An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 73–90

KUROSH-AMITSUR RADICAL THEORY FOR GROUPS

B. J. Gardner

Abstract

This is a survey of some aspects of Kurosh-Amitsur radical theory for groups which touches on history, some links and contrasts with ring radicals and some recent appearances of group radicals in the mathematical literature.

1 Introduction

Radical theory for groups has not been studied intensively by radical theorists since the 1960s, which is unfortunate. There are striking similarities between radicals of groups and associative rings – **(ADS)** is universally valid, the lower radical construction stops at or before step ω and the intermediate classes have a neat description by accessible subgroups, every class defines an upper radical – and sometimes the group proofs are easier. In some ways life is harder in groups: for example we have nothing like an Andrunakievich Lemma. For whatever reason, there are some things we don't know about groups though the corresponding questions have long been answered for associative rings. We don't know whether there is a lower radical construction over a class of groups which requires infinitely many steps. We don't even know how many steps are required over the class of abelian groups: we only know that at least three are required. With such basic questions unanswered, and given the current state of scholarly activity in abstract radical theory, it seems like an appropriate time to re-direct some attention on to the group case. Further encouragement comes from a few relatively recent appearances of group radicals in other parts of mathematics.

Received: February, 2010
Accepted: September, 2010

This is not a complete survey of radical theory for groups. We have taken a few topics where we can illustrate the similarities with rings as well as some contrasts both in the availability of methods and the comparative paucity of answers to natural questions. We describe also the recent occurrences of group radicals in the literature mentioned above and in the final section we give an account of a method of representing group radicals by actions analogous to the representation of ring radicals by modules introduced by Andrunakievich and Ryabukhin. There are a few proofs, included for various reasons, but mostly we give references. On the whole we give references to primary sources (with some supplementary references) but for some more peripheral results we have been content to give secondary sources only. For further reading [23] and [47] are recommended.

There is some mention of the role of groups in the history of radical theory, and the intriguing connections between the prime radical and the lower radical over abelian groups disussed by Baer [6] (implicitly) and Shchukin [50], [53] (explicitly).

2 The standard results

Semi-simple classes of groups are hereditary and the lower radical construction terminates in at most ω steps.These and other things are more easily proved for groups than for rings partly because normality is defined by a type of automorphic invariance. (**ADS**) is almost immediate: if $\mathcal{R}$ is a radical class and $N \triangleleft G$ then then for each $g \in G$ we have $g\mathcal{R}(N)g^{-1} = \mathcal{R}(gNg^{-1}) = \mathcal{R}(N)$. In general if $K \triangleleft N \triangleleft G$ then the normal subgroup K^* generated by K is generated by normal (in K) subgroups isomorphic to K, namely the gKg^{-1}, $g \in G$. This leads to a characterization of the classes $\mathcal{M}_n$ in the Kurosh lower radical construction over a homomorphically closed class $\mathcal{M}$: G is in $\mathcal{M}_n$ if and only if for every non-trivial homomorphic image $\overline{G}$ we have $H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft H_{n-1} \triangleleft \overline{G}$ for some non-trivial $H_1 \in \mathcal{M}$. This characterization and the termination result are due to Shchukin [51]. For more details and a comparison with the corresponding proofs for associative rings (and other structures) see §2.1 of [23].

3 Some history

The usual account of the genesis of abstract radical theory begins in ring theory and proceds along the Wedderburn-Artin-Jacobson path, leaps to Amitsur and Kurosh in the 1950s and then describes how the general theory, once established, spread to other algebraic structures besides rings (though its separate incarnation for modules as torsion theory is also acknowledged). It is a little more complicated than this, and the role of group theory should be recognized.

Although Kurosh's fundamental paper on radicals of groups [35] appeared nine years after his rings and algebras paper [34], the influence of his earlier group-theoretic work and that of others of the Moscow School is significant. For instance, in a 1935 paper [33] concerning a generalization of the Jordan-Hölder Theorem to infinite groups, Kurosh introduced the notion of a normal series accessible in itself: a transfinite series

$$N_0 \triangleleft N_1 \triangleleft \ldots \triangleleft N_\alpha \triangleleft \ldots \triangleleft N_\gamma = G$$

such that for each α there are finitely many indices $\beta_1, \beta_2, \ldots, \beta_n$ such that

$$N_\alpha \triangleleft N_{\beta_1} \triangleleft N_{\beta_2} \triangleleft \ldots \triangleleft N_{\beta_n} \triangleleft G.$$

This concept was used in [35] and in subsequent papers on radicals of multioperator groups rings and algebras (see, e.g., [48], [4]). We note that [33] appeared about halfway between the seminal papers of Artin [5] and Jacobson [30].

It is reasonable to say that general radical theory grew out of the need to find generalizations, useful in larger classes of groups and rings, of, respectively, solvability and nilpotence in finite groups and nilpotence in rings with some sort of finiteness condition. (Of course it grew a long way!) Looked at appropritely, the two tasks can be seen as very closely related, a circumstance conducive to a parallel development in the group and ring cases, but there are important differences, which we might almost describe as "cultural" and which make it less surprising that radical theory for groups (at least in the Kurosh-Amitsur sense) went very quiet after the 1960s (though it had become "mainstream" enough to appear in Robinson's book [47] in 1972 and has made several more recent appearance in the literature).

There is a wide-ranging analogy between groups and rings in which the group operation (which we'll call multiplication) corresponds to the ring addition and the commutator operation $[\bullet, \bullet]$ to ring multiplication. The analogy involves the following correspondences:

normal subgroup $\leftrightarrow$ ideal
centre of group $\leftrightarrow$ two sided annihilator of ring
centralizer of element $\leftrightarrow$ two sided annihilator of element
nilpotent $\leftrightarrow$ nilpotent
hypercentral $\leftrightarrow$ (two sided) T-nilpotent
abelian group $\leftrightarrow$ zeroring.

(As the commutator is anticommutative, there is no "handedness" on the group side. A subgroup N is normal in a group G if and only if $[g, n] \in N$ for all

$n \in N, g \in G$ and $[n, g] \in N$ if and only if $[g, n] \in N$ and so it goes with other things, so we can associate two sided ring concepts with group concepts.) Because of associativity of ring multiplication, solvability doesn't correspond to anything useful. Things are different though for non-associative rings where the analogy still holds. (In Lie rings, practitioners even use terms like "centre" and "abelian" rather than "annihilator" and "zeroring".)

As noted, abelian groups correspond to zerorings, i.e.in effect to themselves, or, if you like,

$$\text{"}\{\text{groups}\} \cap \{\text{rings}\}=\{\text{abelian groups}\}\text{"}.$$

There is a certain curiosity value in knowing how close groups can be to rings without being zerorings. The following seems to be due to Levi [36].

Proposition 3.1. *In a group* $[\bullet, \bullet]$ *is distributive over multiplication if and only if* $[[x, y], z] = 1$ *for all* x, y, z.

Proof. Assuming distributivity, we have $xyzx^{-1}z^{-1}y^{-1} = x(yz)x^{-1}(yz)^{-1} = [x, yz] = [x, y][x, z] = xyx^{-1}y^{-1}xzx^{-1}z^{-1}$, so $zx^{-1}z^{-1}y^{-1} = x^{-1}y^{-1}xzx^{-1}z^{-1}$ and hence $1 = (zx^{-1}z^{-1}y^{-1})^{-1}x^{-1}y^{-1}xzx^{-1}z^{-1} = yzxz^{-1}x^{-1}y^{-1}xzx^{-1}z^{-1} = y[z, x]y^{-1}[x, z] = [y, [z, x]]$, and conversely, if $[y, [z, x]] = 1$ then working backwards we get $(zx^{-1}z^{-1}y^{-1})^{-1}(x^{-1}y^{-1}xzx^{-1}z^{-1}) = 1$, so $x^{-1}y^{-1}xzx^{-1}z^{-1} = zx^{-1}z^{-1}y^{-1}$ and thus $[x, y][x, z] = xyx^{-1}y^{-1}xzx^{-1}z^{-1} = xyzx^{-1}z^{-1}y^{-1} = [x, yz]$. □

Thus we have a nearring if and only if the group is nilpotent of class ≤ 2.(In fact the commutator is associative if and only if $[x, [y, z]] = 1$ for all x, y, z.)

Although nilpotent groups and rings correspond to each other they have been viewed rather differently. Nilpotent rings were seen as something like a contamination to be eliminated, whence the search for more general properties which could be factored out to produce a ring containing no trace of the new property and hence no nilpotence. Nilpotent (and solvable) groups were seen as nice. Even abelian groups were seen as nice but largely irrelevant – the proper way to view them is as modules – in contrast to zerorings which are utterly trivial from the point of view of ring theory. It is not really surprising then that group theorists attach great importance to the "Hirsch-Plotkin radical" – the largest locally nilpotent normal subgroup – though it is not a radical in the Kurosh-Amitsur sense. (We'll see shortly that it comes fairly close.)

4 Examples

We present a few examples of radical classes of groups in this section. Some classes we mention have more than one standard name and notation. We'll use "descriptive script" notation for classes and choose a name for each but occasionally indicate an alternative. We denote the lower radical class defined by a class $\mathcal{M}$ by $L(\mathcal{M})$; the classes $\mathcal{M}_\alpha$ are the classes from the Kurosh lower radical construction.

Example 4.1.

Let $\mathcal{AB}$ denote the class of abelian groups. Then $L(\mathcal{AB})$ is the class of *subsolvable* groups first discussed by Baer [6]. It was proved independently by Chang Wang-Hao [11] and Phillips and Combrink [45] that $L(\mathcal{AB}) \neq \mathcal{AB}_2$. Apart from its radical theoretic implication this discovery answered a question in group theory by showing that two classes of generalized solvable groups are distinct. We note that $L(\mathcal{AB})$ is strongly hereditary, since $\mathcal{AB}$ is.

Example 4.2.

Let S be a set of primes, $\mathcal{T}_S$ the class of S-*torsion* groups whose elements have orders in the monoid generated by S. When S is the set of all primes we just write $\mathcal{T}$ (the class of *torsion* groups). It is quite straightforward to show that all these classes are radical classes. In general it is far from true that $\mathcal{T}(G)$ is the set of elements with finite order (and the corresponding statements are true for each $\mathcal{T}_S$). These classes are strongly hereditary and a group belongs to a class $\mathcal{T}_S$ if and only if its finitely generated subgroups do so.

Example 4.3.

Let $\check{\mathcal{C}}\mathcal{C}$ be the class of *Chernikov complete* groups, those groups which are generated by *nth* powers for each $n \in \mathbb{Z}^+$. (As a generalization for each set S of primes we can consider the class $\check{\mathcal{C}}\mathcal{C}_S$ of groups generated by their *nth* powers where n varies over the monoid generated by S.) If $N \triangleleft G$ and $N, G/N \in \check{\mathcal{C}}\mathcal{C}$, then for $x \in G$ there exist $a_1, a_2, \dots, a_k \in G$ with $xN = {a_1}^n {a_2}^n \dots {a_k}^n N$ and then as $x({a_1}^n {a_2}^n \dots {a_k}^n)^{-1} \in N$ there are elements $b_1, b_2, \dots, b_m \in N$ with $x({a_1}^n {a_2}^n \dots {a_k}^n)^{-1} = {b_1}^n {b_2}^n \dots {b_m}^n$ and thus $x = {b_1}^n \dots {a_k}^n$. Hence G belongs to $\check{\mathcal{C}}\mathcal{C}$ which is therefore closed under extensions. The rest of the proof that $\check{\mathcal{C}}\mathcal{C}$ is a radical class uses similarly straightforward arguments and the same goes for the classes $\check{\mathcal{C}}\mathcal{C}_S$. These radical classes are closed under unions of directed sets of subgroups but are not hereditary as can be seen from the divisible abelian groups.

Example 4.4.

Let $\mathcal{AF}$ denote the class of *antifinite* groups, those without proper subgroups of finite index. A group is antifinite if and only if it has no proper normal subgroup of finite index (see, e.g., [29], p.44) so $\mathcal{AF}$ is the upper radical defined by the class of finite groups. It is strict. It is clear that $\check{\mathcal{C}}\mathcal{C} \subseteq \mathcal{AF}$.

In the same way each $\check{\mathcal{C}}\mathcal{C}_S$ is a subclass of the upper radical class defined by the groups with orders in the monoid generated by S. The first inclusion is proper as is the second if S contains a prime ≥ 665. These statements follow from the known existence of infinite simple groups of prime exponent. See [47], Part 2, pp.123-124, but note the improved bound 665 arising from the improved solution to the Burnside problem [2]. The existence of such simple groups can also be obtained from [1]. If $S = \{2\}$ or $\{3\}$ the second inclusion is *not* proper. Commutators are products of squares, so if $G(2)$ is the (normal) subgroup of G generated by the squares and $G \neq G(2)$ then $G/G(2)$ is an abelian 2-group and so has a finite homomorphic image. The second Engel word $[[x, y], y]$ is a product of cubes and so $G/G(3)$ (with the obvious meaning) is a 2-Engel group. These are nilpotent of class ≤ 3 (see [47], Part2, p.45) and it follows that likewise $\check{\mathcal{C}}\mathcal{C}$ coincides with the upper radical class defined by the finite 3-groups. Explicit formulae for the products of powers alluded to are given in [25], for example.

Example 4.5.

The class $\mathcal{LF}$ of *locally finite* groups is a (strongly hereditary) radical class. The tricky part is proving that $\mathcal{LF}$ is closed under extensions (Shmidt [55], Theorem 6). A proof is given in [23], p. 22.

5 Wreath products

An important source of examples and counterexamples and of demonstrations that groups are "radically different" from rings is the *wreath product*, a standard construction in group theory which may not be universally familiar and which we'll therefore briefly describe.

What we discuss here is called the *regular* wreath product to distinguish it from other variants, but we'll suppress the adjective. Let G and H be groups and let $B = \prod G_h, h \in H$, be a direct (cartesian) product of copies G_h of G. Then H acts on itself as index set *via* the left regular representation $x \mapsto hx$ and hence on B: $(g_x)_{x\in H}$ is sent to the element which has g_x as its hx- coordinate and hence $g_{h^{-1}x}$ as its x-coordinate, i.e. we have $(g_x)_{x\in H} \mapsto (g_{h^{-1}x})_{x\in H}$. The wreath product $G \wr H$ is a semidirect product of B and H with

$$((a_x)_{x\in H}, h)((b_x)_{x\in H}, k) = ((a_x)_{x\in H}(b_{h^{-1}x})_{x\in H}, hk) = ((a_x b_{h^{-1}x})_{x\in H}, hk).$$

The identity is $(1, 1)$ and $((a_x)_{x\in H}, h)^{-1} = ((a^{-1}{}_{hx})_{x\in H}, h^{-1})$. The *restricted wreath product* is obtained if we replace the cartesian product with the restricted product $\{(a_x)_{x\in H} : a_x = 1 \text{ for almost all } x\}$.

We use the wreath product to show that there is nothing in group theory analogous to the Andrunakievich Lemma or the following results.

Accessible idempotent subrings are ideals.

If $I \triangleleft J \triangleleft A$ (rings) and J/I is semiprime, then $I \triangleleft A$.

Proposition 5.1. *For every group G with $|G| > 1$ there is a chain $H \triangleleft K \triangleleft M$ of groups with $K/H \cong G$ and $H \ntriangleleft M$.*

Proof. Let $\langle a \rangle = \{a, e\}$ be a cyclic group of order 2. In $G \wr \langle a \rangle$ we have, for $g \neq 1$, $((1,1),a)((g,1),e)((1,1),a)^{-1} = ((1,1),a)((g,1),e)((1,1),a^{-1}) = ((1,g),a)((1,1),a^{-1}) = ((1,g),e)$. Thus $G_a \triangleleft G_a \times G_e = B \triangleleft G \wr \langle a \rangle$, but $G_a \ntriangleleft G \wr \langle a \rangle$, while $B/G_a \cong G_e \cong G$. $\square$

This demonstrates a big difference between groups and associative rings. There appears to be nothing like the wreath product in the latter case, though for non-associative rings and algebras the situation could be more complicated: there is a kind of wreath product for Lie algebras [44]. It should be pointed out that consideration of (group) wreath products provided a large part of the motivation for the non-associative ring constructions in [18].

6 Radicals (and non-radicals) related to nilpotence

There are many varieties of non-associative rings in which the locally nilpotent rings do not form a Kurosh-Amitsur radical class. For groups the story is the same. Even finite nilpotent groups are not closed under extensions (look at dihedral groups for example) so naturally a class of "generalized nilpotent" groups, a class whose finite members are the finite nilpotent groups, will not be a radical class, though we can ask how close it comes to being one. One measure of closeness is the number of steps required in its lower radical construction. If we take the p-groups in such a class we might hope for better luck.

A group is

locally nilpotent if its finitely generated subgroups are nilpotent;

Baer nilpotent [6] if every finite homomorphic image of every subgroup is nilpotent.

Let $\mathcal{LN}, \mathcal{BN}$ denote, respectively, the classes of locally nilpotent and Baer nilpotent groups . Neither of these is a radical class.

The class of p-groups in $\mathcal{LN}$ *is* a radical class. This is a consequence of

Proposition 6.1. *A p-group is locally nilpotent if and only if it is locally finite.*

Proof. Let G be a nilpotent p-group with upper central series

$$1 \subseteq Z(G) \subseteq Z_2(G) \subseteq \ldots \subseteq Z_n(G) = G.$$

Then each $Z_{i+1}(G)/Z_i(G)$ is an abelian p-group and hence locally finite, so by **4.5** G is locally finite. It follows that every locally nilpotent p-group is locally finite. Conversely, in a locally finite p-group every finitely generated subgroup is a finite p-group and hence nilpotent. □

Using **4.2** and **4.5** we get

Corollary 6.2. *The class of locally nilpotent p-groups is a radical class for all primes p.*

Every p-group is Baer nilpotent, so trivially the Baer nilpotent p-groups form a radical class. We have

$$\mathcal{LN} \subset \mathcal{BN}$$

(proper inclusion: see [47], Part 2, p. 9). Although $\mathcal{LN}$ is not a radical class we have $L(\mathcal{LN}) = \mathcal{LN}_2$. This was in effect proved by Plotkin [46]. His argument shows that

$L(\mathcal{M}) = \mathcal{M}_2$ *for every homomorphically closed class* $\mathcal{M}$ *of groups such that*

$$N, K \lhd G \textit{ and } N, K \in \mathcal{M} \Rightarrow NK \in \mathcal{M} \textit{ and}$$
$$\mathcal{M} \textit{ is closed under unions of directed sets of subgroups.}$$

For a proof with this generality see [23], p.89.

The effect of radicals on nilpotent groups and the nature of the intersection of a radical class with the class of nilpotent groups is completely controlled by abelian groups. This is made more precise in our next result. (Note that we can take the nilpotent groups as a universal class for radical theory.)

Theorem 6.3. (Warfield [58]; see also [23], p.89.) *For a radical class $\mathcal{R}$ of abelian groups, let $\tilde{\mathcal{R}}$ denote the class of nilpotent groups with a finite ascending invariant series with factors in $\mathcal{R}$. Then*

(i) the correspondence $\mathcal{R} \mapsto \tilde{\mathcal{R}}$ defines a bijection from the radical classes of abelian groups to those of nilpotent groups and

(ii) $\tilde{\mathcal{R}} = L(\mathcal{R})$ (constructed in the class of nilpotent groups) for all $\mathcal{R}$. Also

(iii) if N is nilpotent and $\mathcal{U}$ is a radical class of nilpotent groups, then $N \in \mathcal{U}$ if and only if $N/[N, N] \in \mathcal{U}$.

If a radical class $\mathcal{U}$ of nilpotent groups contains a cyclic group or $\mathbb{Q}$ then fairly clearly it contains a non-abelian group. In the contrary case, all abelian groups in $\mathcal{U}$ are divisible torsion groups. It turns out then that all groups in $\mathcal{U}$ are abelian. We'll deduce this shortly from something more general. The class

$\check{\mathcal{C}}\mathcal{C}$ of Chernikov complete groups contains the class $\mathcal{C}$ of *complete* groups, those in which every element is an nth power, for all n. A *hypercentral* group is one with an ascending central series. It seems likely that the behaviour towards hypercentral groups of radicals in general and the Chernikov complete radical in particular might be tractable but complicated enough to be interesting. The following results of Chernikov are pertinent.

(i) All hypercentral Chernikov complete groups are complete ([12], Theorem 10).

(ii) All hypercentral antifinite groups are Chernikov complete ([12], Theorems 1,2).

(iii) Every (Chernikov) complete hypercentral torsion group is abelian ([12], Theorem 5).

The ring analogue of hypercentrality is T-nilpotence. Radicals of T-nilpotent rings are, like those of nilpotent rings, determined by zerorings [17], so we have an analogue for rings of Warfield's theorem which can be generalized to T-nilpotent rings. It's not clear whether we can generalize Warfield's theorem itself in group theory, but (iii) is an important result pointing in this direction (and it generalizes the result stated above concerning radical classes of nilpotent groups defined by divisible torsion groups).

In the same paper [12] Chernikov gives an example of a Chernikov complete group which is not complete, based on a construction of Shmidt [54]. This group is an extension of $\mathbb{Z}(p^\infty)$ by itself. Robinson [47], Part 2, pp. 123-124 uses $\mathbb{Z}(p^\infty) \wr \mathbb{Z}(p^\infty)$ to the same effect. Thus even the lower radical class $L(\mathbb{Z}(p^\infty))$ contains groups which are not hypercentral. *This* is not very ring-like: the lower radical class defined by the zeroring on $\mathbb{Z}(p^\infty)$ contains only the divisible p-zerorings. An interesting problem: **sort out the relationship between the following radical classes:** $\check{\mathcal{C}}\mathcal{C}, L(\mathcal{C}), L(\{\mathbb{Q} \cup \{\mathbb{Z}p^\infty) : p \text{ prime }\})$. One can also ask about universal classes in which the complete groups form a radical class; cf. [59].

While the classes $\check{\mathcal{C}}\mathcal{C}$ and $\mathcal{C}$ can be viewed as generalizations of divisible abelian groups, they are very much more general. For example there are are finitely generated complete groups [22]. (Complete and Chernikov complete groups are often called *radicable* and *semiradicable* respectively in the literature.)

It is natural that there should be a degree of similarity between some generalizations of nilpotent rings and some generalizations of solvable and nilpotent groups. We end this section with a look at the prime radical and $L(\mathcal{AB})$. Inasmuch as abelian groups correspond to zerorings their lower radical classes could be said to correspond also. There are other ways of describing the prime radical however, and we shall consider two here. Both of them have correspondents for groups.

In 1949 McCoy [38] considered the prime radical as the intersection of all prime ideals and obtained a description in terms of m-systems. Two years later Levitzki [35] used the related idea of an *m-sequence*: a sequence $a_1, a_2, \dots, a_n, \dots$ such that there is another sequence $b_1, b_2, \dots, b_n, \dots$ with $a_{n+1} = a_n b_n a_n$ for each n. An element a belongs to the prime radical if and only if every m-sequence starting with a becomes zero, so that in particular a ring is in the prime radical class if and only if all its m-sequences become zero. Four years after this Baer [6] showed that (in our terminology) a group is in $\mathcal{AB}_2$ if and only if for all sequences $a_1, a_2, \dots, a_n, \dots$ for which there is another sequence $b_1, b_2, \dots, b_n, \dots$ such that $a_{n+1} = [a_n, [a_n, b_n]]$ for every n, eventually $a_n = 1$. We have preserved the original notation, but the difference in appearance is superficial (the commutator operation being anticommutative). Baer's sequences correspond to m-sequences under our group-ring correspondence. Baer characterizes $\mathcal{AB}_2$ by these sequences, but of course the prime radical class *is* the second step class over the zerorings.

The prime radical is the intersection of the prime ideals (or the whole ring). Shchukin [50], following on from work of Schenkman [49] and Murata [40], sought a "prime radical" for groups, using a concept of prime normal subgroup in accord with the group-ring analogy. Thus a normal subgroup P of G is *prime* if for $A, B \triangleleft G$ with $[A, B] \subseteq P$ we must have $A \subseteq P$ or $B \subseteq P$. This also led to $\mathcal{AB}_2$. Again the parallel with rings is striking. This story has, in a sense, a happy ending. Later Shchukin [53] showed that all $L(\mathcal{AB})$-semi-simple groups are subdirect products of prime $L(\mathcal{AB})$-semi-simple groups, so that $L(\mathcal{AB})$ is "special". **Are there any other interesting group radicals which are special in this sense?**

7 Verbal radicals, amenability, categories ...

A variety which is closed under extensions is a radical class (in contexts where everything makes sense) and sometimes a semi-simple class as well. There are no non-trivial extension-closed varieties of groups: one source of this result is the Neumanns' results [41] on the structure of the set of group varieties with respect to the Mal'tsev product. There are not even any non-trivial group varieties $\mathcal{V}$ satisfying the weaker condition that if N and K are in $\mathcal{V}$ and both are normal in some group G then $NK \in \mathcal{V}$. This result is due to Shores [56]. (Thus no group variety satisfies the conditions in the middle of p.7.) Nevertheless there are ways of getting group radicals which are related to varieties. We'll examine two such.

Every variety determines a strict upper radical class and these were studied by Shchukin [52]. Let $\mathcal{V}$ be a group variety, $\mathcal{R}_\mathcal{V}$ its upper radical class. A radical arising in this way is called a *verbal radical*. Let $G(\mathcal{V})$ denote the verbal subgroup of G with respect to $\mathcal{V}$. Let $G(\mathcal{V}^0) = G$, $G(\mathcal{V}^{\alpha+1}) = G(\mathcal{V}^\alpha)(\mathcal{V})$ for all ordinals α and let $G(\mathcal{V}^\beta) = \bigcap_{\gamma<\beta} G(\mathcal{V}^\gamma)$ for limit ordinals β. For each G there is some ordinal λ for which $G(\mathcal{V}^{\lambda+1}) = G(\mathcal{V}^\lambda)$ and then $\mathcal{R}_\mathcal{V}(G) = G(\mathcal{V}^\lambda)$ [52]. An example of a verbal radical is the class $\mathcal{R}_{\mathcal{AB}}$ of *perfect groups*. The radicals defined by atomic varieties recently introduced by Martynov [39] in a general setting and studied by Kornev [31] and by Kornev and Pavlova [32] in groups, rings and group rings also fit in here.

Verbal radicals have arisen more recently in algebraic topological contexts. Berrick [7] has found a significance for surjective homomorphisms $f : G \to H$ such that $f(\mathcal{R}_{\mathcal{AB}}(G)) = \mathcal{R}_{\mathcal{AB}}(H)$. Casacuberta, Rodríguez and Scevenels [10] established a connection between verbal radicals and homotopy theory, in the process establishing a rather interesting result concerning the former. *For every variety $\mathcal{V}$ of groups there is a locally free group $L_\mathcal{V} \in \mathcal{R}_\mathcal{V}$ such that for every group G, $\mathcal{R}_\mathcal{V}(G)$ is generated by the images of the homomorphisms $L_\mathcal{V} \to G$.* See also [9].

We'll call a radical class *local* if it satisfies the condition

$$G \in \mathcal{R} \Leftrightarrow \text{each finitely generated subgroup of } G \text{ is in } \mathcal{R}.$$

Then $\mathcal{R}$ is local if and only if it is strongly hereditary and closed under unions of directed sets of subgroups. E.g. $\mathcal{T}$ is local.

A group G is *amenable* if it has a finitely additive left invariant measure μ such that $\mu(G) = 1$. The notion of amenability arose from the Banach-Tarski "paradox". Amenable groups G are the "non-paradoxical" ones and the measure of G can be transferred to sets on which G acts, which are accordigly not paradoxical with respect to G. For background and full details of all this see Wagon [57]. Its significance for our present discussion comes from

Theorem 7.1. *The class $\mathcal{AG}$ of amenable groups is a local radical class.*

This theorem is an amalgam of results of von Neumann [42], Day [15] and Følner [16]. It is known that finite and abelian groups are amenable, while free groups of rank > 1 are not. Now it's clear that intersections of local radical classes are local radical classes, so there is a smallest local radical class containing the finite and abelian groups. Its members are called *elementary groups* and we'll call the class $\mathcal{EG}$. The groups without free subgroups of rank > 1 also form a local radical class [21]. Following Day [15], we'll call this class $\mathcal{NF}$. We have

$$\mathcal{EG} \subset \mathcal{AG} \subset \mathcal{NF}$$

and each inclusion is known to be strict: again see [57] for details. Ching Chou [13] gave a transfinite construction of $\mathcal{EG}$. In [19] there is a transfinite construction for the smallest local radical class containing a given class of rings and this transfers straightforwardly to groups (cf. [22]). Recently Osin [43] has considered local radical classes of groups under the name *elementary classes*, using the Ching Chou construction to build them. (This construction differs from that of [19].) It was proved in [19] that local radical classes are the same things as extension-closed *locally equational classes* in the sense of Hu [28]. This is the other link between radicals and extension-closed varieties adverted to at the beginning of this section.

We note finally that there has been much work in recent years on radicals in categories with a strong emphasis on the extension to non-abelian categories of torsion theory as previously developed in certain abelian categories. (For example see [8], [14] and their references.) Groups form a semi-abelian category and might thus be expected to feature at least as a source of examples in this work (and the same goes for rings, for that matter). However there is a culturally imposed restriction that the assignment of radicals should be functorial, and this requires strictness. Now many of the more familiar radical classes of groups satisfy extra conditions: thus $\mathcal{T}, L(\mathcal{AB})$ and $L(\mathcal{LN})$ are strongly hereditary. There is never going to be any interaction between these and "torsion theory" as the only strongly hereditary strict radical classes of groups (in fact the only hereditary strict ones [20]) are the class of all groups and the class of one element groups. Note, though, that verbal radicals *are* strict.

8 Radicals and representations

Andrunakievich and Ryabukhin [3] showed that every radical of rings could be represented by a class of modules . In this final section we briefly describe how group radicals can similarly be represented in terms of group actions. We omit the proofs: they can be adapted fairly mechanically from those of the corresponding ring results which are given in detail in [25], pp.118 ff.

For a group G, a $G-set$ is a set $E \neq \varnothing$ on which G acts by permutations, i.e. for which there is a homomorphism $F : G \to S_E$, the group of permutations of E, but for $g \in G, x \in E$ we write gx rather than $F(g)(x)$. If E is a G-set, we let $Ker_G(E) = \{g \in G : gx = x \forall x \in E\}$. If E is a G-set, $N \triangleleft G$ and $N \subseteq Ker_G(E)$ then E is a G/N-set with respect to $gNx = gx$ and $Ker_{G/N}(E) = Ker_G(E)/N$. If $N \triangleleft G$ and E is a G/N-set, then by defining $gx = gNx$ we make E a G-set with $N \subseteq Ker_G(E)$.

For each group G let Σ_G be a class of G-sets, $\Sigma = \bigcup_G \Sigma_G$. Let $Ker(\Sigma_G) = \bigcap\{Ker_G(E) : E \in \Sigma_G\}$. Here are some conditions Σ might satisfy.

(M1) If $E \in \Sigma_{G/N}$, then $E \in \Sigma_G$.

(M2) If $E \in \Sigma_G$ and $N \subseteq Ker_G(E)$, then $E \in \Sigma_{G/N}$.

(M3) If $Ker(\Sigma_G) = \{1\}$ then $\Sigma_N \neq \varnothing$ for every $N \triangleleft G$ with $|N| > 1$.

(M4) If $\Sigma_N \neq \varnothing$ for each $N \triangleleft G$ with $|N| > 1$, then $Ker(\Sigma_G) = \{1\}$.
(The notation in (M1) and (M2) has the same meaning as above.)

Let $F(\Sigma) = \{G : \exists E \in \Sigma_G \text{ with } Ker_G(E) = \{1\}\}$. If Σ satisfies (M3), then $F(\Sigma)$ is a regular class.

Proposition 8.1. *For a class Σ satisfying* (M1), (M2) and (M3)*, let $\mathcal{R}_\Sigma = \{G : \Sigma_G = \varnothing\}$. Then $\mathcal{R}_\Sigma$ is the upper radical class defined by $F(\Sigma)$.*

Proposition 8.2. *If Σ satisfies* (M1), (M2), (M3) and (M4)*, then*

(i) $\mathcal{R}_\Sigma(G) = 1$ if and only if G is a subdirect product of groups in $F(\Sigma)$ and

(ii) $\mathcal{R}_\Sigma(G) = Ker(\Sigma_G)$ for every group G.

As in the ring case, for every radical we can find a defining Σ but it is too big to be practically useful. Note that for the proof of this claim we need a faithful G-set for each group G and the regular action of G on itself will serve. In the ring case the corresponding role of a universal faithful module was played by the standard unital extension.

We do not have any useful examples of classes satisfying (M1)-(M4), but this matter has not been explored very much. Probably (M4) will prove to be elusive, but there may be interesting things to find among the classes satisfying (M1)-(M3) as in [22]. Another possibility would be to look at classes in which the G-sets have some extra structure and the action is defined by a homomorphism from G into automorphisms of some kind. Here is one example of this kind of thing.

Let $\mathcal{K}$ be a class of groups and for each group G let Σ_G consist of all groups in $\mathcal{K}$ with all possible actions of G by automorphisms. Then

$$\mathcal{R}_\Sigma = \{G : \text{ every } K \rtimes G \text{ with } K \in \mathcal{K} \text{ is a direct product } \}.$$

Acknowledgement. The author is grateful to Laci Kovács for helpful correspondence.

References

[1] S. I. Adyan, *The simplicity of periodic products of groups* (in Russian), Dokl. Akad. Nauk SSSR 241 (1978), 745-748.

[2] S. I. Adian, *The Burnside Problem and Identities of Groups*, Berlin-Heidelberg-New York: Springer 1979.

[3] V. A. Andrunakievich and Yu. M. Ryabukhin, *Modules and radicals*, Soviet Math. Dokl. 3(1964), 728-732.

[4] V. A. Andrunakievich and Yu. M. Ryabukhin, *Torsion and Kurosh chains in algebras*, Trans. Moscow Math. Soc. 29(1973), 17-47.

[5] E. Artin, *Zur Theorie der hyperkomplexen Zahlen*, Abh. Math. Sem. Univ. Hamburg 5(1927), 251-260.

[6] R. Baer, *Nilgruppen*, Math. Z. 62(1955), 402-437.

[7] A. J. Berrick, *Group epimorphisms preserving perfect radicals, and the plus-construction*, Lecture Notes in Math. 1046 (*Algebraic K-theory,number theory, geometry and analysis (Bielefeld, 1982)*, 1-12.

[8] D. Bourn and M. Gran, *Torsion theories in homological categories*, J. Algebra 305(2006), 18-47.

[9] C. Casacuberta and A. Descheemaeker, *Relative group completions*, J. Algebra 285(2005), 451-469.

[10] C. Casacuberta, J. L. Rodríguez and D. Scevenels, *Singly generated radicals associated with varieties of groups*, *London Math. Soc. Lecture Note Ser. 260 (Groups St. Andrews 1997 in Bath, I)* , 202-210.

[11] Chang Wang-Hao, *On the minimal radical class over a [sic] class of abelian groups*, Soviet Math. Dokl. 4(1963), 552-555.

[12] S. N. Chernikov, *Complete groups having an ascending central series* (in Russian), Mat. Sb. 18(60)(1946), 397-422.

[13] Ching Chou, *Elementary amenable groups*, Illinois J. Math. 24(1980), 396-407.

[14] M. M. Clementino, D. Dikranjan, W. Tholen, *Torsion theories and radicals in normal categories*, J. Algebra 305(2006), 98-129.

[15] M. M. Day, *Amenable semigroups*, Illinois J. Math. 8(1964), 100-111.

[16] E. Følner, *On groups with full Banach mean value*, Math. Scand. 3(1955), 243-254.

[17] B. J. Gardner, *Some aspects of T-nilpotence*, Pacific J. Math. 53(1974), 117-130.

[18] B. J. Gardner, *Some degeneracy and pathology in non-associative radical theory*, Annales Univ. Sci. Budapest. Sect. Math. 22-23(1979-80), 65-74.

[19] B. J. Gardner, *Radical properties defined locally by polynomial identities I*, J. Austral. Math. Soc. Ser. A 27(1979), 257-273.

[20] B. J. Gardner, *A remark on strict radical classes of groups*, Acta Math. Acad. Sci. Hungar. 38(1981), 61.

[21] B. J. Gardner, *Radical properties defined by the absence of free subobjects*, Annales Univ. Sci. Budapest. Sect. Math. 25(1982), 53-60.

[22] B. J. Gardner, *Radicals and varieties*, *Colloq. Math. Soc. János Bolyai* 38(1985), 93-133.

[23] B. J. Gardner, *Radical Theory*, Harlow: Longman, 1989.

[24] B. J. Gardner, *Strict radicals and endomorphism rings*, Acta Math. Hung. 124 (2009), 371-383.

[25] B. J. Gardner and R. Wiegandt, *Radical Theory of Rings*, New York: Marcel Dekker, 2004.

[26] V. S. Guba, *A finitely-generated complete group* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. 50(1986), 883-924.

[27] G. Havas, *Commutators in groups expressed as products of powers*, Comm. Algebra 9(1981), 115-129.

[28] Tah-kai Hu, *Locally equational classes of universal algebras*, Chinese J. Math. 1(1973), 143-165.

[29] I. M. Isaacs, *Algebra A graduate Course*, Pacific Grove: Brooks/Cole, 1993.

[30] N. Jacobson, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. 67(1945), 300-320.

[31] A. I. Kornev, *Complete radicals of some group rings*, Siberian Math. J. 48(2007), 857-862.

[32] A. I. Kornev and T. V. Pavlova, *Characterization of one radical of group rings over finite prime fields*, Siberian J. Math. 45(2004), 504-510.

[33] A. Kurosch, *Eine Verallgemeinerung des Jordan-Hölderschen Satzes*, Math. Ann. 11(1935), 13-18.

[34] A. G. Kurosh, *Radicals of rings and algebras*, *Colloq. Math. Soc. János Bolyai* 6(1971), 297-314.

[35] A. G. Kurosh, *Radicals in the theory of groups*, *Colloq. Math. Soc. János Bolyai* 6(1971), 271-296.

[36] F. W. Levi, *Groups in which the commutator operation satisfies certain algebraic conditions*, J. Indian Math. Soc. 6(1942), 87-97.

[37] J. Levitzki, *Prime ideals and the lower radical*, Amer. J. Math. 73(1951), 25-29.

[38] N. H. McCoy, *Prime ideals in general rings*, Amer. J. Math. 71(1949), 823-833.

[39] L. M. Martynov, *On primary and reduced varieties of monoassociative algebras*, Siberian Math. J. 42(2001), 91-98.

[40] K. Murata, *On nilpotent-free multiplicative systems*, Osaka Math. J. 14(1962), 53-70.

[41] B. H. Neumann, H. Neumann and P. M. Neumann, *Wreath products and varieties of groups*, Math. Z. 80(1962), 44-62.

[42] J. von Neumann, *Zur allgemeinen Theorie des Masses*, Fund. Math. 13(1929), 73-116.

[43] D. V. Osin, *Elementary classes of groups*, Math. Notes 72(2002), 84-93.

[44] V. M. Petrogradsky, Yu. P. Razmyslov and E. O. Shishkin, *Wreath products and Kaluzhnin-Krasner embedding for Lie algebras*, Proc. Amer. Math. Soc. 135(2007), 625-636.

[45] R. E. Phillips and C. R. Combrink, *A note on subsolvable groups*, Math. Z. 92(1966), 349-352.

[46] B. I. Plotkin, *Radical groups*, Amer. Math. Soc. Transl. (2)17, 9-28.

[47] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Berlin-Heidelberg-New York: Springer, 1972.

[48] Yu. M. Ryabukhin, *Radicals of* Ω*-groups I* (in Russian),Mat. Issled. 3, vyp. 2(1968), 123-160.

[49] E. Schenkman, *The similarity between the properties of ideals in commutative rings and the properties of normal subgroups of groups*, Proc. Amer. Math. Soc. 9(1958), 375-381.

[50] K. K. Shchukin, *The* RI^**-solvable radical* (in Russian), Mat. Sb. 52(94)(1960), 1021-1031.

[51] K. K. Shchukin, *On the theory of radicals in groups* (in Russian), Sibirsk. Mat. Zh. 3(1962), 932-942.

[52] K. K. Shchukin, *On verbal radicals of groups*, Kishinevsk. Gosuniv. Uchenye Zapiski 82(1965), 97-99.

[53] K. K. Shchukin,*Approximation by prime groups* (in Russian), *Studies in General Algebra, No. 1 (in Russian)*, Kishinev. Gosuniv., 1968, 110-119.

[54] O. Yu. Shmidt, *On infinite special groups* (in Russian), Mat.Sb. 8(50)(1940), 363-375.

[55] O. Yu. Shmidt, *Infinite solvable groups* (in Russian), Mat. Sb. 17(59)(1945), 145-162.

[56] T. S. Shores, *A note on products of normal subgroups*, Canad. Math. Bull. 12(1969), 21-23.

[57] S. Wagon, *The Banach-Tarski Paradox*, Cambridge, etc.: Cambridge Uni. Press, 1985.

[58] R. B. Warfield, *Nilpotent Groups*, Berlin-Heidelberg-New York: Springer, 1976.

[59] B.A.F. Wehrfritz, *The divisible radical of a group*, Cent. Eur. J. Math. 7 (2009), 387-394.

University of Tasmania,
Private Bag 37, Hobart Tas. 7001, Australia
e-mail: gardner@hilbert.maths.utas.edu.au