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MINIMAL POINTS IN PRODUCT SPACES

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Abstract

Some technical extensions of the minimal point statements due to Goepfert, Tammer and Zălinescu [7] are given. The basic tool for such a device is a lot of abstract ordering principles obtained under the lines in Turinici [14].

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1. Introduction

Let (X, d) be a *complete metric space*; and Y, some (real) separated *locally convex space*. By a *convex cone* in Y we mean, as usually, any part L of Y with

(1D1) $L + L \subseteq L$; $\lambda L \subseteq L$, for all $\lambda > 0$; $0 \in L$.

In this case, the relation $\leq (\mod L)$ on Y defined as

(1D2) $y_1 \leq y_2 \ (\mod L)$ if and only if $y_2 - y_1 \in L$

is reflexive and transitive; hence a *quasi-order*. Moreover, it is *compatible* with the linear structure of Y, in the sense

(1.1) $$\begin{cases} y_1 \leq y_2 \ (\mod L), \quad y \in Y, \ \lambda \geq 0 \Longrightarrow \\ y_1 + y \leq y_2 + y \ (\mod L), \ \lambda y_1 \leq \lambda y_2 \ (\mod L). \end{cases}$$

Assume further that K is a convex cone in Y and pick some k^0 in K. We introduce a quasi–order $(\preceq) = (\preceq_K^{k^0})$ over $X \times Y$ by the convention

(1D3) $(x_1, y_1) \preceq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \leq y_2 - y_1 (\mod K)$.

Finally, take some nonempty part A of $X \times Y$. For a number of both practical and theoretical reasons, it would be useful to determine sufficient conditions under which the quasi–ordered structure $(A, \preceq)$ should have points with certain Zorn type *minimality* properties. A basic result in this direction obtained by Goepfert, Tammer and Zălinescu [7, Theorem 1], deals with convex cones K taken according to

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(1H1) $K \setminus (-\mathrm{cl}(K))$ is nonempty [where "cl" is the closure operator];

and with elements $k^0 \in K \setminus (-\mathrm{cl}(K))$. The crucial assumption used by the quoted authors may be written as

(1H2) $\begin{cases} \text{if } ((x_n, y_n)) \subseteq A \text{ is } \preceq\text{–descending and } x_n \to x \text{ then } x \in P_X(A) \\ \text{and there exists } y \in A(x) \text{ such that } (x, y) \preceq (x_n, y_n), \text{ for all } n. \end{cases}$

[Here, for each $(x, y) \in A$, $A(x)$ (resp., $A(y)$) stands for the *x–section* (resp., *y–section*) of (the relation) A; and P_X, P_Y are the *projection* operators from $X \times Y$ to X and Y respectively]. And the specific one is

(1H3) $P_Y(A)$ is bounded below ($\mathrm{mod}\ \mathrm{cl}(K)$) [$\exists \widetilde{y} \in Y : P_Y(A) \subseteq \widetilde{y} + \mathrm{cl}(K)$].

The announced result may now be stated as follows

THEOREM 1.1 *Suppose that* (1H2) *and* (1H3) *are in force. Then, for each* $(x_0, y_0) \in A$ *there exists* $(\bar{x}, \bar{y}) \in A$ *in such a way that*

(1.2) $(\bar{x}, \bar{y}) \preceq (x_0, y_0)$; *and, moreover,*

(1.3) *if* $(x', y') \in A$ *fulfils* $(x', y') \preceq (\bar{x}, \bar{y})$ *then* $x' = \bar{x}$.

[As a matter of fact, the original formulation of (1H3) is with K in place of $\mathrm{cl}(K)$. But, a simple inspection shows that the argument developed there also works in this relaxed setting].

This result extends a related statement in this area due to Loridan [10]; and, as such, it includes the (classical by now) Ekeland's variational principle [6]. So, a technical development of its basic lines would be not without profit. In this direction, we note that Theorem 1.1 may be equally viewed as a *maximality* statement, with respect to the *dual* quasi–order $(\succeq) = (\succeq_K^{k^0})$ (on $X \times Y$):

(1D4) $(x_1, y_1) \succeq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \leq y_1 - y_2 (\mod K)$.

Moreover, denote again by d the *semi–metric* (i.e.: non–sufficient metric) over $X \times Y$:

(1D5) $d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2), \quad (x_1, y_1), (x_2, y_2) \in X \times Y.$

The last conclusion of the result above may be then written as

(1.4) $(\bar{x}, \bar{y}) \succeq (x', y') \Longrightarrow d((\bar{x}, \bar{y}), (x', y')) = 0.$

This suggests us a possible deduction of Theorem 1.1 from a related ordering principle in Turinici [12]. (We refer to Section 2 for its exact formulation). It is our aim in the following to show (in Section 3) that this approach is effective: Theorem 1.1 is a particular case of the quoted statement. The reduction method to be used allows us giving in Section 5 a technical enlargement of this result involving *archimedean* cones and *gauge* functions (cf. Section 4). Finally, the possibility of deriving *genuine* Zorn minimality principles from such results is analyzed in Section 6. The obtained statements are comparable with the contributions in this area due to Goepfert, Tammer and Zălinescu [8, Theorem 1]. Some other aspects will be discussed elsewhere.

2. Abstract ordering principles

Let M be a nonempty set; and $\leq$ be a *quasi-order* over it. Further, let ρ be some *semi-metric* over M. For an easy reference, we shall write the working hypothesis to be used:

$$\text{(2H1)} \quad \begin{cases} (\rho, \leq) \text{ is normal [each } (\leq)\text{-ascending sequence in } M \text{ is a} \\ \rho\text{-Cauchy one, bounded from above].} \end{cases}$$

The following ordering principle established in Turinici [12] is our starting point.

THEOREM 2.1. *Suppose that* (2H1) *holds. Then, for each* $a_0 \in M$ *there exists* $\bar{a} \in M$ *with*

(2.1) $a_0 \leq \bar{a}$; *and, moreover,*

(2.2) *if* $a' \in M$ *fulfils* $\bar{a} \leq a'$, *then* $\rho(\bar{a}, a') = 0$.

Note that, if the structure $(M, \leq, \rho)$ fulfils the extra assumption

(2H2) $a_1, a_2 \in M,\ a_1 \leq a_2,\ \rho(a_1, a_2) = 0 \Longrightarrow a_2 \leq a_1$,

the point $\bar{a}$ described by (2.2) is a *maximal* one (in the usual sense); and Theorem 2.1 becomes a variant of the well known Zorn maximality principle (cf. Bourbaki [2]). But, in the following, this will be *not accepted*. For a number of related aspects we refer to Altman [1].

A useful version of this result may be given under the lines below. Let $\varphi : M \to \bar{R} = R \cup \{-\infty, +\infty\}$ be a function. The basic hypothesis to be considered about this object is

(2H3) φ is $\leq$–decreasing ($a_1 \leq a_2 \Longrightarrow \varphi(a_1) \geq \varphi(a_2)$).

THEOREM 2.2. *Suppose that* (2H1) *and* (2H3) *hold. Then, for each* $a_0 \in M$ *there exists* $\bar{a} \in M$ *fulfiling* (2.1), *as well as*

(2.3) $a' \in M,\ \bar{a} \leq a' \Longrightarrow \rho(\bar{a}, a') = 0,\quad \varphi(\bar{a}) = \varphi(a')$.

Proof. Without any loss, one may assume that (in addition to (2H3))

(2H4) φ is bounded in R ($-\infty < \inf \varphi(M) \leq \sup \varphi(M) < +\infty$).

For, otherwise, let χ be an *order isomorphism* between $\bar{R}$ and some *bounded* interval of R; such as, e.g.,

(2D1) $\chi(t) = \text{arctg}(t),\ t \in R;\quad \chi(-\infty) = -\pi/2,\ \chi(+\infty) = \pi/2$.

The composed function (from M to R)

(2D2) $\varphi_1(x) = \chi(\varphi(x))$, $x \in M$ (in short: $\varphi_1 = \chi \circ \varphi$)

fulfils (2H3) and (2H4). And, if the conclusion of Theorem 2.2 holds for φ_1 it will be also retainable for φ. Define another semi–metric $\sigma = \sigma_\varphi$ over M by the convention

(2D3) $\sigma(x, y) = \max\{\rho(x, y), |\varphi(x) - \varphi(y)|\}$, $x, y \in M$.

Let (a_n) be some $(\leq)$–ascending sequence in M. By (2H1), (a_n) is a ρ–Cauchy sequence, bounded from above. On the other hand, (2H3)+(2H4) tell us that $(\varphi(a_n))$ is a descending and bounded sequence in R; hence a Cauchy one. Summing up, (a_n) is a σ–Cauchy sequence (bounded from above, as already said); and from this,

(2.4) $(\sigma, \leq)$ is normal (i.e., (2H1) holds).

The conclusion to be derived is now a consequence of Theorem 2.1 applied to the structure $(M, \leq, \sigma)$. ■

This result extends the one due to Brezis and Browder [3]. As far as we know, the idea of handling general (unbounded) functions goes back to Carja and Ursescu [4]. In general, Theorem 2.2 cannot be reduced to the Zorn maximality principle, unless our data are taken so as

(2H5) $a_1 \leq a_2,\ \rho(a_1, a_2) = 0,\ \varphi(a_1) = \varphi(a_2) \Longrightarrow a_2 \leq a_1$.

But, in what follows, conditions of this type are *not accepted*. So,we may ask of which is the relevance of this result in getting the quoted principle. As we shall see, a positive answer is available with respect to a certain *order* (i.e.: antisymmetric quasi–order) on M induced by our data. Precisely, denote by $\prec$ the relation (over M):

(2D4) $a_1 \prec a_2$ iff $a_1 \leq a_2$ and $\varphi(a_1) > \varphi(a_2)$.

(Note that, the alternative of $\prec$ having an empty graph in M^2 cannot be avoided, in general). The following facts are almost evident. (So, we omit the details).

LEMMA 2.1 *The introduced relation is a strict order; i.e.,*

(2.5) $a \not\prec a$, *for each* $a \in M$ *(irreflexive)*

(2.6) $a_1 \prec a_2,\ a_2 \prec a_3 \Longrightarrow a_1 \prec a_3$ *(transitive).*

As a consequence, the relation $(\preceq)$ over M, defined as

(2D5) $a_1 \preceq a_2$ iff either $a_1 \prec a_2$ or $a_1 = a_2$

is an *order* on M, which in addition is *coarser* than $(\leq)$:

(2.7) $a_1, a_2 \in M$, $a_1 \preceq a_2 \Longrightarrow a_1 \leq a_2$

and fulfils the sufficiency property

(2.8) $a_1, a_2 \in M,\ a_1 \preceq a_2,\ \varphi(a_1) = \varphi(a_2) \Longrightarrow a_1 = a_2$.

The usefulness of this construction is to be judged from

THEOREM 2.3. *Let the conditions* (2H1)+(2H3) *be in force. Then, for each* $a_0 \in M$*, there exists* $\bar{a} \in M$ *with*

(2.9) $a_0 \preceq \bar{a}$*;* *and, moreover,*

(2.10) *if* $a' \in M$ *fulfils* $\bar{a} \preceq a'$ *then* $\bar{a} = a'$.

(In other words: $(\preceq)$ *is a Zorn ordering over* M*).*

Proof. We show that, in the precised setting, $(\rho, \preceq)$ is normal [i.e.: (2H1) holds, with $(\preceq)$ in place of $(\leq)$]. In fact, let (a_n) be an $(\preceq)$–ascending sequence in M. By (2.7), this sequence is $(\leq)$–ascending; so, from (2H1), (a_n) is ρ–Cauchy and bounded from above [modulo $(\leq)$]:

(2.11) $\exists\, a \in M: \ a_n \leq a_m \leq a,$ provided $n \leq m$.

This, along with (2H3), tells us that the (extended real) sequence $(\varphi(a_n))$ is descending and bounded from below:

(2.12) $\varphi(a_n) \geq \varphi(a_m) \geq \varphi(a),$ whenever $n \leq m$.

If $(\varphi(a_n))$ is constant then, by (2.8), so is (a_n); and the conclusion is clear. Otherwise, we have relations like

(2.13) for each n there exists $m > n$ with $\varphi(a_n) > \varphi(a_m)$ (hence $a_n \prec a_m$).

But then (cf. Lemma 2.1 above)

(2.14) $\varphi(a_n) > \varphi(a)$ (hence $a_n \prec a$), for each n;

and the conclusion is again clear. On the other hand, (2.7) tells us that (2H3) holds [modulo $(\preceq)$]. Summing up, Theorem 2.2 is applicable to $((M, \preceq, \rho); \varphi)$. Hence, for each $a_0 \in M$ there exists $\bar{a} \in M$ fulfiling the properties (2.1)+(2.3) [with $(\preceq)$ in place of $(\leq)$]. And this, along with (2.8), yields the conclusion we need. ■

Remark. The core of our argument is the implication

(2.15) $(\rho, \leq)$ is normal $\Longrightarrow$ $(\rho, \preceq)$ is normal.

A natural question is of whether or not is this reversible. The answer is negative, in general. For, let the couple $((M, \leq, \rho); \varphi)$ be such that

(2H6) $(\rho, \leq)$ is not normal (i.e., (2H1) fails) and φ =constant.

The strict quasi–order $(\prec)$ attached to these data has an empty graph; so

(2.16) $a_1 \preceq a_2$ if and only if $a_1 = a_2$.

In other words, $(\rho, \preceq)$ is normal; but [cf. (2H6)] $(\rho, \leq)$ is not. Hence the claim.

3. Proof of Theorem 1.1 via Theorem 2.1

Let the working conditions of Theorem 1.1 be admitted. Without loss, one may assume that (1H3) is to be written as

(3H1) $P_Y(A) \subseteq \mathrm{cl}(K)$ [i.e.: $\widetilde{y} = 0$ in that condition].

For, otherwise, passing to the subset $\widetilde{A}$ of $X \times Y$ defined as

(3D1) $(x, y) \in \widetilde{A}$ if and only if $(x, \widetilde{y} + y) \in A$,

the requirement (3H1) is fulfilled, as well as (1H2). And, if the conclusion of Theorem 1.1 is retainable for $\widetilde{A}$, it will remain as such for the initial subset A. The following auxiliary fact will be useful for us.

LEMMA 3.1. *Let $((x_n, y_n))$ be a sequence in A which is $(\succeq)$–ascending [that is, $(\preceq)$- descending]:*

(3H2) $k^0 d(x_n, x_m) \le y_n - y_m \ (\bmod K)$, *if* $n \le m$.

Then, (x_n) is a d-Cauchy sequence in $P_X(A)$.

Proof. Suppose that this would be not true. Then, there exists an $\varepsilon > 0$ in such a way that

(3.1) for each n, there exists $m > n$ with $d(x_n, x_m) \ge \varepsilon$.

Inductively, one may construct a subsequence $(u_n = x_{p(n)})$ of (x_n) such that

(3.2) $d(u_n, u_{n+1}) \ge \varepsilon$, for all $n \ge 1$.

This in turn yields, for the corresponding subsequence $(v_n = y_{p(n)})$ of (y_n), an evaluation like

(3.3) $k^0 \varepsilon \le k^0 d(u_n, u_{n+1}) \le v_n - v_{n+1} \ (\bmod K)$, $n = 1, 2, \ldots$.

But then [cf. (3H1)], one derives a relation like

(3.4) $k^0 q\varepsilon \le v_1 - v_{q+1} \le v_1 (\bmod \mathrm{cl}(K))$ [hence $k^0 - \frac{1}{q\varepsilon} v_1 \in -\mathrm{cl}(K)$], $q \ge 1$.

Passing to limit as $q \to \infty$, one gets $k^0 \in -\mathrm{cl}(K)$, contradiction. Consequently, (x_n) is d–Cauchy, as claimed. ■

Proof of Theorem 1.1. Let $((x_n, y_n))$ be a $(\succeq)$–ascending (that is, $(\preceq)$–descending) sequence in A. By Lemma 3.1, (x_n) is a d–Cauchy sequence in $P_X(A)$; hence, by completeness,

(3.5) $x_n \to x$ as $n \to \infty$, for some $x \in X$.

This, along with (1H2), assures us that $x \in P_X(A)$ and there exists an element $y \in A(x)$ such that

(3.6) $(x, y) \preceq (x_n, y_n)$ [i.e.: $(x_n, y_n) \succeq (x, y)$], for all n.

Summing up, $(d, \succeq)$ is normal over A (in the sense of (2H1)). And then, from Theorem 2.1, we are done. ■

4. Conical gauge functions

Let Y be a (real) *vector space*; and L be some *convex cone* in Y. By a convention in Cristescu [5,ch.5,Sect.1], we say that L is *archimedean*, provided

(4D1) $k, y \in Y$ and $[\lambda k \leq y(\mod L)$, for all $\lambda \geq 0] \Longrightarrow k \in -L$.

Assume in the following that

(4H1) L is an archimedean cone; and also,
(4H2) $L \setminus (-L) \neq \emptyset$ (i.e.: L is not a linear subspace of Y).

Fix a certain $k^0 \in L \setminus (-L)$. Define the function (from Y to $\bar{R}$) as: for each $y \in Y$,

(4D2) $\gamma(y) = \sup \Gamma(y)$, where $\Gamma(y) = \{s \in R;\ k^0 s \leq y(\mod L)\}$.

(As usually, $\sup(\emptyset) = -\infty$). This will be referred to as the *gauge function* attached to the (convex) cone L and the (nonzero) element k^0 (of L). It is our aim in what follows to study a few basic properties of this function. (Their usefulness will become clear in the next sections).

(A) We start our developments by showing that

(4.1) $+\infty \notin \gamma(Y)$ (hence $\gamma(Y) \subseteq R \cup \{-\infty\}$).

To verify this note that, for each $y \in Y$, the real subset $\Gamma(y)$ fulfils a hereditary property like
(4.2) $s \in \Gamma(y),\ s' < s \Longrightarrow s' \in \Gamma(y)$.
Now, assume by contradiction that

(4H3) $\gamma(y_0) = +\infty$ (hence $\Gamma(y_0) = R$), for some $y_0 \in Y$.

By the remark above, one has evaluations like

$$k^0 s \leq y_0(\mod L), \quad \text{for all } s \in R.$$

This, along with (4H1), yields $k^0 \in -L$; in contradiction to (4H2); hence the claim. Note that, the alternative property

(AP) $-\infty \in \gamma(Y)$ [i.e.: $\gamma(y_1) = -\infty$, for some $y_1 \in Y$]

cannot be avoided. So, we may ask of what can be said about the finite values of these functions. For a partial answer, note that
(4.3) $-\infty < \gamma(y) < +\infty$, for all $y \in k^0 R + L$.
Hence, in particular, one gets the useful fact
(4.4) $0 \leq \gamma(y) < +\infty$, for all $y \in L$.
The global counterpart of it is to be given under the extra requirement

(4H4) $\mathrm{aint}(L) \neq \emptyset$ (where "aint"=the *algebraic interior*).

Note that (4H2) implies a regularity condition like

(4H5) $0 \in Y$ is not an element of $\mathrm{aint}(L)$.

Conversely, this last requirement [and (4H4)] yields (4H2); because, in such a case, $\mathrm{aint}(L) \subseteq L \setminus (-L)$. [The last assertion follows at once from the (set) relations

(4.5) $L + \mathrm{aint}(L) \subseteq \mathrm{aint}(L)$ (hence $L + \mathrm{aint}(L) = \mathrm{aint}(L)$);

we do not give details]. Now, assume that k^0 is taken according to $k^0 \in \mathrm{aint}(L)$ (hence $k^0 \in L \setminus (-L)$). We claim that, necessarily,

(4.6) $-\infty \notin \gamma(Y)$ (hence $\gamma(Y) \subseteq R$).

In fact, let $y \in Y$ be arbitrary fixed. By the choice of k^0, there must be some $\varepsilon = \varepsilon(y) > 0$ in such a way that $k^0 + \lambda y \in L$, for each $\lambda \in [-\varepsilon, \varepsilon]$. In particular, when $\lambda = \varepsilon$, this gives

$y \in -\frac{1}{\varepsilon}k^0 + L$ (wherefrom $\gamma(y) \geq -\frac{1}{\varepsilon}$); and the assertion is proved.

(B) Return to the general setting of (4H2). It is easy to see that the multifunction $y \vdash \Gamma(y)$ (from Y to R) has the k^0–translation property

(4.7) $\Gamma(y + k^0 t) = \Gamma(y) + t$, for all $(y, t) \in Y \times R$.

This yields a k^0– translation property for its associated gauge function γ:

(4.8) $\gamma(y + k^0 t) = \gamma(y) + t$, for all $(y, t) \in y \times R$.

In addition, by the very definition of this object, one has (via (4H2))

(4.9) $\gamma(k^0 t) = t$, $\forall t \in R$ (hence, in particular, $\gamma(0) = 0$);

So, (combining with a previous conclusion) γ is a *proper* function from Y to $R \cup \{-\infty\}$.

(C) A useful property relating the couple (Γ, γ) is

(4.10) $\gamma(y) \in \Gamma(y)$, whenever $\gamma(y) > -\infty$.

Indeed, by the hereditary property (4.2) above, one has

$k^0\gamma(y) - y \leq k^0 t(\mod L)$, for all $t > 0$; wherefrom
$s(k^0\gamma(y) - y) \leq k^0(\mod L)$, for all $s \geq 0$.

This, along with (4H1), establishes the assertion.

(D) We close these developments with the monotonicity properties of the gauge function γ. For example, one has

(4.11) $y_1 \leq y_2(\mod L) \Longrightarrow \gamma(y_1) \leq \gamma(y_2)$.

[The verification is immediate, by definition; we do not give details]. Further aspects may be delineated under the regularity condition (4H4). Precisely, $\mathrm{aint}(L)$ is a convex cone without origin; i.e., (1D1) holds without its last part. As a consequence, the object

(4D3) $\mathrm{Aint}(L) = \{0\} \cup \mathrm{aint}(L)$

is a convex cone of Y, with the extra property (cf. (4.5))

(4.12) $\mathrm{Aint}(L) \cap (-\mathrm{Aint}(L)) = \{0\}$ (*pointedness*).

Let $<(\mod \operatorname{aint}(L))$ stand for the relation

(4D4) $y_1 < y_2(\mod \operatorname{aint}(L))$ if and only if $y_2 - y_1 \in \operatorname{aint}(L)$.

This is a *strict order* (on Y) in the sense described by Lemma 2.1. Moreover, it is *compatible* with the linear structure of Y, in the sense

$$(4.13)\quad \begin{cases} y_1 < y_2(\mod \operatorname{aint}(L)), \quad y \in Y,\ \lambda > 0 \Longrightarrow \\ y_1 + y < y_2 + y(\mod \operatorname{aint}(L)), \quad \lambda y_1 < \lambda y_2(\mod \operatorname{aint}(L)). \end{cases}$$

Likewise, $\leq (\mod \operatorname{Aint}(L))$ is an order on Y, compatible with its linear structure (cf. (1.1)). In fact, it is nothing but the object attached to $<(\mod \operatorname{aint}(L))$ under the model of (2D5); namely

(4.14) $y_1 \leq y_2(\mod \operatorname{Aint}(L))$ iff either $y_1 < y_2(\mod \operatorname{aint}(L))$ or $y_1 = y_2$.

The following statement is now available.

LEMMA 4.1. *Let the precised conditions be in use. Then, γ is strictly $<(\mod \operatorname{aint}(L))$-increasing on $k^0R + L$:*

(4.15) $y_1, y_2 \in k^0R + L,\ y_1 < y_2(\mod \operatorname{aint}(L)) \Longrightarrow \gamma(y_1) < \gamma(y_2)$.

Proof. By the very definition of the algebraic interior,

$y_2 - y_1 \in k^0\varepsilon + L$, for some $\varepsilon > 0$ (small enough).

On the other hand, $\gamma(y_1) > -\infty$ (cf. (4.3)), yields (via (4.10))

$y_1 \in k^0\gamma(y_1) + L$; so, by simply adding to the above
$y_2 \in k^0(\varepsilon + \gamma(y_1)) + L$; hence $\gamma(y_2) \geq \varepsilon + \gamma(y_1) > \gamma(y_1)$.

The proof is thereby complete. ■

Remark The finiteness condition involved in (4.15) cannot be removed. Indeed, let $y_2 \in Y$ be such that $\gamma(y_2) = -\infty$. If $y_1 \in Y$ fulfils $y_1 < y_2(\mod \operatorname{aint}(L))$ then, by (4.11), $\gamma(y_1) \leq \gamma(y_2)$; hence $\gamma(y_1) = -\infty$. This proves our claim.

Finally, by taking (4.4) into account, it follows that the restriction of γ to L is strictly $<(\mod \operatorname{aint}(L))$–increasing:

(4.16) $y_1, y_2 \in L,\ y_1 < y_2(\mod \operatorname{aint}(L)) \Longrightarrow \gamma(y_1) < \gamma(y_2)$.

Some related facts may be found in Goepfert, Tammer and Zălinescu [7, Section 3].

5. Main results

The informations offered by the conclusion of Theorem 1.1 are, in a certain sense, *incomplete*. For, the assertion (1.3) of this conclusion deals only with the relationships between the points $\bar{x}, x'$ of the couples $(\bar{x}, \bar{y}), (x', y')$. It is therefore natural getting the "dual" relationships between the points $\bar{y}, y'$ of these couples. To do this, we need some conventions and auxiliary facts. Let Y be a (real) vector space. The notion of *archimedean* (convex) *cone* was already introduced in Section 4. Note that

(5.1) $\begin{cases} \text{the intersection of any (nonempty) family of} \\ \text{archimedean cones is an archimedean cone.} \end{cases}$

So, for each (nonempty) part M of Y,
(5D1) $\mathrm{arch}(M) = \cap \{L; M \subseteq L = \text{archimedean cone}\}$
is an archimedean cone including M, and minimal with these properties; we shall term it, the *archimedean closure* of M. Let (X, d) be a complete metric space; and $\{K, H\}$, a pair of convex cones in Y with
(5H1) $K \subseteq H = \text{archimedean cone}$.
[For example, a good candidate for H is (cf. the above) $H = \mathrm{arch}(K)$. Moreover, if Y is taken as in Section 1, then another candidate is $H = \mathrm{cl}(K)$; because any closed (convex) cone is archimedean]. Pick some $k^0 \in K$ and introduce the quasi–order $(\preceq) = (\preceq_K^{k^0})$ on $X \times Y$ by (1D3). Finally, take some nonempty part A of $X \times Y$. As in Section 1, we are interested to get sufficient conditions upon our data under which the quasi–ordered structure $(A, \preceq)$ should have points with certain Zorn type minimality properties. A basic answer to this problem is available for convex cones K taken according to
(5H2) $K \setminus (-H)$ is nonempty [hence $K \neq \{0\}$];
and for elements $k^0 \in K \setminus (-H)$. [Note that, if Y is taken as in Section 1, then (cf. a previous remark) (1H1) is a particular case of this condition]. The basic working hypothesis of these developments is again (1H2). And, the specific assumption to be used is formulated in terms of

(5D2) $\gamma =$ the gauge functions attached to H and k^0.

Precisely, this may be written as
(5H3) $\gamma(P_Y(A))$ is a subset of R, bounded from below (in R).
The announced result may now be stated as

THEOREM 5.1. *Let the conditions* (1H2) *and* (5H3) *be in use. Then, for each* $(x_0, y_0) \in A$, *there exists* $(\bar{x}, \bar{y}) \in A$ *such that*
(5.2) $(\bar{x}, \bar{y}) \preceq (x_0, y_0)$; *and moreover*
(5.3) *if* $(x', y') \in A$ *fulfils* $(x', y') \preceq (\bar{x}, \bar{y})$ *then* $x' = \bar{x}$, $\gamma(y') = \gamma(\bar{y})$.

Proof. Let $((x_n, y_n))$ be a $(\succeq)$–ascending (that is, $(\preceq)$–descending) sequence in A; i.e., (3H2) is being accepted. (Here, $(\succeq)$ is the *dual* of $(\preceq)$). By the choice (5H1) of H, one gets

(5.4) $k^0 d(x_n, x_m) \le y_n - y_m (\mod H)$, whenever $n \le m$.

This, along with the finiteness, k^0–translation and monotonicity properties of γ (cf. Section 4), yields

(5.5) $d(x_n, x_m) \le \gamma(y_n) - \gamma(y_m)$, if $n \le m$.

The (real) sequence $(\gamma(y_n))$ is descending and (by (5H3)) bounded from below (in R); hence, a Cauchy sequence. This, added to (5.5), shows that (x_n) is d–Cauchy; and, as such, $x_n \to x$, for some $x \in X$. Combining with (1H2) yields $x \in P_X(A)$ and there exists an element $y \in A(x)$ with the property (3.6). In other words, $(d, \succeq)$ is normal over A (in the sense of (2H1)). Further, let the function $\Phi : X \times Y \to \bar{R}$ be introduced as

(5D3) $\Phi(x, y) = \gamma(y)$, $(x, y) \in X \times Y$ (i.e., $\Phi = \gamma \circ P_Y$).

Again by the monotonicity of γ, it follows that Φ is $\preceq$–increasing (or, equivalently, $\succeq$–decreasing) over A. Summing up, Theorem 2.2 is applicable to the couple $((A, \succeq, d); \Phi)$. This firstly proves (5.2) (via (2.1)); and, secondly, (5.3) follows from (2.3). Hence the conclusion. ■

Concerning the relationships with Theorem 1.1, it would be useful getting concrete situations (comparable with (1H3) above) under which the regularity condition (5H3) be fulfilled. The basic one may be written as

(5H4) $P_Y(A)$ is bounded below ($\mod H$) [$\exists \widetilde{y} \in Y : P_Y(A) \subseteq \widetilde{y} + H$].

The following particular version of Theorem 5.1 is then available.

THEOREM 5.2. *Assume that* (1H2) *and* (5H4) *hold. Then, conclusions of Theorem 5.1 are necessarily retainable.*

Proof. By the same way as in Section 3, it is no loss in generality if (5H4) would be written as

(5H5) $P_Y(A) \subseteq H$ [i.e.: $\widetilde{y} = 0$ in that condition].

But then, the (finite) positivity of γ over H (cf. Section 4) shows that (5H3) must be true. In other words, Theorem 5.1 applies to these data and this ends the argument. ■

Now, as conclusion (5.3) above includes also relationships between the points $\bar{y}, y'$ of the couples $(\bar{x}, \bar{y})$, (x', y'), it is clear that Theorem 5.2 (hence, a fortiori, Theorem 5.1) appears as a *strict* extension of Theorem 1.1. But, even if this were ignored, the logical inclusion between these results is retainable; because, if Y is taken as in Section 1, the choice $H = \mathrm{cl}(K)$ is allowed in (5H2). For a number of related aspects we refer to Isac [9] and Nemeth [11].

6. Zorn minimal points

The results we just derived are *not genuine* Zorn minimality principles (as in Bourbaki [2]); because the quasi–orders appearing there are *not anti-symmetric* in general. So, it is natural to ask of whether or not is this removable. As we shall see below, such a device is possible, under the model of Theorem 2.3. Further aspects occasionated by these developments are also discussed.

Let the structures $\{X, Y\}$ be taken as in Section 5; and $\{K, H\}$, be a pair of (convex) cones in Y, fulfiling (5H1)+(5H2). Further, pick some $k^0 \in K\setminus(-H)$; and construct the quasi–order $(\preceq) = (\preceq_K^{k^0})$ as in (1D3). Given the (nonempty) part A of $X \times Y$, we may ask of which are the conditions upon our data so that *coarser* than $(\preceq)$ orders over A be available with the *standard* minimal Zorn property. For an appropriate answer, assume that (1H2) holds, as well as (5H3); where, as precised in that place, γ is the *gauge* function attached to H and k^0. Let also $\Phi : X \times Y \to \bar{R}$ stand for the function introduced as in (5D3). The relation $(\sqsubset) = (\sqsubset_K^{k^0})$ over $X \times Y$ defined as

(6D1) $(x_1, y_{)} \sqsubset (x_2, y_2)$ iff $(x_1, y_1) \preceq (x_2, y_2)$ and $\Phi(x_1, y_1) < \Phi(x_2, y_2)$

is a *strict order* (cf. Lemma 2.1). Let $\sqsubseteq$ stand for its associated *order* (on $X \times Y$)

(6D2) $(x_1, y_2) \sqsubseteq (x_2, y_2)$ if either $(x_1, y_1) \sqsubset (x_2, y_2)$ or $(x_1, y_1) = (x_2, y_2)$.

For the moment, $\sqsubseteq$ is *coarser* than $\preceq$ (over A), in the sense

(6.1) $(x_1, y_1), (x_2, y_2) \in A,\ (x_1, y_1) \sqsubseteq (x_2, y_2) \Longrightarrow (x_1, y_1) \preceq (x_2, y_2)$.

Concerning the converse inclusion, the following statement is true.

LEMMA 6.1. *Assume that*

(6H1) $y_1, y_2 \in P_Y(A),\ y_1 \leq y_2(\mod K),\ y_1 \neq y_2 \Longrightarrow \gamma(y_1) < \gamma(y_2)$.

Then, $\preceq$ is coarser than $\sqsubseteq$ over A; so, these relations are identical (over A).

Proof. Let $(x_1, y_1), (x_2, y_2)$ be a couple of points in A with $(x_1, y_1) \preceq (x_2, y_2)$. We thus have (in particular) $y_1, y_2 \in P_Y(A)$ and $y_1 \leq y_2(\mod K)$. If $y_1 = y_2$, a relation like $d(x_1, x_2) \neq 0$ yields (by the choice of our data) $k^0 \in -K \subseteq -H$, contradiction. So, necessarily, $d(x_1, x_2) = 0$; wherefrom $(x_1, y_1) = (x_2, y_2)$. If $y_1 \neq y_2$ one has, (by (6H1)) $\gamma(y_1) < \gamma(y_2)$ [hence $\Phi(x_1, y_1) < \Phi(x_2, y_2)$]. This, combined with our starting hypothesis, yields $(x_1, y_1) \sqsubset (x_2, y_2)$. The proof is complete. ■

Let us now return to the initial framework (in which (6H1) is excluded). The following Zorn (minimality) principle is available.

THEOREM 6.1. *Let the precised conditions be admitted. Then, for each $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$ such that*

(6.2) $(\bar{x}, \bar{y}) \sqsubseteq (x_0, y_0)$; *and,moreover,*

(6.3) *if* $(x', y') \in A$ *fulfils* $(x', y') \sqsubseteq (\bar{x}, \bar{y})$ *then* $(x', y') = (\bar{x}, \bar{y})$.

(In other words: $\sqsubseteq$ *is a Zorn ordering on* A*).*

The proof is immediate, via Theorem 2.3, if we note that the *dual* strict order $\sqsupset$ and the *dual* order $\sqsupseteq$ are obtainable from the dual quasi–order $\succeq$ in the way described by (2D4)+(2D5) (with Φ in place of φ). This result may be viewed as an *algebraic* version of the one due to Goepfert, Tammer and Zălinescu [7, Theorem 4]. It tells us that coarser than $\preceq$ orders (on A) with a *standard* (minimal) Zorn property do exist. Moreover, if the (non–degenerate) convex cone K fulfils the regularity condition (6H1), then (cf. Lemma 6.1), conclusions (6.2)+(6.3) may be written with $\preceq$ in place of $\sqsubseteq$. An interesting circumstance of this type is to be described as follows. Assume that the couple of convex cones $\{K, H\}$ in Y (taken as before) fulfils the additional condition

(6H2) $K \subseteq \mathrm{Aint}(H)[= \{0\} \cup \mathrm{aint}(H)]$.

Note that, in such a case (cf. the developments in Section 4)

(6.4) K is pointed (because, so is $\mathrm{Aint}(H)$).

As a consequence, the relation $(\preceq) = (\preceq_K^{k^0})$ given by (1D3) is an *order* (on $X \times Y$). Let $\sqsubseteq$ stand for its associated order (on $X \times Y$) introduced as in (6D2). A useful completion of Lemma 6.1 is now

LEMMA 6.2. *Under the above conventions, the restrictions to* A *of* $\preceq$ *and* $\sqsubseteq$ *are identical; i.e., for* $(x_1, y_1), (x_2, y_2) \in A$,

(6.5) $(x_1, y_1) \preceq (x_2, y_2)$ *if and only if* $(x_1, y_1) \sqsubseteq (x_2, y_2)$.

Proof. It will suffice establishing that (6H1) is fulfilled by our data. In fact, let $y_1, y_2 \in P_Y(A)$ be such that $y_1 \leq y_2(\mod K)$ and $y_1 \neq y_2$. By (6H2), we have $y_1 < y_2(\mod \mathrm{aint}(H))$; and this, combined with Lemma 4.1, yields $\gamma(y_1) < \gamma(y_2)$. Hence the conclusion. ■

Now, by simply adding this to Theorem 6.1, one gets the following practical statement. (The general assumptions of this section prevail).

THEOREM 6.2. *Under the precised setting, it is the case that: for each* $(x_0, y_0) \in A$, *there exists* $(\bar{x}, \bar{y}) \in A$, *in such a way that*

(6.6) $(\bar{x}, \bar{y}) \preceq (x_0, y_0)$; *and, noreover*

(6.7) *if* $(x', y') \in A$ *fulfils* $(x', y') \preceq (\bar{x}, \bar{y})$, *then* $(x', y') = (\bar{x}, \bar{y})$.

(In other words: $\preceq$ *is a Zorn ordering on* A*).*

This result may be viewed as an algebraic *completion* of the one due to Goepfert,Tammer and Zălinescu [8, Theorem 1]. Further aspects will be discussed elsewhere.

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