An. Şt. Univ. Ovidius Constanţa Vol. 10(1), 2002, 135–148

ACTIONS OF GROUPS ON LATTICES

Abstract

The aim of this paper is to study the actions of the groups on lattices and to give some connections between the structure of a group and the structure of its subgroup lattice. Moreover, we shall introduce the concept of direct $\vee$-sum of G-sublattices and we shall present a generalization of a result about finite nilpotent groups.

1 Preliminaries

Let $(G, \cdot, e)$ be a monoid and L be a G–set (relative to an action ρ of G on L; for $(g, \ell) \in G \times L$, we denote by $g \circ \ell$ the element $\rho(g)(\ell) \in L$). If L is a poset (reltive to a partial ordering relation "$\leq$") and, for $\ell, \ell' \in L$, $\ell \leq \ell'$ implies $g \circ \ell \leq g \circ \ell'$, for any $g \in G$, then L is called a *G–poset*. Moreover, if $(L, \leq)$ is a lattice and, for $\ell, \ell' \in L$, we have:

$$g \circ (\ell \wedge \ell') = (g \circ \ell) \wedge (g \circ \ell'),$$
$$g \circ (\ell \vee \ell') = (g \circ \ell) \vee (g \circ \ell'),$$

for any $g \in G$, then L is called a *G–lattice.*

A *G–sublattice* of a G–lattice L is a sublattice L' of L satisfying the property:

$$G \circ L' = \{g \circ \ell' \mid g \in G,\ \ell' \in L'\} \subseteq L'.$$

Let L_1 and L_2 be two G–posets (respectively two G–lattices). A monotone map (respectively a lattice homomorphism) $f : L_1 \longrightarrow L_2$ is called a *G–poset homomorphism* (respectively a *G–lattice homomorphism*) if $f(g \circ \ell_1) = g \circ f(\ell_1)$, for any $(g, \ell_1) \in G \times L_1$. Moreover, if f is one-to-one and onto, then it is called a *G–poset isomorphism* (respectively a *G–lattice isomorphism*).

A *G–congruence* on a G–lattice L is a congruence relation "$\sim$" on L which has the property that $\ell \sim \ell'$ $(\ell, \ell' \in L)$ implies $g \circ \ell \sim g \circ \ell'$, for any $g \in G$.

Let L be a G–lattice and "$\sim$" be a G–congruence on L. Then the quotient lattice $L/\sim\ = \{[\ell] \mid \ell \in L\}$ of L modulo "$\sim$" is a G–lattice, where $g \circ [\ell] = [g \circ \ell]$, for any $(g, \ell) \in G \times L$.

If $f : L_1 \longrightarrow L_2$ is a G–lattice homomorphism, then the sublattice $\operatorname{Im} f = \{f(\ell_1) \mid \ell_1 \in L_1\}$ of L_2 is a G–lattice and there exists a G–congruence "$\sim$" on L_1 such that the G–lattices $L_1/\sim$ and $\operatorname{Im} f$ are isomorphic.

Let L be a lattice having the initial element 0. On L is well defined the *height* function: for $\ell \in L$, let $h_L(\ell)$ denote the length of a longest maximal chain in $[0, \ell]$ if there is a finite longest maximal chain; otherwise put $h_L(\ell) = \infty$. If L is of finite length, then the following conditions are equivalent:

i) L is modular.

ii) The height function h_L on L satisfies the property:
$h_L(\ell) + h_L(\ell') = h_L(\ell \wedge \ell') + h_L(\ell \vee \ell')$, for any $\ell, \ell' \in L$.

2 Main results

2.1 Finite G–lattice

Let $(G, \cdot, e)$ be a monoid.

Proposition 1. *Let $(L, \leq)$ be a complete lattice such that L is a G–poset. Then we have:*

$$G = \bigcup_{\ell \in L} \operatorname{Stab}_G(\ell).$$

Proof. Let $g \in G$ and $L_g = \{\ell \in L \mid g \circ \ell \geq \ell\}$. We have $L_g \neq \emptyset$ (L_g contains the initial element of L). Since L is complete, there exists $\bar{\ell} = \vee L_g$. We have $\ell \leq g \circ \ell \leq g \circ \bar{\ell}$, for any $\ell \in L_g$, therefore:

$$\bar{\ell} \leq g \circ \bar{\ell}. \tag{1}$$

Using the relation (1), we obtain that $g \circ \bar{\ell} \leq g \circ (g \circ \bar{\ell})$, thus $g \circ \bar{\ell} \in L_g$. Since $\bar{\ell} = \vee L_g$, it results:

$$g \circ \bar{\ell} \leq \bar{\ell}. \tag{2}$$

The relations (1) and (2) give us $g \circ \bar{\ell} = \bar{\ell}$, so that $g \in \operatorname{Stab}_G(\bar{\ell})$. Thus $G = \bigcup_{\ell \in L} \operatorname{Stab}_G(\ell)$.

Corollary. (The Fixed–Point Theorem of complete lattice)
Any monotone map of a complete lattice L into itself has a fixed point.

Proof. The set G' of all monotone maps of L into itself is a monoid. Moreover, L is a G'-poset, where $f \circ \ell = f(\ell)$, for any $(f, \ell) \in G' \times L$. From

Proposition 1, we obtain $G' = \bigcup_{\ell \in L} \mathrm{Stab}_{G'}(\ell)$, therefore, for any $f \in G'$, there exists $\ell \in L$ such that $f \in \mathrm{Stab}_{G'}(\ell)$, i.e. $f(\ell) = f \circ \ell = \ell$.

In the followings we suppose that $(G, \cdot, e)$ is a group and we denote by $L(G)$ (respectively by $L_0(G)$) the lattice of subgroups of G (respectively the lattice of normal subgroups of G).

Proposition 2. *Let L be a complete G–lattice such that* $\mathrm{Stab}_G(\ell) = \{e\}$*, for any $\ell \in L$. Then the group G is abelian.*

Proof. Let g_1, g_2 be two elements of G and $f_{g_1,g_2} : L \longrightarrow L$ be the map defined by $f_{g_1,g_2} = [g_1, g_2] \circ \ell$, for any $\ell \in L$ (where $[g_1, g_2]$ is the commutator of g_1 and g_2). We have $f_{g_1,g_2}(\ell \wedge \ell') = [g_1, g_2] \circ (\ell \wedge \ell') = ([g_1, g_2] \circ \ell) \wedge ([g_1, g_2] \circ \ell') = f_{g_1,g_2}(\ell) \wedge f_{g_1,g_2}(\ell')$, for any $\ell, \ell' \in L$, thus f_{g_1,g_2} is a monotone map. From the above corollary, we obtain that there exists $\ell_0 \in L$ such that $f_{g_1,g_2}(\ell_0) = \ell_0$. It results $[g_1, g_2] \in \mathrm{Stab}_G(\ell_0)$, i.e. $[g_1, g_2] = e$.

Since any ordered latticeal group G is a G–lattice, from Proposition 2 we obtain the following result:

Corollary. *Any ordered latticeal group complete as lattice is abelian.*

Let L be a finite G–lattice, 0 be the initial element of L and 1 be the final element of L.

Remark. If $L = \{\ell_1 = 0, \ell_2, ..., \ell_m = 1\}$ and $H_i = \mathrm{Stab}_G(\ell_i)$, $i = \overline{1, m}$, then from Proposition 1, we have $G = \bigcup_{i=1}^{m} H_i$. Let I be a maximal subset of $\{1, 2, ..., m\}$ with the property:

$$\begin{cases} G = \bigcup_{i \in I} H_i \\ H_j \not\subseteq \bigcup_{i \in I \setminus \{j\}} H_i, \quad \text{for any } j \in I. \end{cases}$$

Then, for any $g \in G$, there exists $n_g \in \mathbb{N}^*$ such that $g^{n_g} \in \bigcap_{i \in I} H_i$. Since, for any $\ell, \ell' \in L$, $\mathrm{Stab}_G(\ell) \cap \mathrm{Stab}_G(\ell') \subseteq \mathrm{Stab}_G(\ell \wedge \ell')$, we obtain that there exists $\ell_0 \in L$ such that every element of G has a natural power in $\mathrm{Stab}_G(\ell_0)$.

We suppose that G is a finite group, $\mathrm{Stab}_G(0) = \mathrm{Stab}_G(1) = G$ and let $f_L : L \longrightarrow L$ be the map defined by $f_L(\ell) = \bigwedge_{g \in G} g \circ \ell$, for any $\ell \in L$.

Proposition 3. *The map f_L is a G-poset homomorphism which has the following properties:*

a) $f_L(\ell) \leq \ell$, *for any* $\ell \in L$.

b) $\operatorname{Im} f_L = \operatorname{Fix}_G(L)$, *where* $\operatorname{Fix}_G(L) = \{\ell \in L \mid g \circ \ell = \ell$, *for any* $g \in G\}$.

c) $f_L^2 = f_L$.

Proof. a) Since $e \circ \ell = \ell$, we obtain $f_L(\ell) = \ell \wedge \left(\bigwedge_{g \in G \setminus \{e\}} g \circ \ell \right) \leq \ell$, for any $\ell \in L$.

b) Let $\ell' \in \operatorname{Im} f_L$. Then there exists $\ell \in L$ such that $\ell' = f_L(\ell)$. For any $g' \in G'$, we have:

$$g' \circ \ell' = g' \circ f_L(\ell) = g' \circ \left(\bigwedge_{g \in G} g \circ \ell \right) = \bigwedge_{g \in G} g' \circ (g \circ \ell) = \bigwedge_{g \in G} (g'g) \circ \ell = f_L(\ell) = \ell',$$

therefore $\ell' \in \operatorname{Fix}_G(L)$.

Conversely, let $\ell' \in \operatorname{Fix}_G(L)$. Then $g \circ \ell' = \ell'$, for any $g \in G$. It results $f_L(\ell') = \bigwedge_{g \in G} g \circ \ell' = \bigwedge_{g \in G} \ell' = \ell'$, thus $\ell \in \operatorname{Im} f_L$.

c) We have $f_L^2(\ell) = f_L(f_L(\ell)) = \bigwedge_{g \in G} g \circ f_L(\ell) = \bigwedge_{g \in G} f_L(\ell) = f_L(\ell)$, for any $\ell \in L$. Thus $f_L^2 = f_L$.

Now, the fact that f_L is a G-poset homomorphism is obvious.

Remark. If L is a fully ordered G–lattice, then f_L is a G–lattice homomorphism. Moreover, the binary relation"$\sim$" on L defined by $\ell \sim \ell'$ if and only if $f_L(\ell) = f_L(\ell')$ is a G-congruence. Therefore, we obtain the G–latice isomorphism:

$$L/\!\sim\; \cong \operatorname{Fix}_G(L).$$

Let $n = |\operatorname{Fix}_G(L)|$ and $C_1, C_2, ..., C_n$ be the equivalence classes modulo "$\sim$". If $(\ell'_i)_{i=\overline{1,n}}$ is a set of representatives for the equivalence classes $(C_i)_{i=\overline{1,n}}$ then $C_i = \{\ell \in L \mid f_L(\ell) = f_L(\ell'_i)\} \neq \emptyset$, $i = \overline{1,n}$, $C_i \cap C_j = \emptyset$, for $i \neq j$ and $L = \bigcup_{i=1}^{n} C_i$. Moreover, for any $i \in \{1, 2, ..., n\}$, we have:

$$G \circ \ell'_i = \{g \circ \ell'_i \mid g \in G\} \subseteq C_i.$$

It results that:

$$|G \circ \ell'_i| = \frac{|G|}{|\mathrm{Stab}_G(\ell'_i)|} \leq |C_i|,\ i = \overline{1,n}.$$

This implies the following inequality:

$$(*) \qquad |G| \sum_{i=1}^{n} \frac{1}{|\mathrm{Stab}_G(\ell'_i)|} \leq \sum_{i=1}^{n} |C_i| = |L|.$$

Let $C_{i_1}, C_{i_2}, ..., C_{i_r}$ be the classes having an unique element (i.e. $c_{i_j} = \{\ell'_{i_j}\}$, $j = \overline{1,r}$, where $r \leq n$, $i_r = n$ and $\ell'_n = 1$). Then, for each $s \in \{1, 2, ..., n\} \setminus \{i_1, i_2, ..., i_r\}$, we can suppose that $\ell'_s \notin \mathrm{Fix}_G(L)$. We obtain $|G \circ \ell'_s| \neq 1$, therefore

$$\frac{|G|}{|\mathrm{Stab}_G(\ell'_s)|} \geq p,$$

where p is the smallest prime divisor of $|G|$. Using the inequality $(*)$, it results that:

$$|L| \geq pn - (p-1)r.$$

Taking the particular case $L = L(G)$, it obtains the following results:

Corollary 1. *If G is a finite group and r is the number of equivalence classes modulo "$\sim$" having a unique element, then:*

$$|L(G)| \geq p|L_0(G)| - (p-1)r,$$

where p is the smallest prime divisor of $|G|$.

Corollary 2. *If G is a nonabelian simple finite group, then:*

$$|L(G)| \geq p + 1,$$

where p is the smallest prime divisor of $|G|$.

Remark. Let $\mathrm{Min}(L)$ be the set of all minimal elements of L and $\mathrm{Ker}\, f_L = \{\ell \in L \mid f_L(\ell) = 0\}$. Then the following relations hold:

$$(**) \qquad \mathrm{Min}(L) \subseteq \mathrm{Ker}\, f_L \cup \mathrm{Fix}_G(L).$$

Indeed, if $\ell \in \mathrm{Min}(L)$ and $f_L(\ell) \neq 0$, then, from the inequalities $0 \leq f_L(\ell) \leq \ell$, we obtain $f_L(\ell) = \ell$, i.e. $\ell \in \mathrm{Fix}_G(L)$.

Let k be the length of the finite G-lattice L.

Definition 1. We say that L is *regular* if it satisfies the following conditions:

(i) All maximal chains of L have the same length.

(ii) For any $\ell \in L \setminus (\operatorname{Ker} f_L \cup \{1\})$ with $h_L(\ell) = p$, the equivalence class modulo "$\sim$" of ℓ has at most $k - p$ elements.

Definition 2. A family $U = (u_i)_{i=\overline{1,k}}$ of elements of L is called a *k-independent minimal system* if it has the properties:

(i) $U \subseteq \operatorname{Min}(L),\ U \cap \operatorname{Fix}_G(L) \neq \emptyset$.

(ii) For any distinct numbers $i_1, i_2, ..., i_k \in \{1, 2, ..., k\}$, we have:

$$\begin{aligned}
&|\{u_{i_1} \vee u_j \mid j \neq i_1\}| = k - 1,\\
&|\{u_{i_1} \vee u_{i_2} \vee u_j \mid j \notin \{i_1, i_2\}\}| = k - 2,\\
&\vdots\\
&|\{u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_{k-2}} \vee u_j \mid j \notin \{i_1, i_2, ..., i_{k-2}\}\}| = 2.
\end{aligned}$$

(iii) For any distinct numbers $i_1, i_2, ..., i_k \in \{1, 2, ..., k\}$ (where $p \in \mathbb{N}^*$, $p \leq k$), if $\{u_{i_1}, u_{i_2}, ..., u_{i_p}\} \cap \operatorname{Fix}_G(L) \neq \emptyset$, then $h_L(u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_p}) = p$.

Proposition 4. *Let L be a finite G–lattice of length k. If L is regular and it has a k-independent minimal system, then there exists a maximal chain of L:*

$$0 = a_0 < a_1 < \cdots < a_k = 1,$$

with $a_i \in \operatorname{Fix}_G(L)$, for any $i = \overline{0,k}$.

Proof. We prove the statement by induction on k. If $k \leq 1$, the statement is trivial. Let us assume the statement to hold for $k-1$ and let $U = (u_i)_{i=\overline{1,k}}$ be a k-independent minimal system of L. Since $U \cap \operatorname{Fix}_G(L) \neq \emptyset$, we can suppose that $u_k \in \operatorname{Fix}_G(L)$. Let $L' = [u_k, 1] = \{\ell \in L \mid u_k \leq \ell \leq 1\}$. L' is a finite G-lattice of length $k - 1$. For any $\ell \in L' \setminus (\operatorname{Ker} f_{L'} \cup \{1\})$ with $h_{L'}(\ell) = p$, we have $h_L(\ell) = p + 1$, therefore the equivalence class modulo "$\sim$" of ℓ has at most $k - 1 - p$ elements. It results that L' is regular.

Now we prove that $V = (v_i)_{i=\overline{1,k-1}}$, where $v_i = u_i \vee u_k$ for any $i = \overline{1,k-1}$, is a $(k-1)$-independent minimal system of L'.

Since $u_k \in \operatorname{Fix}_G(L)$, we have $h_L(v_i) = 2$, $i = \overline{1,k-1}$, thus $h_{L'}(v_i) = 1$, $i = \overline{1,k-1}$, i.e. $V \subseteq \operatorname{Min}(L')$. If we suppose $V \cap \operatorname{Fix}_G(L') = \emptyset$, then, using the remark $(**)$, we obtain that V is containing in the equivalence class modulo "$\sim$" of u_k. It results that the equivalence class modulo "$\sim$" of u_k has at least k elements (u_k and v_i, $i = \overline{1,k-1}$). This contradicts the assumption that L

is regular. The fact that V satisfies the property (ii) of Definition 2 is obvious. For the property (iii), let the distinct numbers $i_1, i_2, ..., i_p \in \{1, 2, ..., k-1\}$ (where $p \in \mathbb{N}^*$, $p \leq k-1$). We have $h_{L'}(v_{i_1} \vee v_{i_2} \vee \cdots \vee v_{i_p}) = h_{L'}(u_k \vee u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_p}) = h_L(u_k \vee u_{i_1} \vee u_{i_2} \vee \cdots \vee u_{i_p}) - 1 = (p+1) - 1 = p$.

From inductive hypothesis, it results that there exists a maximal chain of L':

$$u_k = a_1 < a_2 < \cdots < a_k = 1,$$

with $a_i \in \mathrm{Fix}_G(L'),\ i = \overline{1,k}$. Thus

$$0 = a_0 < a_1 < \cdots < a_k = 1$$

is a maximal chain of L, with $a_i \in \mathrm{Fix}_G(L),\ i = \overline{0,k}$.

Corollary. *The symmetric group of degree* 3 Σ_3 *and the dihedral group of order* 8 D_8 *have principal series of subgroups.*

Proof. We have $\Sigma_3 = \{e, \sigma_1, \sigma_2, \sigma_3, \tau, \tau^2\}$ (where $\sigma_1 = (2\ 3)$, $\sigma_2 = (1\ 3)$, $\sigma_3 = (1\ 2)$ and $\tau = (2\ 3\ 1)$) and $D_8 = \{1, \rho, \rho^2, \rho^3, \varepsilon, \rho\varepsilon, \rho^2\varepsilon, \rho^3\varepsilon\}$ (where $\rho^4 = \varepsilon^2 = 1$ and $\varepsilon\rho = \rho^3\varepsilon$). We obtain $L(\Sigma_3) = \{H_0 = \{e\},\ H_1 = \{e, \sigma_1\},\ H_2 = \{e, \sigma_2\},\ H_3 = \{e, \sigma_3\},\ H_4 = \{e, \tau, \tau^2\},\ H_5 = \Sigma_3\}$ and $L(D_8) = \{H'_0 = \{1\},\ H'_1 = \{1, \varepsilon\},\ H'_2 = \{1, \rho^2\varepsilon\},\ H'_3 = \{1, \rho^2\},\ H'_4 = \{1, \rho\varepsilon\},\ H'_5 = \{1, \rho^3\varepsilon\},\ H'_6 = \{1, \rho^3, \rho\varepsilon, \rho^3\varepsilon\},\ H'_7 = \{1, \rho, \rho^2, \rho^3\},\ H'_8 = \{1, \rho^2, \rho\varepsilon, \rho^3\varepsilon\},\ H'_9 = D_8\}$. It is a simple exercise to verify that the Σ_3–lattice $L(\Sigma_3)$ (respectively the D_8–lattice $L(D_8)$) is regular and that $U = \{H_3, H_4\}$ (respectively $U' = \{H'_2, H'_3, H'_4\}$) is a 2-independent minimal system of $L(\Sigma_3)$ (respectively a 3-independent minimal system of $L(D_8)$). Now the statement results from Proposition 4.

2.2 On a property of finite nilpotent groups

Let $(G, \cdot, e)$ be a group.

Definition 1. Let L be a G–lattice having the initial element 0 and $(L_i)_{i \in I}$ be a finite family of G–sublattices of L. We say that L is the *direct* $\vee$*–sum* of the family $(L_i)_{i \in I}$ (and we denote this by $L = \overset{\vee}{\bigoplus_{i \in I}} L_i$) if the following two equalities hold:

i) $L = \bigvee_{i \in I} L_i$.

ii) $L_j \wedge \left(\bigvee_{\substack{i \in I \\ i \neq j}} L_i \right) = \{0\}$, for any $j \in I$.

Examples. 1) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, be the decomposition of the natural number n as a product of prime factors. If, for any $m \in \mathbb{N}^*$, we denote by L_m the lattice of all natural divisors of m and we consider the set $G = \{\sigma \in \text{Aut}(L_n) \mid \sigma(L_{p_i^{\alpha_i}}) = L_{p_i^{\alpha_i}},\ i = \overline{1,k}\}$, then G is a group, L_n is a G–lattice (where $\sigma \circ d = \sigma(d)$, for any $(\sigma, d) \in G \times L_n$) and $L_{p_i^{\alpha_i}}$ is a G–sublattice of L_n, $i = \overline{1,k}$. It is easy to see that L_n is the direct $\vee$–sum of the family $(L_{p_i^{\alpha_i}})_{i=\overline{1,k}}$.

2) Let $m \geq 2,\ n \geq 2$ be two natural numbers, $f : \mathbb{Z}_m \longrightarrow \text{Aut}(\mathbb{Z}_n)$ be a group homomorphism and $\hat{k}_0 = f(\bar{1})(\hat{1})$. We denote by S the semidirect product of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ with respect to the homomorphism f and by G, respectively H the images of $\mathbb{Z}_m$, respectively $\mathbb{Z}_n$ through the group homomorphisms:

$$\sigma_1 : \mathbb{Z}_m \longrightarrow S,\ \sigma_1(\bar{x}) = (\bar{x}, \hat{0}),\ \text{for any } \bar{x} \in \mathbb{Z}_m,$$

respectively

$$\sigma_2 : \mathbb{Z}_n \longrightarrow S,\ \sigma_2(\hat{y}) = (\bar{0}, \hat{y}),\ \text{for any } \hat{y} \in \mathbb{Z}_n.$$

If $L(S), L(G)$ and $L(H)$ are the subgroup lattices of S, G, respectively H, then we have $L(S) = L(G) \overset{\vee}{\oplus} L(H)$ if and only if $(m, n) = 1$ and $k_0 \equiv 1 (\text{mod}\, n)$ (see [10], Proposition 3).

Proposition 1. *If L is a distributive G–lattice having the initial element* 0 *and $(L_i)_{i \in I}$ is a finite family of G–sublattices of L such that $L_j \wedge \left(\bigvee_{\substack{i \in I \\ i \neq j}} L_i \right) = \{0\}$, for any $j \in I$, then the following two conditions are equivalent:*

i) $L = \overset{\vee}{\bigoplus_{i \in I}} L_i$.

ii) *Every element $\ell \in L$ can be written uniquely as $\bigvee_{i \in I} \ell_i$, where $\ell_i \in L_i$, for any $i \in I$.*

Proof. i)$\Longrightarrow$ii) Since $L = \overset{\vee}{\bigoplus_{i \in I}} L_i$, we have $L = \bigvee_{i \in I} L_i$, therefore every element $\ell \in L$ can be written as $\bigvee_{i \in I} \ell_i$, where $\ell_i \in L_i$, $i \in I$. If $\ell = \bigvee_{i \in I} \ell_i = \bigvee_{i \in I} \ell'_i$ with $\ell_i, \ell'_i \in L_i$, $i \in I$, then, for any $j \in I$, we have $\ell'_j = \ell'_j \wedge \ell = \ell'_j \wedge \left(\bigvee_{i \in I} \ell_i \right) =$

$$\ell'_j \wedge \left[\ell_j \vee \left(\bigvee_{\substack{i\in I\\ i\neq j}} \ell_i\right)\right] = (\ell'_j \wedge \ell_j) \vee \left[\ell'_j \wedge \left(\bigvee_{\substack{i\in I\\ i\neq j}} \ell_i\right)\right] = \ell'_j \wedge \ell_j,$$ thus $\ell'_j \leq \ell_j$. In the same way, we obtain $\ell_j \leq \ell'_j$, therefore $\ell'_j = \ell_j$, $j \in I$.

ii)$\Longrightarrow$i) Obvious.

Next aim is to establish connections between the direct product of G–lattices and the direct $\vee$-sum of G-sublattices.

Proposition 2. *If $(L_i)_{i\in I}$ is a finite family of G–lattices having initial elements (denoted all by 0), $0 \in \mathrm{Fix}_G(L_i)$, $i \in I$, and L is the direct product of the family $(L_i)_{i\in I}$, then there exists a family $(L'_i)_{i\in I}$ of G–sublattices of L which satisfies the following properties:*

i) $L = \overset{\vee}{\bigoplus\limits_{i\in I}} L'_i$.

ii) $L'_i \cong L_i$ *(isomorphism of G–lattices), for any $i \in I$.*

Proof. It is easy to see that the sets $L'_i = \{(a_j)_{j\in I} \in L \mid a_j = 0$, for any $j \in I \setminus \{i\}\}$, $i \in I$, are G-sublattices of L and $L = \overset{\vee}{\bigoplus\limits_{i\in I}} L'_i$. Moreover, the maps

$$\begin{aligned}&f_i : L_i \longrightarrow L'_i\\ &f_i(\ell_i) = (a_j)_{j\in I},\ \text{where } a_i = \ell_i \text{ and } a_j = 0,\ \text{for } j \neq i,\end{aligned}$$

are isomorphism of G–lattices, $i \in I$.

Let L be a finite G–lattice with the initial element denoted by 0 such that $0 \in \mathrm{Fix}_G(L) = \{\ell \in L \mid g \circ \ell = \ell$, for any $g \in G\}$. If $(\ell_i)_{i=\overline{1,k}}$ is a family of elements of L, then we make the following notations:

$$\begin{aligned}L_i &= [0, \ell_i] = \{\ell \in L \mid 0 \leq \ell \leq \ell_i\},\\ G \circ L_i &= \{g \circ \ell \mid g \in G,\ \ell \in L_i\},\end{aligned}$$

where $i \in \{1, 2, ..., k\}$.

Definition 2. The family $(\ell_i)_{i=\overline{1,k}}$ is called a *maximal system* of L if it satisfies the properties:

i) $L = \bigvee\limits_{i=1}^{k} G \circ L_i$.

ii) $G \circ L_j \wedge \left(\bigvee_{\substack{i=I \\ i \neq j}}^{k} G \circ L_i \right) = \{0\}$, for any $j = \overline{1,k}$.

Remark. If $(\ell_i)_{i=\overline{1,k}}$ is a maximal system of L, then, for $i \in \{1, 2, ..., k\}$, the sublattice L_i of L is not necessarily a G–sublattice. A sufficient condition for this fact holds is $\ell_i \in \mathrm{Fix}_G(L)$. In the case when $(\ell_i)_{i=\overline{1,k}} \subseteq \mathrm{Fix}_G(L)$, we have $G \circ L_i = L_i$ for any $i = \overline{1,k}$ and $L = \overset{\vee}{\bigoplus_{i=\overline{1,k}}} L_i$.

Definition 3. Let $U, V \in L(G)$.

(i) We say that U and V form a *permutable pair* if $[U \cup V] = UV = VU$ (where $[U \cup V]$ denotes the subgroup of G generated by $U \cup V$).

(ii) We say that U and V form a *modular pair* if

$W \cap [U \cup V] = [U \cup (W \cap V)]$ for any $W \in L(G)$ with $U \subseteq W$

and

$W \cap [U \cup V] = [V \cup (W \cap U)]$ for any $W \in L(G)$ with $V \subseteq W$.

Remarks. 1) Any permutable pair of subgroups is a modular pair (see [7], Theorem 5, page 5).

2) If the group G is finite and it satisfies the property tht any two subgroups $U, V \in L(G)$ with $(|U|, |V|) = 1$ form a permutable pair, then, for any $H_1, H_2, ..., H_k \in L(G)$ with $(|H_i|, |H_j|) = 1$, $i \neq j$, we have:

$$H_1 H_2 ... H_k = \left[\bigcup_{i=1}^{k} H_i \right] \in L(G)$$

and

$$|H_1 H_2 ... H_k| = \prod_{i=1}^{k} |H_i|.$$

Proposition 3. *For a finite group G which satisfies the property that any two subgroups $U, V \in L(G)$ with $(|U|, |V|) = 1$ form a permutable pair, the G–lattice $L(G)$ has a maximal system.*

Proof. Let $n = |G|$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, is the decomposition of n as a product of prime factors, then, for any $i = \overline{1,k}$, let H_i be Sylow p_i–subgroup of G.

We prove that $\{H_1, H_2, ..., H_k\}$ is a maximal system for $L(G)$. Let $H \in L(G)$ and $m = |H|$. Then m/n, therefore there exist the numbers $\beta_i \in \mathbb{N}$, $\beta_i \leq \alpha_i$, $i = \overline{1,k}$, such that $m = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}$. For any $i = \overline{1,k}$, let U_i be a Sylow p_i–subgroup of H and, using the Theorems of Sylow, let $x_i \in G$ such that $U_i \leq H_i^{x_i}$, i.e. $U_i^{x_i^{-1}} \in [\{e\}, H_i]$ (where e is the identity of G). From Remark 2), we obtain $H = U_1 U_2 ... U_k = \left(U_1^{x_1^{-1}}\right)^{x_1} \left(U_2^{x_2^{-1}}\right)^{x_2} ... \left(U_k^{x_k^{-1}}\right)^{x_k}$, thus:

$$L(G) = \bigvee_{i=1}^{k} G \circ [\{e\}, H_i].$$

Let $j \in \{1, 2, ..., k\}$ and $K \in G \circ [\{e\}, H_j] \wedge \left(\bigvee_{\substack{i=1 \\ i \neq j}}^{k} G \circ [\{e\}, H_i] \right)$. Then $K = V_j^{x_j} \wedge \left(\bigvee_{\substack{i=1 \\ i \neq j}}^{k} V_i^{x_i} \right)$ (where $V_s \leq H_s$ and $x_s \in G$, for any $s = \overline{1,k}$).

Since $\left(|V_j^{x_j}|, \left| \bigvee_{\substack{i=1 \\ i \neq j}}^{k} V_i^{x_i} \right| \right) = 1$, it follows that $K = \{e\}$, thus:

$$G \circ [\{e\}, H_j] \wedge \left(\bigvee_{\substack{i=1 \\ i \neq j}}^{k} G \circ [\{e\}, H_i] \right) = \{e\}.$$

Remark. Let G be a finite group of order n, $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ be the decomposition of n as a product of prime factors and H_i be a Sylow p_i–subgroup of G, $i = \overline{1,k}$. If $(H_i)_{i=\overline{1,k}}$ is a maximal system of $L(G)$, then, for any $x_i \in G$, $i = \overline{1,k}$, $(H_i^{x_i})_{i=\overline{1,k}}$ is a maximal system of $L(G)$.

Let L be a modular finite G–lattice with the initial element denoted by 0 and $(\ell_i)_{i=\overline{1,k}}$ be a maximal system of L.

Lemma. *The following equality holds:*

$$h_L \left(\bigvee_{i=1}^{k} \ell_i \right) = \sum_{i=1}^{k} h_L(\ell_i).$$

Proof. We prove the above equality by induction on k. For $k = 2$, we have $h_L(\ell_1 \vee \ell_2) = h_L(\ell_1) + h_L(\ell_2) - h_L(\ell_1 \wedge \ell_2) = h_L(\ell_1) + h_L(\ell_2) - h_L(0) = h_L(\ell_1) + h_L(\ell_2)$. Let us assume the equality to hold for $k = 1$. We obtain $h_L\left(\bigvee_{i=1}^{k} \ell_i\right) =$
$h_L\left(\left(\bigvee_{i=1}^{k-1} \ell_k\right) \vee \ell_j\right) = h_L\left(\bigvee_{i=1}^{k-1} \ell_i\right) + h_L(\ell_k) - h_L\left(\left(\bigvee_{i=1}^{k-1} \ell_i\right) \wedge \ell_k\right) =$
$= \sum_{i=1}^{k-1} h_L(\ell_i) + h_L(\ell_k) - h_L(0) = \sum_{i=1}^{h} h_L(\ell_i)$.

For any $i=\overline{1,k}$, let $\alpha_i = h_L(G \circ L_i)$, (i.e. $\alpha_i = \max\{h_L(g \circ \ell) \mid g \in G, \ell \in L_i\}$), $\mathbb{N}_{\alpha_i}=\{0, 1, ..., \alpha_i\}$ and $h_i : G \circ L_i \to \mathbb{N}_{\alpha_i}$ be the restriction of the height function h_L on the set $G \circ L_i$. We suppose that is well defined the function:

$$h' : L \longrightarrow \bigtimes_{i=1}^{k} \mathbb{N}_{\alpha_i},$$

$$h'\left(\bigvee_{i=1}^{k} g_i \circ \ell_{ii}\right) = (h_1(\ell_{11}), h_2(\ell_{22}), ..., h_k(\ell_{kk})),$$

where $g_i \in G$, $\ell_{ii} \in L_i$, for any $i = \overline{1,k}$ (it is easy to see that a sufficient condition for this fact holds is "L = distributive lattice").

Proposition 4. *The function h' is onto. Moreover, for any $(\beta_1, \beta_2, ..., \beta_k) \in \bigtimes_{i=1}^{k} \mathbb{N}_{\alpha_i}$, we have:*

$$(h')^{-1}(\beta_1, \beta_2, ..., \beta_k) \cap \left\{\ell \in L \mid h_L(\ell) = \sum_{i=1}^{k} \beta_i\right\} \neq \emptyset.$$

Proof. For each $i \in \{1, 2, ..., k\}$, the function h_i is onto.

Let $(\beta_1, \beta_2, ..., \beta_k) \in \bigtimes_{i=1}^{k} \mathbb{N}_{\alpha_i}$ and $\ell_{ii} \in G \circ L_i$ such that $h_i(\ell_{ii}) = \beta_i$, $i = \overline{1,k}$. Using the above lemma, it is a simple exercise to verify that

$$\bigvee_{i=1}^{k} \ell_{ii} \in (h')^{-1}(\beta_1, \beta_2, ..., \beta_k) \cap \left\{\ell \in L \mid h_L(\ell) = \sum_{i=1}^{k} \beta_i\right\}.$$

Proposition 5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ be the decomposition of the natural number n as a product of prime factors, L_n be the lattice of all natural divisors of n and G be a finite group of order n which satisfies the following properties:*

(i) *$L(G)$ is a modular lattice.*

(ii) *There exists a maximal system $(H_i)_{i=\overline{1,k}}$ of $L(G)$ such that H_i is a Sylow p_i–subgroup of G, $i=\overline{1,k}$.*

(iii) *For any $H \in L(G)$, $|H| = \prod_{i=1}^{k} p_i^{x_i}$ implies $h_{L(G)}(H) \geq \sum_{i=1}^{k} x_i$.*

Then the function $\text{ord} : L(G) \to L_n$, $\text{ord}(H) = |H|$, *for any $H \in L(G)$, is onto.*

Proof. If $m \in L_n$, then $m = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}$, where $\beta_i \in \mathbb{N}$, $\beta_i \leq \alpha_i$, $i = \overline{1,k}$. For each $i \in \{1,2,...,k\}$, let $U_i \subseteq H_i$ be a subgroup of G having order $p_i^{\beta_i}$. Since $(H_i)_{i=\overline{1,k}}$ is a maximal system of $L(G)$, it results that $h_{L(G)}\left(\bigvee_{i=1}^{k} U_i\right) = h_{L(G)}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) = \sum_{i=1}^{k} h_{L(G)}(U_i) = \sum_{i=1} \beta_i$. This fact implies the equality $\text{ord}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) = \prod_{i=1}^{k} p_i^{\beta_i} = m$. Indeed, if we suppose that $\text{ord}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) \neq m$, then $\text{ord}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) = \prod_{i=1}^{k} p_i^{\gamma_i}$, where $\beta_i \leq \gamma_i \leq \alpha_i$, for any $i = \overline{1,k}$ and there exists $i_0 \in \{1,2,...,k\}$ such that $\beta_{i_0} < \gamma_{i_0}$ (this fact holds because $U_q \leq \left[\bigcup_{i=1}^{k} U_i\right]$ (so that $|U_q| / \left|\left[\bigcup_{i=1}^{k} U_i\right]\right|$) for any $q = \overline{1,k}$ and $(|U_q|, |U_{q'}|{=}1$ for $q \neq q'$). From property (iii), we obtain $h_{L(G)}\left(\left[\bigcup_{i=1}^{k} U_i\right]\right) \geq \sum_{i=1}^{k} \gamma_i \geq \sum_{i=1}^{k} \beta_i + 1$; contradiction.

Corollary 1. *For any finite group G of order n which satisfies the property that any two subgroups $U, V \in L(G)$ with $(|U|,|V|) = 1$ form a permutable pair, the function* $\text{ord} : L(G) \longrightarrow L_n$ *is onto.*

Proof. The statement results from Proposition 3 and Proposition 5 or, directly, making the next reasoning.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ be the decomposition of n as a product of prime factors and $m \in L_n$. Then m/n, therefore $m = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}$, where $\beta_i \in \mathbb{N}$, $\beta_i \leq \alpha_i$, $i = \overline{1,k}$. For each $i \in \{1,2,...,k\}$, let U_i be a subgroup of G having the order $p_i^{\beta_i}$. From hypothesis, it results tht the subgroup $\left[\bigcup_{i=1}^{k} U_i\right] \in L(G)$ has the order m.

Corollary 2. *For any finite nilpotent group G of order n, the function* ord : $L(G) \longrightarrow L_n$ *is onto.*

Proof. The statement results from Corollary 1, using the fact that, for a finite nilpotent group, any two subgroups of relative prime orders form a permutable pair.

Corollary 3. *For any finite abelian group G of order n, the function* ord : $L(G) \longrightarrow L_n$ *is onto.*

Proof. Since any abelian group is nilpotent, the statement results from Corollary 2.

References

[1] Birkhoff, G., *Lattice theory*, Amer. Math. Soc., Providence, R.I., 1967.

[2] Davis, A.C., *A characterization of complete lattices*, Pacific J. Math., n. 5, 1955, 311–319.

[3] Grätzer, G., *General lattice theory*, Academic Press, New York, 1978.

[4] Ore, O., *Structures and group theory*, I., Duke Math. J., n. 3, 1937, 149–174.

[5] Ore, O., *Structures and group theory*, II, Duke Math. J., n. 4, 1938, 247–269.

[6] Suzuki, M., *Group theory*, I, II, Springer Verlag, Berlin, 1980, 1985.

[7] Suzuki, M., *Structure of a group and the structure of its lattice of subgroups*, Springer Verlag, Berlin, 1956.

[8] Ştefănescu, M., *Introduction to group theory* (Romanian), Editura Universităţii "Al.I. Cuza", Iaşi, 1993.

[9] Tărnăuceanu, M., *On the subgroup lattice of an abelian finite group*, accepted for publication in Ratio Matematica.

[10] Tărnăuceanu, M., *On the subgroup lattice of a semidirect product of finite cyclic groups*, accepted for publication in Memoriile Ştiinţifice ale Academiei Române.

[11] Tărnăuceanu, M., *Fundamental group lattices*, accepted for publication in Mathematics and Informatics, Theory and Applications (F. Eugeni, H. Luchian eds.) , Editura Panfilius, Iaşi, 2003.