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ON SOME SUFFICIENT CONDITIONS FOR UNIVALENCE

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Abstract

In this paper the method of subordination chains is used to establish some sufficient conditions for univalence for analytic functions defined in the open unit disk.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

In order to prove our main result we need a brief summary of the method of subordination chains.

A function $L(.,t) : \mathbb{U} \to \mathbb{C}, t \geq 0$ is said to be a *subordination chain* or a *Loewner chain* if:

(i) $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$.

(ii) $L(z,t) \prec L(z,s)$ for all $0 \leq t \leq s < \infty$, where the symbol "$\prec$" stands for subordination.

The following result is due to Ch. Pommerenke [6].

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Theorem 1.1. *Let $L(z,t) = a_1(t)z + \dots$ be an analytic function in $\mathbb{U}$ for all $t \geq 0$. Suppose that:*

(i) $L(z,t)$ is a locally absolutely continuous function of $t \in [0,\infty)$, locally uniform with respect to $z \in \mathbb{U}$;

(ii) $a_1(t)$ is a complex valued continuous function on $[0,\infty)$ such that $a_1(t) \neq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and

$$\left\{ \frac{L(.,t)}{a_1(t)} \right\}_{t\geq 0}$$

is a normal family of functions in $\mathbb{U}$;

(iii) there exists an analytic function $p : \mathbb{U} \times [0,\infty) \to \mathbb{C}$ satisfying $\Re p(z,t) > 0$ for all $(z,t) \in \mathbb{U} \times [0,\infty)$ and

$$\frac{\partial L}{\partial t}(z,t) = p(z,t) z \frac{\partial L}{\partial z}(z,t)\ ,\ z \in \mathbb{U}\ ,\ a.e\ \ t \geq 0.$$

Then, for all $t \geq 0$, the function $L(z,t)$ is a subordination chain.

2 Sufficient conditions for univalence

In this section, making use of Theorem 1.1, we obtain various conditions for univalence which generalize some known results.

Theorem 2.1. *Consider $f \in \mathcal{A}$. Let m be a positive real number and let α be a complex number such that $\alpha \neq 1$, $\left| \frac{\alpha}{1-\alpha} \right| < 1$. If the inequalities*

$$\left| \frac{\alpha}{f'(z) - \alpha} - \frac{m-1}{2} \right| < \frac{m+1}{2} \tag{2}$$

and

$$\left| \frac{\alpha |z|^{m+1} + (1 - |z|^{m+1}) z f''(z)}{f'(z) - \alpha} - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3}$$

are satisfied for all $z \in \mathbb{U}$, then the function f is univalent in $\mathbb{U}$.

Proof. Define the function $L(.,t) : \mathbb{U} \to \mathbb{C}\ , t \geq 0$

$$L(z,t) = f(e^{-t}z) + \left(e^{mt}z - e^{-t}z\right)\left(f'(e^{-t}z) - \alpha\right). \tag{4}$$

We will prove that the function $L(z,t)$ satisfies the conditions of Theorem 1.1.

Since the function $f(e^{-t}z)$ is analytic in $\mathbb{U}$, it is easy to see that the function $L(z,t)$ is also analytic in $\mathbb{U}$ for all $t \geq 0$. We have

$$\frac{\partial L}{\partial t}(z,t) = -e^{-t}z\left[\alpha + (e^{mt} - e^{-t})zf''(e^{-t}z)\right] + me^{mt}z\left[f'(e^{-t}z) - \alpha\right].$$

It follows that $\left|\frac{\partial L}{\partial t}(z,t)\right|$ is bounded on $[0,T]$, for any fixed $T > 0$ and $z \in \mathbb{U}$. Therefore, the function $L(z,t)$ is locally absolutely continuous on $[0,\infty)$, locally uniform with respect to $z \in \mathbb{U}$.

Elementary calculations give

$$a_1(t) = e^{mt}[\alpha e^{-(m+1)t} + 1 - \alpha].$$

From $\alpha \neq 1$ and $\left|\frac{\alpha}{1-\alpha}\right| < 1$, it follows easily that $a_1(t) \neq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$.

Let $r \in (0,1)$ and let $K = \overline{\{z \in \mathbb{C} : |z| \leq r\}}$. Since the function $L(z,t)$ is analytic in $\mathbb{U}$, there exists $M > 0$ such that $|L(z,t)| \leq Me^{mt}$ for $z \in K$ and $t \geq 0$. Also, for $t \geq 0$, it is easy to see that there exists $N > 0$ such that $|a_1(t)| > Ne^{mt}$. It follows that $\left|\frac{L(z,t)}{a_1(t)}\right| \leq \frac{M}{N}$ for $z \in K$ and $t \geq 0$. Thus, $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ is a normal family in $\mathbb{U}$.

Consider the function $p(z,t)$ defined by

$$p(z,t) = \frac{\partial L}{\partial t}(z,t)/z\frac{\partial L}{\partial z}(z,t).$$

In order to prove that the function $p(z,t)$ is analytic and has positive real part in $\mathbb{U}$, we will show that the function

$$w(z,t) = \frac{1 - p(z,t)}{1 + p(z,t)} \tag{5}$$

is analytic in $\mathbb{U}$ and

$$|w(z,t)| < 1\,, \text{for all } z \in \mathbb{U}\,, t \geq 0. \tag{6}$$

Elementary calculations give

$$w(z,t) = \frac{2}{m+1}F(z,t) - \frac{m-1}{m+1},$$

where

$$F(z,t) = e^{-(m+1)t} \cdot \frac{\alpha + (e^{mt} - e^{-t})zf''(e^{-t}z)}{f'(e^{-t}z) - \alpha}.$$

The inequality (2.5) is therefore equivalent to

$$\left|F(z,t)-\frac{m-1}{2}\right|<\frac{m+1}{2}\ ,z\in\mathbb{U}\ ,t\geq 0. \tag{7}$$

If $t=0$ the last inequality yields

$$\left|\frac{\alpha}{f'(z)-\alpha}-\frac{m-1}{2}\right|<\frac{m+1}{2}.$$

Define $G(z,t)=F(z,t)-\dfrac{m-1}{2}$.

Since $|e^{-t}z|\leq e^{-t}<1$ for all $z\in\bar{\mathbb{U}}=\{z\in\mathbb{C}:|z|\leq 1\}$ and $t>0$ it follows that $G(z,t)$ is an analytic function in $\bar{\mathbb{U}}$. Making use of the maximum modulus principle we obtain that for each fixed $t>0$, there exists $\theta\in\mathbb{R}$ such that :

$$|G(z,t)|<\max_{|z|=1}|G(z,t)|=|G(e^{i\theta},t)|\ ,z\in\mathbb{U}.$$

Let $u=e^{-t}e^{i\theta}$. We have $|u|=e^{-t}$ and $e^{-(m+1)t}=(e^{-t})^{m+1}=|u|^{m+1}$. Therefore,

$$|G(e^{i\theta},t)|=\left|\frac{\alpha|u|^{m+1}+(1-|u|^{m+1})uf''(u)}{f'(u)-\alpha}-\frac{m-1}{2}\right|.$$

Inequality (2.2), from the hypothesis, yields

$$|G(e^{i\theta},t)|\leq\frac{m+1}{2}. \tag{8}$$

From (2.1) and (2.7) it follows that the inequality (2.6) is satisfied for all $z\in\mathbb{U}$ and $t\geq 0$.

Since all the conditions of Theorem 1.1 are satisfied we obtain that the function $L(z,t)$ is a subordination chain. If $t=0$, we have $L(z,0)=f(z)$ and thus, the function f is univalent in $\mathbb{U}$. □

Remark 2.1. *Some particular cases of Theorem 2.1 are the following:*

(i) When $m=1$ *and* $\alpha=0$ *inequality (2.2) becomes*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right|\leq 1\ ,z\in\mathbb{U}$$

which is Becker's condition of univalence [2].

(ii) A result due to N. N. Pascu [4] is also obtained when $m=1$.

The condition (2.2) of Theorem 2.1 can be replaced with a simpler one.

Corollary 2.1. *Consider $f \in \mathcal{A}$. Let m be a positive real number and let α be a complex number such that $\alpha \neq 1$ and $\left|\frac{\alpha}{1-\alpha}\right| < 1$. If*

$$\left|\frac{\alpha}{f'(z)-\alpha} - \frac{m-1}{2}\right| < \frac{m+1}{2}\ , z \in \mathbb{U}$$

and

$$\left|\frac{zf''(z)}{f'(z)-\alpha} - \frac{m-1}{2}\right| \leq \frac{m+1}{2}\ , z \in \mathbb{U} \tag{9}$$

then the function f is univalent in $\mathbb{U}$.

Proof. Making use of (2.1) and (2.8) we obtain

$$\left|\frac{\alpha|z|^{m+1} + (1-|z|^{m+1})zf''(z)}{f'(z)-\alpha} - \frac{m-1}{2}\right| =$$

$$= \left||z|^{m+1}\left(\frac{\alpha}{f'(z)-\alpha} - \frac{m-1}{2}\right) + (1-|z|^{m+1})\left(\frac{zf''(z)}{f'(z)-\alpha} - \frac{m-1}{2}\right)\right| <$$

$$< |z|^{m+1}\frac{m+1}{2} + (1-|z|^{m+1})\frac{m+1}{2} = \frac{m+1}{2}.$$

The conditions of Theorem 2.1 being satisfied it follows that the function f is univalent in $\mathbb{U}$. □

Remark 2.2. *Consider $\alpha < 0$. By elementary calculations we obtain that the inequality (2.1) is equivalent to*

$$\Re f'(z) > \frac{m}{\alpha(m+1)}|f'(z)|^2\ , z \in \mathbb{U}.$$

If in the last inequality we let $\alpha \to -\infty$ we obtain that

$$\Re f'(z) \geq 0.$$

Since (2.8) holds true for $\alpha \to -\infty$ it follows from Corollary 2.1 that the function f is univalent in $\mathbb{U}$.

Therefore, we can conclude that the univalence criterion due to Alexander-Noshiro-Warschawski [1], [3], [8] is a limit case of Corollary 2.1.

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