets analogously as in the first case that (ZC 2) is valid. Note for this that the orders of the centralizers and the class lengths are the same as in the case when $q \equiv 1 \bmod 4$.

4.9. Survey on (ZC 2) for simple groups.

The methods described above on cyclic blocks and decomposition numbers together with the action of normalized group ring automophisms of RG on tensor-products of Brauer characters the following results were obtained. Most of them are due to my student F. Bleher and proved in her thesis. It should be noted that no results on series of non-abelian simple groups were known before. The only result in that area known before was that (ZC 2) holds for the symmetric groups [37].

Computer algebra and its use should be regarded in this context under two aspects. Certainly it is true that the results on $\mathrm{PSL}(2,p)$, where p denotes a rational prime, have been discovered looking at the Atlas of finite groups and calculating examples for small primes p first. Clearly the Atlas is a product of computer algebra. The discovery of a general recipe looking at examples is therefore precisely in the sense of statement (b) of section 1. But in order to get more results the use of generic character tables or of other generic properties on the series of simple groups has to be used. This is again a topic of computer algebra. This finally demonstrates clearly the possibility to prove theorems with the aid of computers. One could criticize that generic methods give only results on infinitely many examples. However, if one takes into account, that classifications consist usually of such lists, then a complete statement about a class of mathematical objects may be made. For example the present knowledge with respect to (ZC 2) covers all minimal simple groups, see below.

According to the classification we consider simple or almost simple groups of **Lie Type** first.

4.10 Theorem [4, section 3] The Zassenhaus Conjecture (ZC 2) holds for the following groups of Lie type.

$$\mathrm{SL}(2,p^f), \mathrm{PSL}(2,p^f), {}^2B_2(2^{2m+1}), {}^2G_2(3^{2m+1}),$$
$${}^2F_4(2^{2m+1}), \mathrm{SL}(3,3^m), \mathrm{SU}(3,3^{2m}), \mathrm{Sp}(4,2^m), G_2(p^m), {}^3D_4(p^{3m}),$$

where p always denotes a rational prime.

As a consequence one obtains that (ZC 2) holds for any minimal simple group, for any simple group with abelian Sylow 2 – subgroups and for any simple Zassenhaus group.

Alternating and sporadic groups. With respect to the alternating groups A_n it is unknown whether normalized group ring automorphisms admit a Zassenhaus decomposition. It is proved for $n \le 10$ and A_{12}, cf. [4]. $\mathrm{Aut}_n(RG)$ does not act only on $\mathbb{C}G$ it acts also on $\mathbb{Q}G$. At the level of the rational group algebra $\mathbb{Q}G$ the following is true.

4.11. Theorem [23, Satz 5.9] For alternating groups A_n each $\sigma \in \mathrm{Aut}_n(\mathbb{Z}\mathrm{A}_n)$ may be modified by $\tau \in \mathrm{Aut}(\mathrm{A}_n)$ such that $\tau \cdot \sigma$ fixes each block of the Wedderburn decomposition of $\mathbb{Q}\mathrm{A}_n$. In other words each irreducible $\mathbb{Q}$ - character of A_n is fixed by $\tau \cdot \sigma$.

For the proof of this result it is shown that for $n \neq 6$ a bijection of the set of conjugacy classes of A_n which respects the length of the classes as well as the order of a representative fixes each conjugacy class of A_n which is invariant under conjugation in S_n. Note that a normalized automorphism of RG always induces such a bijection on the set of conjugacy classes of G. Since the conjugacy class of an even permutation π of a symmetric group S_n splits restricted to A_n if and only if the cycle type of π consists of odd cycles of pairwise different length, the following follows.

4.12. Corollary For A_n the variation $(\mathrm{ZC}\ )_p$ holds for each prime p.

Proof: For a given degree n and a given prime p there is at most one sum $\sum_{j=1}^k p^{i_j} = n$ with $0 \le i_1 < \ldots < i_k$. Thus there is at most one conjugacy class of a p-element in S_n which splits into two classes restricted to A_n. These two classes are linked by an outer automorphism τ of A_n. Consequently, if $\sigma \in \mathrm{Aut}_n(R\mathrm{A}_n)$ does not fix all conjugacy classes of p-elements, the composition of σ with τ fixes all these classes. Thus $(\mathrm{ZC}\ )_p$ is established. q.e.d.

With respect to the sporadic groups the following is known.

4.13. Theorem [7], [4], [3] (ZC 2) is valid for the sporadic groups

$$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, HS,$$

$$Co_3, Co_2, Co_1, HN, Th, J_1, J_2, Ru, B.$$

For the other sporadic groups $(ZC\ 2)_p$ is true for each prime p except possibly $p = 23$ for J_4.

For results on (ZC 2) with respect to soluble groups we refer to [43]. For surveys with respect to recent developments on the isomorphism problem, the Zassenhaus Conjectures and related topics see [41], [44], [46], [25], [49], [50] and [52].

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```
> for i from 1 to 8 do
> for j from 1 to 8 do
> res := mulperms ( els[i], els[j] );
> member ( res, els, 'k' );
> t[i,j] := k;
> od;
> od;
> print ( t );
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 6 & 1 & 4 & 8 & 3 & 7 \\ 3 & 7 & 1 & 6 & 8 & 4 & 2 & 5 \\ 4 & 1 & 7 & 5 & 2 & 3 & 8 & 6 \\ 5 & 4 & 8 & 2 & 1 & 7 & 6 & 3 \\ 6 & 3 & 2 & 8 & 7 & 1 & 5 & 4 \\ 7 & 8 & 4 & 3 & 6 & 5 & 1 & 2 \\ 8 & 6 & 5 & 7 & 3 & 2 & 4 & 1 \end{bmatrix}$$

Now we are ready to set up the eight equations for the coefficients x_g. We add the linear equation $\sum_{g \in G} x_g = 1$ which restricts the search to so called "normalized" units. The equations are represented as a set of polynomials, the solutions are just the common roots of these polynomials:

```
> gls := array ( 1..9);
```

$$gls := \mathrm{array}(1..9, [\,])$$

```
> for i from 2 to 8 do
> gls[i] := 0:
> od:
> gls[1] := -1:
> for i from 1 to 8 do
> for j from 1 to 8 do
> k := t[i,j];
> gls[k] := gls[k] + x[i]*x[j];
> od;
> od;
> gls[9] := sum(x[l],l=1..8)-1;
```

$$gls_9 := x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 - 1$$

```
> F := convert ( gls, set ):
```

$$F := \{2x_1x_8 + x_2x_6 + 2x_3x_5 + x_4x_7 + x_4x_6 + x_2x_7,\\ 2x_1x_6 + x_2x_3 + x_3x_4 + x_4x_8 + 2x_5x_7 + x_2x_8,\\ 2x_1x_5 + x_2^2 + 2x_3x_8 + x_4^2 + 2x_6x_7,\\ 2x_1x_7 + x_2x_8 + x_2x_3 + x_3x_4 + 2x_5x_6 + x_4x_8,\\ 2x_1x_3 + x_2x_7 + x_4x_6 + 2x_5x_8 + x_2x_6 + x_4x_7,\\ 2x_1x_2 + x_3x_7 + 2x_4x_5 + x_3x_6 + x_7x_8 + x_6x_8,\\ 2x_1x_4 + 2x_2x_5 + x_3x_6 + x_6x_8 + x_3x_7 + x_7x_8,\\ -1 + x_1^2 + 2x_2x_4 + x_3^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2,\\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 - 1\}$$

At this stage we are ready to load the **groebner** package:

```
> with ( grobner );
```

$$[\mathit{finduni}, \mathit{finite}, \mathit{gbasis}, \mathit{gsolve}, \mathit{leadmon}, \mathit{normalf}, \mathit{solvable}, \mathit{spoly}]$$

First of all we may check whether the system is solvable. Since the group contains involutions and the group embeds into the group ring this must be the case:

```
> solvable ( F );
```

$$\mathit{true}$$

The next command solves the system of polynomial equations with Gröbner basis methods over $\mathbb{Q}[x_{g_1}, \ldots, x_{g_n}]$ using lexicographic term ordering:

```
> solF := gsolve ( F );
```

$$\mathit{solF} := [[x_1 - 1, x_5, x_2, x_3, x_4, x_6, x_7, x_8],\\ [x_1, x_5 - 1, x_2, x_3, x_4, x_6, x_7, x_8],\\ [2x_1 - 1, 2x_5 - 1, x_2 + x_4, x_3, 4x_4^2 + 1, x_6, x_7, x_8],\\ [2x_1 + 1, 2x_5 - 1, 2x_2 - 1, x_3, 2x_4 - 1, x_6, x_7, x_8],\\ [2x_1 - 1, 2x_5 + 1, 2x_2 - 1, x_3, 2x_4 - 1, x_6, x_7, x_8],\\ [x_1, x_5, x_2 + x_4 - 1, x_3, 2x_4^2 - 2x_4 + 1, x_6, x_7, x_8],\\ [2x_1 - 1, 2x_5 - 1, x_2 + x_4, x_3 + x_8, 4x_4^2 - 4x_8^2 + 1, x_6, x_7],\\ [x_1, x_5, x_2 + x_4 - 1, x_3 + x_8, 2x_4^2 - 2x_4 - 2x_8^2 + 1, x_6, x_7],\\ [2x_1 - 1, 2x_5 - 1, x_2 + x_4, x_3 + x_8, 4x_4^2 - 4x_7^2 + 1 - 4x_8^2, x_6 + x_7],$$

$\left[x_1, x_5, x_2 + x_4 - 1, x_3 + x_8, 2\,x_4{}^2 - 2\,x_4 - 2\,x_7{}^2 + 1 - 2\,x_8{}^2, x_6 + x_7\right]$, $\Big[$
$4\,x_1 + 1, 4\,x_5 - 3, 4\,x_2 - 1, 4\,x_3 - 1, 4\,x_4 - 1, 4\,x_6 + 1, 4\,x_7 + 1,$
$4\,x_8 - 1\Big]$, $\Big[4\,x_1 - 3, 4\,x_5 + 1, 4\,x_2 - 1, 4\,x_3 - 1, 4\,x_4 - 1, 4\,x_6 + 1,$
$4\,x_7 + 1, 4\,x_8 - 1\Big]$,
$\Big[4\,x_1 - 1, 4\,x_5 - 1, 2\,x_2 + 2\,x_4 - 1, 4\,x_3 - 1,$
$16\,x_4{}^2 - 8\,x_4 + 5, 4\,x_6 + 1, 4\,x_7 + 1, 4\,x_8 - 1\Big]$,
$\Big[4\,x_1 - 1, 4\,x_5 - 1,$
$2\,x_2 + 2\,x_4 - 1, 2\,x_3 + 2\,x_8 - 1, 16\,x_4{}^2 - 8\,x_4 - 32\,x_8{}^2 + 3 + 16\,x_8,$
$2\,x_6 - 2\,x_8 + 1, x_8 + x_7\Big]$, $\Big[4\,x_1 + 1, 4\,x_5 - 3, 4\,x_2 - 1, 4\,x_3 + 1,$
$4\,x_4 - 1, 4\,x_6 - 1, 4\,x_7 - 1, 4\,x_8 + 1\Big]$, $\Big[4\,x_1 - 3, 4\,x_5 + 1, 4\,x_2 - 1,$
$4\,x_3 + 1, 4\,x_4 - 1, 4\,x_6 - 1, 4\,x_7 - 1, 4\,x_8 + 1\Big]$, $\Big[4\,x_1 - 1, 4\,x_5 - 1,$
$2\,x_2 + 2\,x_4 - 1, 4\,x_3 + 1, 16\,x_4{}^2 - 8\,x_4 + 5, 4\,x_6 - 1, 4\,x_7 - 1, 4\,x_8 + 1\Big]$
, $\Big[4\,x_1 - 1, 4\,x_5 - 1, 2\,x_2 + 2\,x_4 - 1, 2\,x_3 + 2\,x_8 + 1,$
$16\,x_4{}^2 - 8\,x_4 - 32\,x_8{}^2 + 3 - 16\,x_8, 2\,x_6 - 2\,x_8 - 1, x_8 + x_7\Big]$, $\Big[4\,x_1 - 1,$
$4\,x_5 - 1, 2\,x_2 + 2\,x_4 - 1, 2\,x_3 + 2\,x_8 - 1,$
$16\,x_4{}^2 - 8\,x_4 - 16\,x_7{}^2 + 3 - 16\,x_8{}^2 - 8\,x_7 + 8\,x_8, 2\,x_6 + 2\,x_7 + 1\Big]$, $\Big[$
$4\,x_1 - 1, 4\,x_5 - 1, 2\,x_2 + 2\,x_4 - 1, 2\,x_3 + 2\,x_8 + 1,$
$16\,x_4{}^2 - 8\,x_4 - 16\,x_7{}^2 + 3 - 16\,x_8{}^2 + 8\,x_7 - 8\,x_8, 2\,x_6 + 2\,x_7 - 1\Big]$, $\Big[$
$4\,x_1 - 3, 4\,x_5 + 1, 4\,x_2 + 1, 4\,x_3 - 1, 4\,x_4 + 1, 4\,x_6 - 1, 4\,x_7 - 1,$
$4\,x_8 - 1\Big]$, $\Big[4\,x_1 + 1, 4\,x_5 - 3, 4\,x_2 + 1, 4\,x_3 - 1, 4\,x_4 + 1, 4\,x_6 - 1,$
$4\,x_7 - 1, 4\,x_8 - 1\Big]$, $\Big[4\,x_1 - 1, 4\,x_5 - 1, 2\,x_2 + 2\,x_4 + 1, 4\,x_3 - 1,$
$16\,x_4{}^2 + 8\,x_4 + 5, 4\,x_6 - 1, 4\,x_7 - 1, 4\,x_8 - 1\Big]$, $\Big[$
$4\,x_1 - 1, 4\,x_5 + 3, 4\,x_2 - 1, 4\,x_3 - 1, 4\,x_4 - 1, 4\,x_6 - 1, 4\,x_7 - 1, 4\,x_8 - 1\Big]$,
$\Big[4\,x_1 + 3, 4\,x_5 - 1, 4\,x_2 - 1, 4\,x_3 - 1, 4\,x_4 - 1, 4\,x_6 - 1, 4\,x_7 - 1, 4\,x_8 - 1\Big]$,

$$\Big[4\,x_1 + 1, 4\,x_5 + 1, 2\,x_2 + 2\,x_4 - 1, 4\,x_3 - 1, 16\,x_4{}^2 - 8\,x_4 + 5,$$
$$4\,x_6 - 1, 4\,x_7 - 1, 4\,x_8 - 1\Big],$$
$$\Big[4\,x_1 - 1, 4\,x_5 - 1, 2\,x_2 + 2\,x_4 + 1,$$
$$2\,x_3 + 2\,x_8 - 1, 4\,x_4{}^2 + 2\,x_4 - 4\,x_8{}^2 + 2\,x_8 + 1, 4\,x_6 - 1, 4\,x_7 - 1\Big],$$
$$\Big[4\,x_1 + 1, 4\,x_5 + 1, 2\,x_2 + 2\,x_4 - 1, 2\,x_3 + 2\,x_8 - 1,$$
$$4\,x_4{}^2 - 2\,x_4 - 4\,x_8{}^2 + 2\,x_8 + 1, 4\,x_6 - 1, 4\,x_7 - 1\Big], \Big[4\,x_1 - 1, 4\,x_5 - 1,$$
$$2\,x_2 + 2\,x_4 + 1, 2\,x_3 + 2\,x_8 - 1,$$
$$16\,x_4{}^2 + 8\,x_4 - 16\,x_7{}^2 + 3 + 8\,x_8 - 16\,x_8{}^2 + 8\,x_7, 2\,x_6 + 2\,x_7 - 1\Big], \Big[$$
$$4\,x_1 + 1, 4\,x_5 + 1, 2\,x_2 + 2\,x_4 - 1, 2\,x_3 + 2\,x_8 - 1,$$
$$16\,x_4{}^2 - 8\,x_4 - 16\,x_7{}^2 + 3 + 8\,x_8 - 16\,x_8{}^2 + 8\,x_7, 2\,x_6 + 2\,x_7 - 1\Big],$$
$$[2\,x_1 + 1, 2\,x_5 - 1, x_2, 2\,x_3 - 1, x_4, x_6, x_7, -1 + 2\,x_8],$$
$$[2\,x_1 - 1, 2\,x_5 + 1, x_2, 2\,x_3 - 1, x_4, x_6, x_7, -1 + 2\,x_8],$$
$$\Big[x_1, x_5, x_2 + x_4, 2\,x_3 - 1, 4\,x_4{}^2 + 1, x_6, x_7, -1 + 2\,x_8\Big], \Big[x_1, x_5, x_2 + x_4,$$
$$x_3 + x_8 - 1, 4\,x_4{}^2 - 8\,x_8{}^2 + 8\,x_8 - 1, 2\,x_6 - 2\,x_8 + 1, 2\,x_7 + 2\,x_8 - 1\Big],$$
$$[2\,x_1 + 1, 2\,x_5 - 1, x_2, x_3, x_4, 2\,x_6 - 1, 2\,x_7 - 1, x_8],$$
$$[2\,x_1 - 1, 2\,x_5 + 1, x_2, x_3, x_4, 2\,x_6 - 1, 2\,x_7 - 1, x_8],$$
$$\Big[x_1, x_5, x_2 + x_4, x_3, 4\,x_4{}^2 + 1, 2\,x_6 - 1, 2\,x_7 - 1, x_8\Big], \Big[$$
$$x_1, x_5, x_2 + x_4, x_3 + x_8, 4\,x_4{}^2 - 8\,x_8{}^2 + 1, 2\,x_6 - 2\,x_8 - 1, 2\,x_7 + 2\,x_8 - 1$$
$$\Big], \Big[x_1, x_5, x_2 + x_4, x_3 + x_8 - 1, x_4{}^2 - x_7{}^2 + x_8 - x_8{}^2, x_6 + x_7\Big],$$
$$\Big[x_1, x_5, x_2 + x_4, x_3 + x_8, x_4{}^2 - x_7{}^2 - x_8{}^2 + x_7, x_6 + x_7 - 1\Big]\Big]$$

For each sublist of the previous list one obtains solutions by computing common roots of the polynomials in the list. Note that the polynomials in those lists are much simpler than the original polynomials and have pairwise as few indeterminates as possible in common. But not all sublists lead to integer solutions as can be seen rather quickly by reducing modulo 2. Therefore we reduce the set by eliminating those lists that contain 1's after reducing modulo 2:

```
> for  sol in solF do
> if not ( member ( 1, sol mod 2 ) ) then
```

```
> intsolF := intsolF union {sol};
> fi;
> od;
> intsolF;
```

$$\Big\{[x_1 - 1, x_5, x_2, x_3, x_4, x_6, x_7, x_8], [x_1, x_5 - 1, x_2, x_3, x_4, x_6, x_7, x_8],$$
$$[x_1, x_5, x_2 + x_4, x_3 + x_8, {x_4}^2 - {x_7}^2 - {x_8}^2 + x_7, x_6 + x_7 - 1],$$
$$[x_1, x_5, x_2 + x_4, x_3 + x_8 - 1, {x_4}^2 - {x_7}^2 + x_8 - {x_8}^2, x_6 + x_7]\Big\}$$

All these remaining lists lead indeed to integer solutions. The first and second list yield two trivial solutions: the identity and the central involution of G. The remaining trivial solutions (exactly one of $X - 3, x_6, x_7, x_8 = 1$ and all other $x_i = 0$) can be derived from the third and fourth list. But the last two lists also describe infinite series of solutions, e.g.

$$(x_1, \ldots, x_8) = (0, \mp n, -n, \pm n, 0, 1, 0, n) \quad \forall n \quad \text{or}$$
$$(x_1, \ldots, x_8) = (0, \mp n, 0, \pm n, 0, -n, n, 1) \quad \forall n.$$

Essentially the solutions given by those last two lists depend only on the quadratic polynomials ${x_4}^2 - {x_7}^2 - {x_8}^2 + x_7$ and ${x_4}^2 - {x_7}^2 + x_8 - {x_8}^2$ respectively, since the remaining conditions are linear and can always be fulfilled once a solution for the quadratic polynomials is known. These quadratic equations can be viewed as hyperboloids of two sheets in three dimensional space. Then integer solutions are just integral points on these surfaces.

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THE LaTeX 2ε CLASS FOR THE "ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII OVIDIUS CONSTANŢA"

Abstract

The LaTeX class file `ovidius.cls` is designed to include the information needed to electronically submit a manuscript for publication in the An. Şt. Univ. Ovidius, Constanţa.

1 Introduction

This is the documentation for a LaTeX class file which can be used when submitting papers electronically to the An. Şt. Univ. Ovidius, Constanţa. This version of the file, called `ovidius.cls` is designed for use with LaTeX 2ε, the new replacement for all older versions of LaTeX, and is compatible with AMS-LaTeX 1.2. If you are haveing a problem with `ovidius.cls`, we recommend you upgrade your LaTeX to LaTeX 2ε.

This file can be viewed as both the documentation for this class and an example of how to use it. All the features of the class mentioned in this document are also used here, so you can look at them in this file to see how they are implemented.

However, an article written in an older version of LaTeX is also acceptable with a few requirements (see Section 5).

Key Words: TeX, LaTeX, style
Mathematical Reviews subject classification: 26A21, 28B16

*Thank you all!

†I'm very grateful!

2 Top Matter

For the most part, the top matter for **ovidius.cls** is the same as the top matter in the ordinary **article.cls**. However, there are a few changes to meet the requirements of the An. Şt. Univ. Ovidius, Constanţa. The following items should be included.

\author{} This macro should contain the names and addresses of the authors of the work. Each author's information should be listed in the form *author, address, (e-mail address)*. Different authors should have their information separated by \and.

\title{} This is the title of the work in all capital letters.

\keywords{} These are one or more key words which describe the important concepts of your paper.

\MathReviews{} This is for the *Mathematical Reviews* classification number(s) of your work. It will often include a primary and secondary classification.

\markboth{}{} This is the usual LaTeX command for marking the headers of facing pages. The left header should contain the name(s) of the author(s) and the right header should contain the title of the work. It is often the case that the author(s) names or the title won't fit on a single line. In this case, you must use shortened versions.

Any macros defined for your paper should be contained in the top matter, and all environment definitions such as \newtheorem or \newenvironment should be in the top matter. In particular, don't \input your macros as a separate file within the final version because more files just make it harder for the editors to keep track of your paper.

3 References, cross and otherwise

The usual LaTeX labels for cross-referencing should be used for theorem-structures, display lines, figures and bibliography entries. This means that referenced display lines and theorem-structures should have a \label{}, which is used in all references to that object. Every bibliographic citation should be referenced with a \cite{} command as documented in [4].

Theorem 3.1 *Theorems, corollaries, lemmas, etc. should be numbered consecutively by section in the same series.*

In the case of an one-section paper, the section number should be omitted from the theorem number.

Corollary 3.2 *Definitions and examples often do not fit well into this scheme, so the numbering in these cases is left up to the author.*

Proposition 3.3 *If you are in doubt about the best way to proceed, the safest course is to contact the editors.*

4 Miscellaneous Suggestions

A general rule is *do not force whitespace into your manuscripts.* By this we mean that you shouldn't include forced page breaks, vertical space commands or forced line breaks. The reason for this is that the margins and page breaks in the final version to appear in the AN. ŞT. UNIV. OVIDIUS, CONSTANŢA will inevitably be different than those on your printout, so your whitespace or breaks will inevitably be in the wrong place.

Define commands for the most common symbols and constructions and place them in the top matter. Use them! This will assure consistent notation throughout, and makes it easier to change notation, if necessary.

If you use BIBTEX to generate your reference list (which we recommend), do it with the `plain.bst` or `AMSplain.bst` style files. Use the same journal abbreviations as *Mathematical Reviews*. For the final version of your paper, insert the contents of the `bbl` file created by BIBTEX into the source for your manuscript in between the commands `\begin{thebibliography}{}` and `\end{thebibliography}`.

5 For those who still use LATEX209

If you have an older version of LATEX, please use the following preamble:

```
\documentstyle[twoside]{article}
\setlength{\textheight}{6.75in}
\advance\textheight by \topskip
\setlength{\textwidth}{4.675in}
...........................
...........................
other items such \newcommand or \newenvironment
....................
....................
\author{} % Here include also the address
\title{} % Write the title in capital letters
```

```
\date{} % no date
\begin{document}
\maketitle
 ......................
............................
\end{document}
```

6 How to Submit Electronically

Electronic submissions can be done by electronic mail or on disk to the addresses given in the most recent issue of the AN. ŞT. UNIV. OVIDIUS, CONSTANŢA.

A manuscript submitted by electronic mail should be one mail message beginning with a cover letter (not necessarily in TEX form) and ending with the actual manuscript. To make sure your mail doesn't get mangled by unfriendly computers in transit, leave no lines in your message longer than 80 characters, or protect yourself with uuencoding or BinHex.

Disk submissions can be made on any standard MS-DOS disk. The disk should be labeled with the name of the paper, the authors and the format (MS-DOS). Please, also include all local packages you have used.

References

[1] *AMS-LATEX Version 1.1 User's Guide*. American Mathematical Society, 1992.

[2] Guidelines for Preparing Electronic Manuscripts: AMS-LATEX, American Mathematical Society, 1992.

[3] Michael Goossens, Frank Mittelbach, Alexander Samarin,The LATEX Companion, Addison-Wesley, 1993.

[4] Leslie Lamport, *LATEX: A Document Preparation System*, Addison-Wesley, 2nd ed., 1994.