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ERRORS ESTIMATION AND THE ASYMPTOTIC DISTRIBUTION OF PROBABILISTIC ESTIMATES

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To Professor Silviu Sburlan, at his 60' anniversary

Abstract

This paper continues the work done in [3]. We present, besides the dynamical demand problem, the estimation of self-correlated errors and the asymptotic distribution of maximal probabilistic estimates in the self - recessive errors model.

I. The dynamical demand model

The mathematical model can be stated as follows

$$y_t = \alpha_o + \alpha_1 x_t + \alpha_2 s_t + u_t, \tag{1}$$

where y_t is the market demand for the commodity; x_t is the relative price variable (the quotient of the price of the commodity and the price of consumption); u_t is the error term, s_t is the individual stock of this commodity. The variable s_t is viewed as "psyological stock" of the commodity owned by the consummer; it grows directly with the consumption, but its importance is time-decreasing. We consider the equation

$$s_t - s_{t-1} = \beta_o s_{t-1} + \beta_1 y_t, \tag{2}$$

where $\beta_o s_{t-1}$ is the dissipation component, $\beta_o < 0$, β_1 is sometimes unitary. Equation (2) expresses the unobservable quantity s_t in terms of the observable function y_t. Equation (1) must be expressed in terms of the observable quantities (without the error term). We replace (1) in (2) and get:

$$\Rightarrow s_t = \frac{\alpha_o^*}{1-\beta} + \frac{\alpha_1^* I}{I - \beta L} x_t + \frac{\alpha_2^* I}{I - \beta L} u_t \tag{3}$$

with $\alpha_0^* = \alpha_0\beta_1/(1-\alpha_2\beta_1)$, $\alpha_1^* = \alpha_1\beta_1(1-\alpha_2\beta_1)$, $\alpha_2^* = \beta_1/(1-\alpha_2\beta_1)$. Then, puting (3) in (1), we get:

$$y_t = \frac{\alpha_o(1-\beta)+\alpha_2\alpha_o^*}{1-\beta} + \frac{(\alpha_1+\alpha_1^*\alpha_2)I-\alpha_1\beta L}{I-\beta L}x_t + \frac{(1+\alpha_2\alpha_2^*)I-\beta L}{I-\beta L}u_t. \quad (4)$$

Now, by applying the inverse operator $I-\beta L$, we reduce (4) to

$$y_t = -\frac{\alpha_o\beta_o}{1-\beta_1\alpha_2} + \frac{\alpha_1}{1-\beta_1\alpha_2}x_t - \alpha_1\beta x_{t-1} + \beta y_{t-1} + \frac{1}{1-\beta_1\alpha_2}u_t - \beta u_{t-1}. \quad (5)$$

The parameters α_2 and β_1 appear in the form $\beta_1\alpha_2$ and they cannot be separately identified. We take $\beta_1 = 1$ and $\alpha_2 \neq 1$.

II. The estimation of the self-correlated errors of the model

We consider the autoregressive scheme:

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad (6)$$

with the expectation of errors and their covariance taken as

$$E(\varepsilon_t) = 0,\ Cov(\varepsilon_{t_1}\varepsilon_{t'}) = \delta_{tt}\cdot\sigma^2,\ \ \forall\, t,t' \text{ and } |\rho| < 1.$$

Let us take a sample of size T on the above model

$$y_t = \sum_{i=0}^{k}\beta_i x_{t_i} + u_t,\ t = 1,2,...,T, \quad (7)$$

where $x_{t_i}, i = \overline{1,n}$ are independent variables on the error term u_t.

We admit that $x_{t_0} = 1$. Then we have

$$u_t = (I-\rho L)^{-1}\varepsilon_t = \sum_{i=0}^{\infty}\rho^i\varepsilon_{t-i} \quad (8)$$

and $E(u_t) = 0$, $cov(u_t, u_{t+\tau}) = \sigma^2\rho^{-\tau}/(1-\rho^2)$.
All the above considerations prove the following result:

Lemma. *Let* $y_t = \sum_{i=0}^{k}\beta_i x_{t_i} + u_t$, $t = \overline{1,T}$, *be a sample of dimension* T *of the above presented model. If* $\bar{u} = (u_1, u_2, ..., u_T)$, *then* $E(\bar{u}) = 0$ *and*

$$cov(\bar{u}) = \frac{\sigma^2}{1-\rho^2}\begin{bmatrix} 1 \ \rho \ ... \ \rho^{T-1} \\ \rho \ 1 \ ... \ \rho^{T-2} \\ ------- \\ \rho^{T-1} \ \rho^{T-2} \ ... \ 1 \end{bmatrix} = \sigma^2 V$$

and we can decompose it as $V^{-1} \equiv M'M$,where

$$M = \begin{bmatrix} \sqrt{1-\rho} & 0 & 0 & ... & 0 & 0 \\ -\rho & 1 & 0 & ... & 0 & 0 \\ 0 & -\rho & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & -\rho & 1 \end{bmatrix}.$$

For ρ and β we obtain the estimation using the least-squares formulation method and $\lim_{T\to\infty}(1-\rho^2)^{\frac{1}{T}} = 1,\ \rho \in (-1,1)$, by minimizing

$$\frac{\hat{\sigma}^2(\hat{\rho})}{(1-\hat{\rho}^2)^{\frac{1}{T}}}. \tag{9}$$

Thus, to globally maximizing the likelihood function is equivalent to globally minimizing (9). From an asymptotic view point, the two above procedures are equivalent, $\forall \rho \in (-1,1)$ and $\lim_{T\to\infty}(1-\rho^2)^{\frac{1}{T}} = 1$.

III. The asymptotic distribution of maximal probabilistic estimations in the self-recessive errors model

Now, we come back at the estimations from Section II and we study asymptotic distribution.

We firstly introduce the notation $\gamma = \left(\sigma^2\rho\beta\right)'$, and we observe that the estimation satisfies the equality $\frac{\partial L}{\partial \gamma} = 0$.
We extend the probabilistic function concerning the relation upon the vector $\bar{\gamma}_o$ as follows

$$\frac{\partial L}{\partial \gamma_o}(\gamma_o) = -\frac{\partial^2 L}{\partial\gamma\partial\gamma_o}(\gamma_o)(\bar{\gamma}-\gamma_o) + \text{ the order 3 terms.} \tag{10}$$

Now, we shall "drop out" the above "order 3 terms", because, in this context, they go to zero. $(\partial L \partial\overline{\gamma})\,(\bar{\gamma}_o)$ is the gradient of the probability function.

We can write $\dfrac{\partial L}{\partial \rho}$ or $\partial^2\partial\overline{\gamma}\partial\overline{\gamma}_o$, (where $\bar{\gamma}_o$ is implicitly understood) and we then observe that:

$$\frac{\partial L}{\partial \sigma^2} = -\frac{1}{2}\frac{T}{\sigma^2} + \frac{1}{2\sigma^2}u'V^{-1}u, \frac{\partial L}{\partial \rho} = -\frac{\rho}{1-\rho^2} =$$

$$= \frac{1}{\sigma^2}\left[\sum_{i=2}^{T}(u_i - \rho u_{i-1})\,u_{i-1} + \rho u_1^2\right], \ \frac{\partial L}{\partial \beta} = \frac{1}{\sigma^2}X'V^{-1}u.$$

By transforming these relations, we have $\frac{\partial L}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\varepsilon'\varepsilon$,and $\frac{\partial L}{\partial \beta} = \frac{1}{\sigma^2}X'M'\varepsilon$,because $Mu = \varepsilon \sim N(0, \sigma^2 I)$.
We get

$$u_t = \sum_{\tau=0}^{\infty}\rho^\tau \varepsilon_{t-T} = \sum_{\tau=0}^{N-2}\rho^\tau \varepsilon_{t-\tau} + \rho^{N-1}\sum_{\tau=0}^{\infty}\rho^\tau \varepsilon_{t-N+1-\tau} = u^{N\ t}\rho^{N-1}u_{t-N+1}.$$

As u_{t-N+1} has the estimation equal to zero and the covariation equal to $\dfrac{\sigma^2}{1-\rho^2}$, it results that $\rho^{N-1}u_{t-N+1}$ has a very small probability for an enough big N. Using the Chebyshev inequality for $\delta > 0$, we then obtain
$P\left\{\left|\rho^{N_1}u_{t-N+1}\right| > \delta\right\} < \frac{Var\left(\rho^{N-1}u_{t-N+1}\right)}{\delta^2} = \frac{\rho^{2(N-1)}d\sigma t\sigma^2}{(1-\rho^2)\delta^2}$, which is very small.We choose an appropriate N and computing

$$\frac{\partial L}{\partial \rho} = \rho\left[\frac{u_1^2}{\sigma^2} - \frac{1}{1-\rho^2}\right] + \sum_{i=1}^{T}\varepsilon_i u_{i-1}^N + \rho^{N-1}\sum_{i=2}^{T}\varepsilon_i u_{i-N},$$

and $\qquad \frac{\partial L}{\partial \gamma} = \sum_{t=1}^{T} w_t + \left[\rho^{N-1}\sum_{i=1}^{T}\varepsilon_i u_{i-N+1}\right],$

where

$$w_1 = \frac{1}{\sigma^2}\begin{bmatrix} \frac{1}{2}\left\{\left(\frac{\varepsilon_1}{\sigma}\right)^2 - 1\right\} \\ \rho\sigma^2\left(\frac{u_1^2}{\sigma^2} - \frac{1}{1-\rho^2}\right) \\ z_1\varepsilon_1\ , \end{bmatrix}, w_t = \frac{1}{\sigma^2}\begin{bmatrix} \frac{1}{2}\left(\frac{\varepsilon_t}{\sigma}\right)^2 - 1 \\ \varepsilon_t u_{t-1}^N \\ z_t\varepsilon_t \end{bmatrix}, t = 2, ..., T.$$

are entries of the t-th column from $X'M'$.
The asymptotic distribution of $\dfrac{\partial L}{\partial \gamma}$ is given by $\sum_{t=1}^{T} w_t$. The vectors w_t are N-dependents.

Because we are interested in the asymptotic distribution $\frac{1}{T^{\frac{1}{2}}}\sum_{t=1}^{T} w_t$,we can neglect w_t and we get $\frac{1}{T^{\frac{1}{2}}}\sum_{t=1}^{T} w_t = \frac{w_1}{T^{1/2}} + \frac{1}{T^{1/2}}\sum_{t=2}^{T} w_t$ and $p\lim_{T\to\infty}\frac{w_1}{T^{1/2}} = 0$, since $E\left(w_1/T^{1/2}\right) = 0$.

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