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EXPRESSIONS OF SOLUTIONS FOR A CLASS OF DIFFERENTIAL EQUATIONS

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Abstract

In this paper we study the solutions of the following class of difference equation

$$x_{n+1} = \frac{x_{n-8}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}}, \quad n = 0, 1, ...,$$

where initial values are non zero real numbers.

1 Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations. This is especially true in the case of Lyapunov theory of stability. Nonetheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation may have a phenomena often called appearance of "ghost" solutions or existence of chaotic orbits that can only happen for higher order differential equations and the theory of difference equations is interesting in itself.

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The applications of the theory of difference equations is rapidly increasing to various fields such as numerical analysis, control theory, finite mathematics and computer science. Thus, there is every reason for studying the theory of difference equations as a well deserved discipline.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example: Aloqeili [2] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [4-6] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Elabbasy et al. [9] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}.$$

Elabbasy et al. [10] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [11] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Simsek et al. [21] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Other related results on rational difference equations can be found in refs. [1–28].

Similar to the references above, in this paper we obtain the solutions of the following rational difference equations

$$x_{n+1} = \frac{x_{n-8}}{\pm 1 \pm x_{n-2} x_{n-5} x_{n-8}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial values $x_{-j},\ (j=0,1,...,8)$ are arbitrary non zero real numbers.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2 First Equation

In this section we give a specific form of Eq. (1) in the form

$$x_{n+1} = \frac{x_{n-8}}{1 + x_{n-2}x_{n-5}x_{n-8}}, \quad n = 0, 1, ..., \tag{3}$$

where the initial values are arbitrary non zero real numbers.

Theorem 2.1. *Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of Eq.(3). Then for $n = 0, 1, ...$*

$$\begin{aligned}
x_{9n-8} &= k\prod_{i=0}^{n-1}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right), \qquad x_{9n-7} = h\prod_{i=0}^{n-1}\left(\frac{1+3iheb}{1+(3i+1)\,heb}\right),\\
x_{9n-6} &= g\prod_{i=0}^{n-1}\left(\frac{1+3iadg}{1+(3i+1)\,adg}\right), \qquad x_{9n-5} = f\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)kfc}{1+(3i+2)\,kfc}\right),\\
x_{9n-4} &= e\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)heb}{1+(3i+2)\,heb}\right), \qquad x_{9n-3} = d\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)adg}{1+(3i+2)\,adg}\right),\\
x_{9n-2} &= c\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)kfc}{1+(3i+3)\,kfc}\right), \qquad x_{9n-1} = b\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)heb}{1+(3i+3)\,heb}\right),\\
x_{9n} &= a\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)adg}{1+(3i+3)\,adg}\right),
\end{aligned}$$

where $x_{-8}=k,\ x_{-7}=h,\ x_{-6}=g,\ x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_{-0}=a.$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned}
x_{9n-17} &= k\prod_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right), \qquad x_{9n-16}=h\prod_{i=0}^{n-2}\left(\frac{1+3iheb}{1+(3i+1)\,heb}\right),\\
x_{9n-15} &= g\prod_{i=0}^{n-2}\left(\frac{1+3iadg}{1+(3i+1)\,adg}\right), \qquad x_{9n-14}=f\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)kfc}{1+(3i+2)\,kfc}\right),\\
x_{9n-13} &= e\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)heb}{1+(3i+2)\,heb}\right), \qquad x_{9n-12}=d\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)adg}{1+(3i+2)\,adg}\right),\\
x_{9n-11} &= c\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)kfc}{1+(3i+3)\,kfc}\right), \qquad x_{9n-10}=b\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)heb}{1+(3i+3)\,heb}\right),\\
x_{9n-9} &= a\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)adg}{1+(3i+3)\,adg}\right).
\end{aligned}$$

Now, it follows from Eq.(3) that

$$\begin{aligned}
x_{9n-8} &= \frac{x_{9n-17}}{1+x_{9n-11}x_{9n-14}x_{9n-17}}\\
&= \frac{k\prod\limits_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right)}{1+c\prod\limits_{i=0}^{n-2}\left(\frac{1+(3i+2)kfc}{1+(3i+3)kfc}\right)f\prod\limits_{i=0}^{n-2}\left(\frac{1+(3i+1)kfc}{1+(3i+2)kfc}\right)k\prod\limits_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+1)kfc}\right)}
\end{aligned}$$

$$= \frac{k\prod\limits_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right)}{1+kfc\prod\limits_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+3)\,kfc}\right)} = \frac{k\prod\limits_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right)}{1+\frac{kfc}{1+(3n-3)\,kfc}}$$

$$= \frac{k\prod\limits_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right)}{\frac{1+(3n-3)\,kfc+kfc}{1+(3n-3)\,kfc}} = \frac{k\prod\limits_{i=0}^{n-2}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right)(1+(3n-3)\,kfc)}{1+(3n-2)\,kfc}.$$

Hence, we have

$$x_{9n-8}=k\prod_{i=0}^{n-1}\left(\frac{1+3ikfc}{1+(3i+1)\,kfc}\right).$$

Similarly

$$\begin{aligned} x_{9n-4} &= \frac{x_{9n-13}}{1+x_{9n-7}x_{9n-10}x_{9n-13}} \\ &= \frac{e\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)heb}{1+(3i+2)\,heb}\right)}{1+h\prod_{i=0}^{n-1}\left(\frac{1+3iheb}{1+(3i+1)heb}\right)b\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)heb}{1+(3i+3)heb}\right)e\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)heb}{1+(3i+2)heb}\right)} \end{aligned}$$

$$= \frac{e\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)heb}{1+(3i+2)\,heb}\right)}{1+\left(\frac{heb}{1+(3n-2)\,heb}\right)} = e\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)heb}{1+(3i+2)\,heb}\right)\left(\frac{1+(3n-2)\,heb}{1+(3n-1)\,heb}\right).$$

Hence, we have

$$x_{9n-4} = e\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)heb}{1+(3i+2)\,heb}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2.2. *Eq.(3) has one equilibrium point which is the number zero.*

Proof: For the equilibrium points of Eq.(3), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^3}.$$

Then we have

$$\overline{x} + \overline{x}^4 = \overline{x},$$

or,

$$\overline{x}^4 = 0.$$

Thus the equilibrium point of Eq.(3) is $\overline{x} = 0$.

Theorem 2.3. *Every positive solution of Eq.(3) is bounded.*

Proof: Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of Eq.(3). It follows from Eq.(3) that

$$x_{n+1} = \frac{x_{n-8}}{1+x_{n-2}x_{n-5}x_{n-8}} \leq x_{n-8}.$$

Then

$$x_{n+1} \leq x_{n-8} \quad \text{for all} \quad n \geq 0.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and so is bounded from above by $M = \max\{x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (3).

Example 1. We assume $x_{-8} = 8,\ x_{-7} = 7,\ x_{-6} = 5,\ x_{-5} = 3,\ x_{-4} = 4,$ $x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 6,\ x_0 = 6$. See Fig. 1.

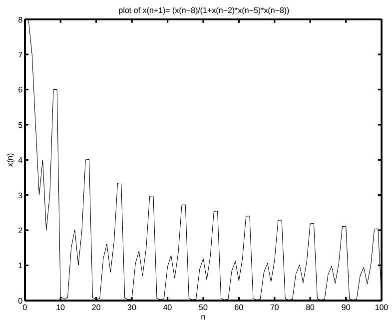

Figure 1: Plot for example 1

Example 2. We assume $x_{-8} = 1,\ x_{-7} = 1.9,\ x_{-6} = -5,\ x_{-5} = 3,\ x_{-4} = -4,\ x_{-3} = 7,\ x_{-2} = 2.1,\ x_{-1} = -1.3,\ x_0 = 1.7$. See Fig. 2.

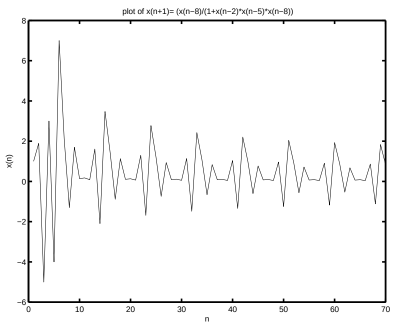

Figure 2: Plot for example 2

3 Second Equation

In this section we obtain the solution of the second equation in the form

$$x_{n+1} = \frac{x_{n-8}}{-1 + x_{n-2}x_{n-5}x_{n-8}}, \quad n = 0, 1, ..., \tag{4}$$

where the initial values are arbitrary non zero real numbers with $x_{-8}x_{-5}x_{-2} \neq 1,\ x_{-7}x_{-4}x_{-1} \neq 1,\ x_{-6}x_{-3}x_0 \neq 1$.

Theorem 3.1. *Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of Eq.(4). Then every solution of Eq.(4) is periodic with period eighteen and for $n = 0, 1, ...$*

$$\begin{aligned}
x_{18n-8} &= k, \quad x_{18n-7} = h, \quad x_{18n-6} = g, \quad x_{18n-5} = f, \quad x_{18n-4} = e, \\
x_{18n-3} &= d, \quad x_{18n-2} = c, \quad x_{18n-1} = b, \quad x_{18n} = a, \\
x_{18n+1} &= \frac{k}{-1+kfc}, \quad x_{18n+2} = \frac{h}{-1+heb}, \quad x_{18n+3} = \frac{g}{-1+adg}, \qquad (5) \\
x_{18n+4} &= f\left(-1+kfc\right), \quad x_{18n+5} = e\left(-1+heb\right), \quad x_{18n+6} = d\left(-1+adg\right), \\
x_{18n+7} &= \frac{c}{-1+kfc}, \quad x_{18n+8} = \frac{b}{-1+heb}, \quad x_{18n+9} = \frac{a}{-1+adg},
\end{aligned}$$

where $x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{18n-26} &= k, \quad x_{18n-25} = h, \quad x_{18n-24} = g, \quad x_{18n-23} = f, \quad x_{18n-22} = e, \\
x_{18n-21} &= d, \quad x_{18n-20} = c, \quad x_{18n-19} = b, \quad x_{18n-18} = a, \\
x_{18n-17} &= \frac{k}{-1+kfc}, \quad x_{18n-16} = \frac{h}{-1+heb}, \quad x_{18n-15} = \frac{g}{-1+adg}, \\
x_{18n-14} &= f\left(-1+kfc\right), \quad x_{18n-13} = e\left(-1+heb\right), \quad x_{18n-12} = d\left(-1+adg\right), \\
x_{18n-11} &= \frac{c}{-1+kfc}, \quad x_{18n-10} = \frac{b}{-1+heb}, \quad x_{18n-9} = \frac{a}{-1+adg}.
\end{aligned}$$

Now, it follows from Eq.(4) that

$$\begin{aligned}
x_{18n-8} &= \frac{x_{18n-17}}{-1 + x_{18n-11}x_{18n-14}x_{18n-17}} = \frac{\frac{k}{-1+kfc}}{-1 + \frac{c}{-1+kfc} f\left(-1+kfc\right)\frac{k}{-1+kfc}} \\
&= \frac{\frac{k}{-1+kfc}}{-1 + \frac{kfc}{-1+kfc}} = \frac{k}{-1\left(-1+kfc\right)+kfc}.
\end{aligned}$$

Hence, we have

$$x_{18n-8} = k.$$

Similarly

$$x_{18n+2} = \frac{x_{18n-7}}{-1 + x_{18n-1}x_{18n-4}x_{18n-7}} = \frac{h}{-1 + heb}.$$

Similarly, one can easily prove the other relations. Thus, the proof is completed.

Theorem 3.2. *Eq.(4) has two equilibrium points which are* $0, \sqrt[3]{2}$.

Proof: For the equilibrium points of Eq.(4), we can write

$$\overline{x} = \frac{\overline{x}}{-1 + \overline{x}^3}.$$

Then we have

$$-\overline{x} + \overline{x}^4 = \overline{x},$$

or,

$$\overline{x}(\overline{x}^3 - 2) = 0.$$

Thus the equilibrium points of Eq.(4) are $0, \sqrt[3]{2}$.

Theorem 3.3. *Eq.(4) has a periodic solution of period nine iff* $kfc = heb = adg = 2$ *and will be taken the form* $\{k, h, g, f, e, d, c, b, a, k, h, g, f, e, d, c, b, a, ...\}$.

Proof: First suppose that there exists a prime period nine solution

$$k, h, g, f, e, d, c, b, a, k, h, g, f, e, d, c, b, a, ...,$$

of Eq.(4), we see from Eq.(5) that

$$\begin{aligned} k &= \frac{k}{-1 + kfc}, \qquad h = \frac{h}{-1 + heb}, \qquad g = \frac{g}{-1 + adg}, \\ f &= f\left(-1 + kfc\right), \qquad e = e\left(-1 + heb\right), \qquad d = d\left(-1 + adg\right), \\ c &= \frac{c}{-1 + kfc}, \qquad b = \frac{b}{-1 + heb}, \qquad a = \frac{a}{-1 + adg}, \end{aligned}$$

or,

$$(-1 + kfc)^n = 1, \quad (-1 + heb)^n = 1, \quad (-1 + adg)^n = 1.$$

Then

$$kfc = heb = adg = 2.$$

Second assume that $kfc = heb = adg = 2$. Then we see from Eq.(5) that

$$\begin{array}{rcllll} x_{18n-8} & = & k, & x_{18n-7} = h, & x_{18n-6} = g, & x_{18n-5} = f, \\ x_{18n-4} & = & e, & x_{18n-3} = d, & x_{18n-2} = c, & x_{18n-1} = b, \\ x_{18n} & = & a, & x_{18n+1} = k, & x_{18n+2} = h, & x_{18n+3} = g, \\ x_{18n+4} & = & f, & x_{18n+5} = e, & x_{18n+6} = d, & x_{18n+7} = c, \\ x_{18n+8} & = & b, & x_{18n+9} = a. & & \end{array}$$

Thus we have a periodic solution of period nine and the proof is complete.
Numerical examples

Example 3. We consider $x_{-8} = 1.3,\ x_{-7} = 1,\ x_{-6} = 2,\ x_{-5} = -3,\ x_{-4} = 1.2,\ x_{-3} = 0.7,\ x_{-2} = 1.6,\ x_{-1} = 1.8,\ x_0 = 1.7$. See Fig. 3.

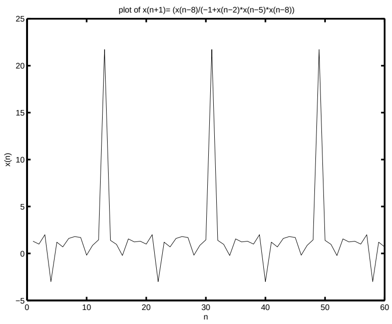

Figure 3: Plot for example 3

Example 4. We assume $x_{-8} = 3,\ x_{-7} = 5,\ x_{-6} = 4,\ x_{-5} = 1.4,\ x_{-4} = 0.2,\ x_{-3} = 0.5,\ x_{-2} = 1/(2.1),\ x_{-1} = 2,\ x_0 = 1$. See Fig. 4.

The following cases can be proved similarly.

4 Third Equation

In this section we get the solution of the third following equation

$$x_{n+1} = \frac{x_{n-8}}{1 - x_{n-2}x_{n-5}x_{n-8}}, \quad n = 0, 1, ..., \tag{6}$$

where the initial values are arbitrary non zero real numbers.

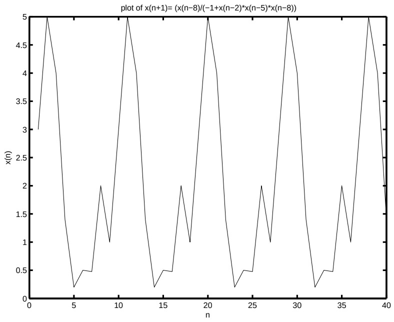

Figure 4: Plot for example 4

Theorem 4.1. *Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of Eq.(6). Then for $n = 0, 1, ...$*

$$x_{9n-8} = k\prod_{i=0}^{n-1}\left(\frac{1-3ikfc}{1-(3i+1)\,kfc}\right), \qquad x_{9n-7} = h\prod_{i=0}^{n-1}\left(\frac{1-3iheb}{1-(3i+1)\,heb}\right),$$

$$x_{9n-6} = g\prod_{i=0}^{n-1}\left(\frac{1-3iadg}{1-(3i+1)\,adg}\right), \qquad x_{9n-5} = f\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)kfc}{1-(3i+2)\,kfc}\right),$$

$$x_{9n-4} = e\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)heb}{1-(3i+2)\,heb}\right), \qquad x_{9n-3} = d\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)adg}{1-(3i+2)\,adg}\right),$$

$$x_{9n-2} = c\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)kfc}{1-(3i+3)\,kfc}\right), \qquad x_{9n-1} = b\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)heb}{1-(3i+3)\,heb}\right),$$

$$x_{9n} = a\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)adg}{1-(3i+3)\,adg}\right),$$

where $x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Theorem 4.2. *Eq.(6) has a unique equilibrium point which is the number zero.*

Example 5. We suppose $x_{-8} = 9,\ x_{-7} = 1.5,\ x_{-6} = 2.4, x_{-5} = 1.4,\ x_{-4} = 0.3,\ x_{-3} = 1.8,\ x_{-2} = 2,\ x_{-1} = 2.3,\ x_0 = 1.7.$ See Fig. 5.

Example 6. We assume $x_{-8} = 9,\ x_{-7} = 5,\ x_{-6} = 4, x_{-5} = 6,\ x_{-4} = 3,\ x_{-3} = 8,\ x_{-2} = 2,\ x_{-1} = -3,\ x_0 = 7.$ See Fig. 6.

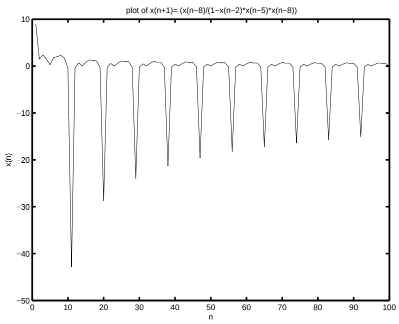

Figure 5: Plot for example 5

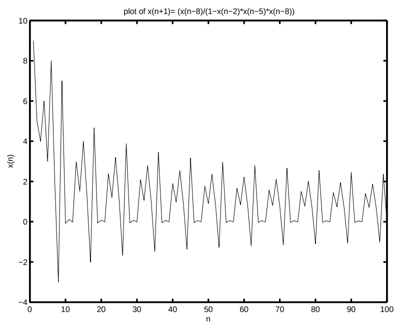

Figure 6: Plot for example 6

5 Fourth Equation

Here we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-8}}{-1 - x_{n-2}x_{n-5}x_{n-8}}, \quad n = 0, 1, ..., \tag{7}$$

where the initial values are arbitrary non zero real numbers with $x_{-8}x_{-5}x_{-2} \neq -1,\ x_{-7}x_{-4}x_{-1} \neq -1,\ x_{-6}x_{-3}x_0 \neq -1.$

Theorem 5.1. *Let $\{x_n\}_{n=-8}^{\infty}$ be a solution of Eq.(7). Then every solution of*

Eq.(7) is periodic with period eighteen and for $n = 0, 1, ...$

$$
\begin{aligned}
x_{18n-8} &= k, \quad x_{18n-7} = h, \quad x_{18n-6} = g, \quad x_{18n-5} = f, \quad x_{18n-4} = e,\\
x_{18n-3} &= d, \quad x_{18n-2} = c, \quad x_{18n-1} = b, \quad x_{18n} = a,\\
x_{18n+1} &= \frac{k}{-1-kfc}, \quad x_{18n+2} = \frac{h}{-1-heb}, \quad x_{18n+3} = \frac{g}{-1-adg},\\
x_{18n+4} &= f\left(-1-kfc\right), \quad x_{18n+5} = e\left(-1-heb\right), \quad x_{18n+6} = d\left(-1-adg\right),\\
x_{18n+7} &= \frac{c}{-1-kfc}, \quad x_{18n+8} = \frac{b}{-1-heb}, \quad x_{18n+9} = \frac{a}{-1-adg},
\end{aligned}
$$

where $x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Theorem 5.2. *Eq.(7) has two equilibrium points which are* $0, \sqrt[3]{-2}$.

Theorem 5.3. *Eq.(7) has a periodic solutions of period nine iff* $kfc = heb = adg = -2$ *and will be taken the form* $\{k, h, g, f, e, d, c, b, a, k, h, g, f, e, d, c, b, a, ...\}$.

Example 7. We suppose $x_{-8} = 1.9,\ x_{-7} = 0.3,\ x_{-6} = 1.4,\ x_{-5} = 3.1,\ x_{-4} = 2.2,\ x_{-3} = 0.2,\ x_{-2} = -1.7,\ x_{-1} = 1.3,\ x_0 = 0.6$. See Fig. 7.
Example 8. We assume $x_{-8} = 11,\ x_{-7} = 4,\ x_{-6} = 14,\ x_{-5} = 1,\ x_{-4} = 4,\ x_{-3} = 0.2,\ x_{-2} = -2/11,\ x_{-1} = -1/8,\ x_0 = -5/7$. See Fig. 8.

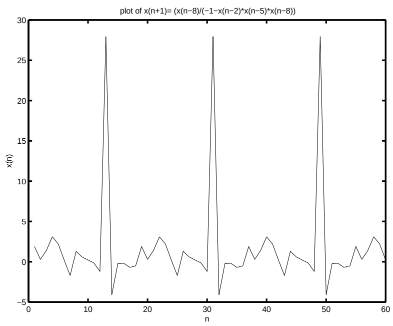

Figure 7: Plot for example 7

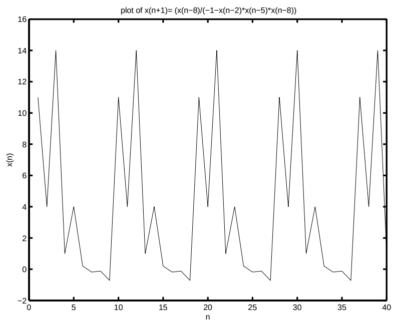

Figure 8: Plot for example 8

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