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LARGE CONSTRUCTION FOR NONSOLVABLE LIE ALGEBRAS

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To Professor Dan Pascali, at his 70's anniversary

Abstract

In this paper we give an explicit construction of nonsolvable Lie algebras, over fields of characteristic zero, in which the ideals are n-element chain. We present one method whose depend on the radical of the Lie algebra .

1. Introduction

We will consider finite dimensional Lie algebras over a field K of characteristic zero. Given such an algebra $\underline{g}, Rad(\underline{g})$ (respectively $Nil(\underline{g})$) denotes the largest solvable (respectively nilpotent) ideal of $\underline{g}$. We denote the terms of the lower central series of $\underline{g}$ by $\underline{g}$=$\underline{g}^1$ and $\underline{g}^i = [\underline{g},\underline{g}^{i-1}]$ for $i > 1$. The center of $\underline{g}$ is denoted by $Z(\underline{g})$. We define the upper central series of $\underline{g}$ by letting $Z_0(\underline{g})= 0$ and $Z_i(\underline{g})$ be the ideal of $\underline{g}$ such that $Z(\underline{g}/Z_{i-1}(\underline{g})) = Z_i(\underline{g})/Z_{i-1}(\underline{g})$ for $i \geq 1$.

Algebra direct sums are denoted by $\oplus$ where as direct sums of vector space structures are denoted by $\dot{+}$.

The problem of determining the Lie algebras in which the ideals are totally ordered by set inclusion was first posed in [Be], where a complete classification was given of the supersolvable algebras in this class; in particular, the solvable Lie algebras whose ideals are in chain are completely classified when the base

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field is algebraically closed. In the case of nonsolvable algebras of this type, it was also obtained the following basic structure result:

Theorem 1.1. [Be]. *Let $\underline{g}$ be a nonsolvable Lie algebra over a field of characteristic zero. Then, the ideals of $\underline{g}$ are in chain if and only if $\underline{g}$ is a simple algebra or a semidirect sum of a nonzero nilpotent ideal $\mathcal{N}$ and a simple algebra $\mathcal{S}$ such that $\mathcal{N}/\mathcal{N}^2$ is a faithful $\mathcal{S}$-module and $Z_i(\mathcal{N})/Z_{i-1}(\mathcal{N})$ are irreducible $\mathcal{S}$-modules via the adjoint representation.*

Moreover, in this case, the terms of the lower central series of $\mathcal{N}$ coincide with the terms of the upper central series and, if n is the nilpotency index of $\mathcal{N}$, the ideals of $\underline{g}$ are in the following $(n+1)$-element chain $0 < \mathcal{N}^{n-1} < .... < \mathcal{N}^i < ... < \mathcal{N} < 1$.

The easier constructions -apart from simple algebras - of nonsolvable Lie algebras in which the ideals are in chain arise from Theorem 1.1. in a very simple way.

Corollary 1.2. *Each nonsolvable Lie algebra over a field of characteristic zero, whose ideals are in a 3-element chain, is a semi-direct sum of an abelian ideal $\mathcal{N}$ of dimension at least 2 and a simple subalgebra $\mathcal{S}$ such that $\mathcal{N}$ is an irreducible $\mathcal{S}$-module.* □

The task of this paper is to give an explicit construction of nonsolvable Lie algebras in which the ideals are 4-element in chain.

2. The construction from the radical

The construction that we will give in this section depends on Lie algebras in which the derived algebra is one-dimensional and equal to the center. Before giving the construction, we need some results about this type of algebras.

Lemma 2.1 *Let $\underline{g}$ be a Lie algebra such that $\dim \underline{g}^2 = 1$, $\underline{g}^2 = Z(\underline{g})$ and u be a nonzero element of $\underline{g}^2$. Then:*

i) For every complementary subspace h of $\underline{g}^2$ in $\underline{g}$, the bilinear form f_h defined in h by $f_h(x,y) = \lambda_{[x,y]}$ where $[x,y] = \lambda_{[x,y]}\, u$ is skew -symmetric and nondegenerate. In particular, the dimension of h is even and there exists a basis $\{e_1, e_2, ..., e_n,\ l_1, l_2, ..., l_n\}$ of h for which $[e_i, l_i] = u$ and all other products are zero.

ii) $\underline{g}$ has a basis $r_1, r_2, ..., r_{2n+1}$ with nonzero products $[r_i, r_{n+i}] = r_{2n+1}$, $1 \le i \le n$.

iii) Suppose $\mathcal{B} = \{r_i\}_{1 \le i \le 2n+1}$ is a basis of $\underline{g}$ as in ii) and $\delta : \underline{g} \to \underline{g}$ is a linear transformation such that the matrix of δ with respect to of $\mathcal{B}$ is of the *form*

$$\begin{pmatrix} & & 0 \\ & M & \vdots \\ \alpha_1 & .... & \alpha_{2n}\alpha \end{pmatrix},$$

where M *is a* $2n \times 2n$ *matrix. Then,* $\delta \in Der(\underline{g})$ *if and only if* $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, *where each* m_j *is an* $n \times n$ *matrix;* ${}^t m_2 = m_2$, ${}^t m_3 = m_3$ *and* ${}^t m_1 + m_4 = \alpha I_n$.

Proof. The assertion in i) is straightforward (see [Ja]) and the part ii) follows from i). To prove iii), consider the vector space

$$h = Kr_1 + Kr_2 + + ... + Kr_{2n}$$

and let f_h be the skew bilinear form defined in *i)* when $u = r_{2n+1}$. We have that $\delta([r_i, r_j]) = \alpha f_h(r_i, r_j) r_{2n+1}$ and the matrix $N = (f_h(r_i, r_j))$is $N = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Clearly, $\delta \in Der(\underline{g})$ if $\delta([r_i, r_j]) = [\delta(r_i), r_j] + [r_i, \delta(r_j)]$ for $i, j = 1, ..., 2n$. If we denote $M = (\gamma_{ij})$, these conditions are that

$$\alpha f_h(r_i, r_j) = \sum_{1 \le k \le 2n} f_h(r_i, r_k)\gamma_{kj} + \sum_{1 \le k \le 2n} \gamma_{ki} f_h(r_k, r_j)$$

or in matrix form

$$\alpha N = NM +^t MN. \tag{2.1}$$

If we partion M in the same way as N, $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ where each m_i is an $n \times n$ matrix, a simple computation shows that (2.1) holds if and only if ${}^t m_2 = m_2$, ${}^t m_3 = m_3$ and ${}^t m_1 + m_4 = \alpha I_n$. □

In the sequel, for each $n \ge 1$, we shall denote by $\underline{g}(n)$ the Lie algebra with basis $\{r_1, r_2, ..., r_{2n+1}\}$ and nonzero products as it is described in *ii)* of Lemma 2.1. Let us consider the symplectic Lie algebra $sp(2p, K)$, which by definition consists of all matrices $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ where each m_i is an $n \times n$ matrix, ${}^t m_2 = m_2$, ${}^t m_3 = m_3$ and ${}^t m_1 + m_4 = 0$ (see [Hu]). This algebra is simple split of type C_n and $(2n^2+n)$ dimensional-for $n = 1$,it coincides with the split simple 3-dimensional, algebra $sl(2, K)$.By means of the homomorphism $M \to M^\rho$ of $sp(2n, K)$ into $gl(\underline{g}(n))$, where M^ρ denotes the linear transformation in $\underline{g}(n)$

with matrix (relatively to the basis $\{r_1, r_2, ..., r_{2n+1}\})$:

$$\begin{pmatrix} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ 0 & ... & 0 & 0 \end{pmatrix}.$$

The algebra $\underline{g}(n)$ can be considered as an $sp(2n,K)$-module. Now let us denote by $\mathcal{L}(n)$ the direct sum of the two vector spaces $\underline{g}(n)$ and $sp(2n,K)$. We introduce in $\mathcal{L}(n)$ a multiplication $[a,b]$ by means of the formula

$$[x+M, y+N] = [x,y] + M^\rho(y) - N^\rho(x) + [M,N].$$

We have the following result:

Theorem 2.2. *Let* $\underline{g}(n)$, $sp(2n,K)$ *and* $\mathcal{L}(n) = \underline{g}(n) \dot{+} sp(n,K)$ *be as in the preceding paragraph. Then:*

i) $\mathcal{L}(n)$ *is a Lie algebra whose ideals are the following 4-element chain:*

$$0 < \underline{g}(n)^2 < \underline{g}(n) < \mathcal{L}(n).$$

ii) Suppose $\underline{p}$ *is a nonsolvable Lie algebra such that* $Rad(\underline{p}) \cong \underline{g}(n)$. *If the ideals of* $\underline{p}$ *are in chain, there exists a monomorphism embedding* $\underline{p}$ *in* $\mathcal{L}(n)$. *In particular the semisimple Levi factors of* $\underline{p}$ *are isomorphic to a simple subalgebra of* $sp(2n,K)$.

Proof. *i)* From Lemma 2.1., it is immediate that $M^\rho \in Der(\underline{g}(n))$, for each $M \in sp(2n,K)$ and therefore $\underline{g}(n)$ is a Lie algebra. Now denote by h the subspace spanned by $r_1, ..., r_{2n}$. If $n=1$, it is clear that h is an $ad_{\underline{g}(n)}sp(2n,K)$-irreducible module. Suppose then $n \geq 2$ and consider f_h the bilinear form in h described in Lemma 2.1, *(i)* for $u = r_{2n+1}$. The Lie algebra of the linear transformations f' in h which are skew with respect to f_h, that is $f_h(f'(x),y) = -f_h(x, f'(y))$, coincides with the set $\{M^\rho|_h : M \in sp(2n,K)\}$.
Then, from [Ja], we get that h is $ad_{\underline{g}(n)}sp(2n,K)$-irreducible and therefore $\underline{g}(n) = h \dot{+} Kr_{2n+1}$ is a decomposition of $\underline{g}(n)$ into $sp(2n,K)$-irreducible modules via the adjoint representation. Now the result follows from Theorem 1.1.

ii) Since the ideals of $\underline{p}$ are in chain, Theorem 1.1 implies that $\underline{p} = Rad(\underline{p}) \dot{+} \mathcal{S}$ where $\mathcal{S}$ is simple and $Rad(\underline{p})/Rad^2(\underline{p})$, $Rad(\underline{p})^2$ are irreducible $ad_{\underline{p}}$ $\mathcal{S}$-modules.

Write $Rad(\underline{p})^2 = Ku$ and note that $[Rad(\underline{p})^2, \mathcal{S}] = 0$. Moreover, there exists an $\mathcal{S}$-irreducible module h such that $Rad(\underline{p}) = h \dot{+} Ku$.

From Lemma 2.1, *i)* we can take $\{e_1, e_2, ..., e_n,\ l_1, l, ..., l_n\}$ as a basis of h such that $[e_i, l_i] = u$ and all other products are zero. For each $s \in \mathcal{S}$, the linear transformation $ad_{\underline{p}}s : x \to [s, x]$ belongs to $Der(Rad(\underline{p}))$ thus applying *iii)* of Lemma 2.1, we have that the matrix of $ad_{\underline{p}}s$ with respect to the basis $\{e_1, ..., e_n, l_1, ..., l_n, u\}$ is of the form:

$$\begin{pmatrix} & & & 0 \\ & P_s & & \vdots \\ & & & 0 \\ 0 & ... & 0 & 0 \end{pmatrix},$$

where $P_s = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$ and each p_i is an $n \times n$ - matrix with ${}^t p_2 = p_2,\ {}^t p_3 = p_3$ and ${}^t p_1 + p_4 = 0$. Then, the mapping $\psi : \underline{p} \to \mathcal{L}(n)$ defined by:

$$\psi(\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n + \alpha_{n+1} l_1 + ... + \alpha_{2n} l_{2n} + \alpha_{2n+1} u + s) =$$

$$= \alpha_1 r_1 + ... + \alpha_{2n+1} r_{2n+1} + P_s$$

is a monomorphism, which proves the result. □

Corollary 2.3. *Up to isomorphism, the unique nonsolvable Lie algebra whose radical is isomorphic to* $\underline{g}(1)$ *and such that the ideals are in chain is* $\underline{g} = Kr_1 + Kr_2 + Kr_3 + Ka + Kb + Kc$, *with nonzero products* $[r_1, r_2] = r_3$; $[a, r_2] = r_1$; $[b, r_1] = r_2$; $[c, r_1] = r_1$; $[c, r_2] = -r_2$; $[a, b] = c$; $[c, a] = 2a$ *and* $[c, b] = -2b$. □

REFERENCES

[Be] Benito, M.P., *Lie algebras in which the lattice formed by the ideals is a chain*, Comm. Alg. 20 (1992), 93-108.

[Di] Dixmier, J., *Enveloping Algebras*, Akademie-Verlag, Berlin, 1977.

[FH] Fulton, W. and Harris, J., *Representation Theory*, Springer-Verlag, 1991.

[HS] Hilton, P.J. and Stammbach, U., *A Course in Homological Algebra*, Springer-Verlag, 1971.

[Hu] Humphreys, J.E., *Introduction to Lie Algebras and Representation Theory*, Springer -Verlag, 1072.

[Ja] Jacobson, N., *Lectures in Abstract Algebras*, Springer-Verlag, 1975.

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