An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 53–58

THE EXISTENCE OF CRITICAL VALUES IN AN ABSTRACT PERTURBATION PROBLEM

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Abstract

The aim of this paper is to obtain conditions for the existence of a critical value for the perturbed function $f^\varepsilon = f + \varepsilon g$ in a given interval, in the presence of Palais-Smale condition.

1 Introduction

It is well-known and often applied the fact that, in the presence of Palais-Smale type conditions, important information concerning the critical points of a function can be obtained. In some problems it is difficult to study the function directly and we need to perturb it.

Let M be a C^2-Finsler manifold and let $f : M \to \mathbb{R}$ be of C^1 class, bounded from below. Perturb it with a function $g : M \to \mathbb{R}$ of C^1 class, such that $g > 0$. Define $f^\varepsilon = f + \varepsilon g$, with $\varepsilon > 0$ enough small. Assume that f and f^ε satisfy the Palais-Smale condition on M and the set of critical values of f has the form $\{s_1, s_2, \ldots, s_k\}$. A basic result in critical point theory is the second deformation theorem, which shows that, in the above conditions, the sublevel set $M_{s_{p-1}}(f)$ is a strong deformation retract of $M_{s_p}(f)$, for $p = \overline{2,k}$. Then the n^{th} relative homology group $H_n(M_{s_p}(f), M_{s_{p-1}}(f)) = 0$, for $p = \overline{2,k}$ and for any n.

With suitable assumptions on the function g, we can obtain conditions such that the n^{th} relative homology group of some sublevel sets of the perturbed

Key Words: homology group, Palais-Smale condition, critical point, deformation retract.
2010 Mathematics Subject Classification: 57R70, 58E05, 46M18.
Received: May, 2009
Accepted: January, 2010

function is nontrivial. This fact implies the existence of a critical value of f^ε in a given interval.

We mention that the use of relative homology of sublevel sets in some perturbation problems comes back to A. Marino and G. Prodi, see [2].

2 Preliminaries

Let M be a C^1-Finsler manifold and $f \in C^1(M, \mathbb{R})$. A point $p \in M$ is critical for f if $df(p) = 0$ and $c \in \mathbb{R}$ is a critical value of f if there exists $p \in M$ such that $df(p) = 0$ and $f(p) = c$. The critical set of f is $C[f] = \{p \in M |\, df(p) = 0\}$ and $C_c[f] = C[f] \cap f^{-1}(c)$ is the critical set of level c of f. We call $f^{-1}(c)$ the set of level c of f and $M_c(f) = \{p \in M | f(p) \leq c\}$ the set of sublevel c. Recall that f satisfies the Palais-Smale condition on M if any sequence (x_n) in M such that $(f(x_n))$ is bounded and $df(x_n) \to 0$ has a convergent subsequence.

In this paper we need the second deformation lemma, see R. Palais [4] and Kc. Chang [1].

Theorem 2.1. *Let M be a C^2-Finsler manifold and let $f : M \to \mathbb{R}$ be of C^1 class. Suppose that f satisfies the Palais-Smale condition on M and the interval $[a, b]$ does not contain critical values of f. Then $M_a(f)$ is a strong deformation retract of $M_b(f)$.*

Denote by $H_n(B, A)$ the n^{th} relative singular homology group of the pair (B, A) with real coefficients, where $A \subset B$. Recall that for a deformation retract A' of A we have, for any n, $H_n(A, A') = 0$. See, for instance, W. Massey [3].

In this section we prove the following:

Lemma 2.1. *Let A, X, B, A', Y, B' be topological spaces such that $A \subset X \subset B \subset A' \subset Y \subset B'$ and $H_n(A', X) = 0$, $H_n(B', A') = 0$, for any n.*

(i) There exists an injective homomorphism $H_n(X, A) \to H_n(B, A)$.

(ii) There exists an injective homomorphism $H_n(A', B) \to H_n(Y, B)$.

(iii) There exists an injective homomorphism $H_n(Y, X) \to H_n(Y, B)$.

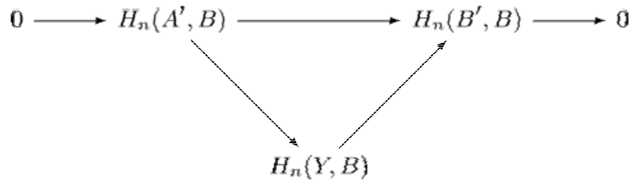
(iv) There exists an injective homomorphism $H_n(A', A) \to H_n(Y, A)$.

Proof. (i) We use the exact sequence of the triple (A', X, A) and the inclusions $X \subset B \subset A'$ and we obtain the following diagram:

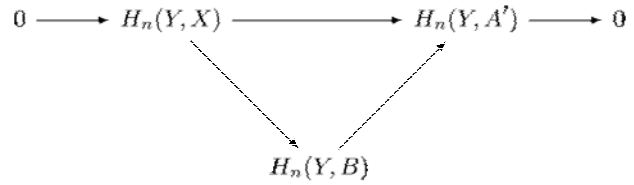
$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(X,A) & \xrightarrow{\quad i_n \quad} & H_n(A',A) & \longrightarrow & 0 \\ & & & \searrow^{\alpha_n} \quad \nearrow^{\beta_n} & & & \\ & & & H_n(B,A) & & & \end{array}$$

Because $\beta_n \circ \alpha_n = i_n$ and i_n is an isomorphism, it follows that α_n is injective.

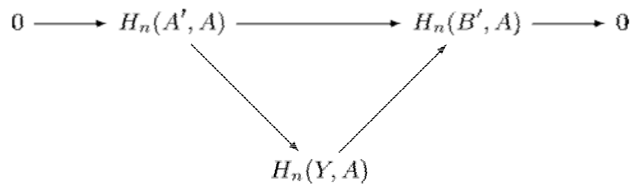
(ii) The conclusion follows by using the exact sequence of (B', A', B) and the inclusions $A' \subset Y \subset B'$:



(iii) We use the exact sequence of the triple (Y, A', X) and the inclusions $X \subset B \subset A'$, which give the following diagram:



(iv) We use the exact sequence of (B', A', A) and the inclusions $X \subset B \subset A'$:



□

3 The main result

Let $f^\varepsilon = f + \varepsilon g$, where M is a C^2-Finsler manifold, $f, g : M \to \mathbb{R}$ are of C^1 class, f is bounded from below, $g > 0$ on M and $\varepsilon > 0$ is enough small. Assume that f and f^ε satisfy the Palais-Smale condition on M and $f(C[f]) = \{s_1, s_2, \ldots, s_k\}$, $a < s_1 < \ldots < s_k < b$.

It is obvious that $M_a(f) \subseteq M_{s_1}(f) \subseteq \ldots \subseteq M_{s_k}(f) \subseteq M_b(f)$ and any interval (s_{p-1}, s_p), $p = \overline{2,k}$, is non-critical for f, i.e. it does not contain critical values. By using Theorem 2.1, the sublevel set $M_{s_{p-1}}(f)$ is a strong deformation retract of $M_{s_p}(f)$, for $p = \overline{2,k}$. Then we get that $H_n(M_{s_p}(f), M_{s_{p-1}}(f)) = 0$, for $p = \overline{2,k}$ and for any n.

In order to simplify the problem, we consider that there exists only one critical value of f into the interval (a, b) and denote it by s. Then $H_n(M_s(f), M_a(f)) = 0$ and $H_n(M_b(f), M_s(f)) = 0$, for any n.

In the sequel, we want to obtain information about $M_a(f^\varepsilon), M_s(f^\varepsilon)$ and $M_b(f^\varepsilon)$.

We work in the presence of the following hypothesis:

$$a + \varepsilon g(x) \leq s \leq b - \varepsilon g(x), \ x \in M.$$

It is easy to see that the following inclusions hold:

$$M_a(f^\varepsilon) \subseteq M_a(f) \subseteq M_s(f^\varepsilon) \subseteq M_s(f) \subseteq M_b(f^\varepsilon) \subseteq M_b(f).$$

Lemma 2.1 implies the following statements:

Proposition 3.1. *There exist the following injective homomorphisms:*
(i) $H_n(M_a(f), M_a(f^\varepsilon)) \to H_n(M_s(f^\varepsilon), M_a(f^\varepsilon))$;
(ii) $H_n(M_s(f), M_s(f^\varepsilon)) \to H_n(M_b(f^\varepsilon), M_s(f^\varepsilon))$;
(iii) $H_n(M_b(f^\varepsilon), M_a(f)) \to H_n(M_b(f^\varepsilon), M_s(f^\varepsilon))$;
(iv) $H_n(M_s(f), M_a(f^\varepsilon)) \to H_n(M_b(f^\varepsilon), M_a(f^\varepsilon))$.

We give now the main result of the paper:

Theorem 3.1. *The following statements hold:*

(i) *If there exists m such that $H_m(M_a(f), M_a(f^\varepsilon)) \neq 0$, then $H_m(M_s(f^\varepsilon), M_a(f^\varepsilon)) \neq 0$; consequently $C[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, s]) \neq \emptyset$.*

(ii) *If there exists m such that $H_m(M_s(f), M_s(f^\varepsilon)) \neq 0$, then $H_m(M_b(f^\varepsilon), M_s(f^\varepsilon)) \neq 0$; consequently $C[f^\varepsilon] \cap (f^\varepsilon)^{-1}([s, b]) \neq \emptyset$.*

(iii) *If there exists m such that $H_m(M_b(f^\varepsilon), M_a(f)) \neq 0$, then $H_m(M_b(f^\varepsilon), M_s(f^\varepsilon)) \neq 0$; consequently $C[f^\varepsilon] \cap (f^\varepsilon)^{-1}([s, b]) \neq \emptyset$.*

(iv) *If there exists m such that $H_m(M_s(f), M_a(f^\varepsilon)) \neq 0$, then $H_m(M_b(f^\varepsilon), M_a(f^\varepsilon)) \neq 0$; consequently $C[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) \neq \emptyset$.*

Proof. (i) Assume that $C[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, s]) = \emptyset$; then $[a, s]$ is a noncritical interval for f^ε and Theorem 2.1 implies that $H_n(M_s(f^\varepsilon), M_a(f^\varepsilon)) = 0$ for any n. But Proposition 3.1 gives $H_m(M_s(f^\varepsilon), M_a(f^\varepsilon)) \neq 0$.

(ii), (iii) and (iv) follows by the same argument. □

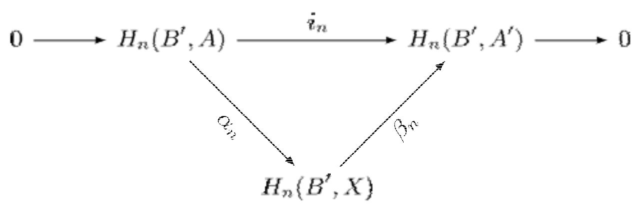
4 An auxiliary result

The following lemma, which was obtained by the author during the preparation of this paper, can be useful in the study of other perturbation problems.

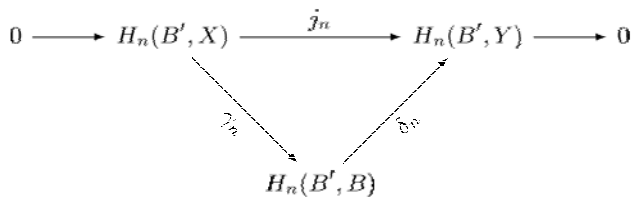
Lemma 4.1. *Let A, X, B, A', Y, B' be topological spaces such that $A \subset X \subset B \subset A' \subset Y \subset B'$. If $H_n(A', A) = H_n(Y, X) = 0$, then there exists an injective homomorphism*

$$H_n(B', A) \to H_n(B', B).$$

Proof. We have the following diagram, which follows from the exact sequence of (B', A', A) and the inclusions $A \subset X \subset A'$:



We conclude that i_n is an isomorphism. On the other hand, we can write $i_n = \beta_n \circ \alpha_n$. Then α_n is injective. The same argument can be used for the diagram



which follows from the exact sequence of (B', Y, X) and the inclusions $X \subset B \subset Y$. Then $j_n = \delta_n \circ \gamma_n$ and the fact that j_n is an isomorphism assure the injectivity of δ_n.

We obtain that $h_n = \gamma_n \circ \alpha_n$ is injective. □

Remark 4.1. The ideea of this lemma in related to the following result of [2]:

if $A \subset X \subset B \subset A' \subset Y \subset B'$ and $H_n(B, A) = H_n(B', A') = 0$, then there exists an injective homomorphism $H_n(A', A) \to H_n(Y, X)$.

The authors use this result in order to prove some stability properties of critical values in presence of Palais-Smale type conditions.

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