An. Şt. Univ. Ovidius Constanţa Vol. 18(1), 2010, 49–62

SOME REMARKS ON PARTITIONING SEMIRINGS

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Abstract

In this article we will introduce the notions of partitioning semirings and strongly zero-sum semirings and study some of their properties. We will analyze possible structures of semidomain and relationships between semirings that share some properties with semidomains, but whose definitions are less restrictive. The main aim of this article is that of extending some results obtained for domain like rinks to the theory of semirings.

1 Introduction

This article is devoted to an exploration of how ideal-theoretic considerations in commutative semirings impact the multiplicative behavior of those elements of the semiring that have additive inverses in the semiring. The general question as to the algebraic nature of these so-called "zero-sums" of a semiring is one of the most central ideas in the theory of semirings. The idea of investigating a mathematical structure via its representations in simpler structures is commonly used and often successfully. The representation theory of semirings has developed greatly in the recent years. It is an area which is very firmly based on the detailed understanding of examples, and there are many powerful techniques for investigating the representations of particular semirings and for relating the representations of one semiring to another. One of the aims of the modern representation theory of semirings is to solve "zero-sum" problem

Key Words: Partitioning semirings, Strongly zero-sum semirings, s-présimplifiable semirings, Semidomainlike semirings
Mathematics Subject Classification: 16Y60
Received: September, 2009
Accepted: January, 2010

for subcategories of semirings. The reader is referred to [12] and [13] for a detailed discussion of "zero-sum" problem and useful computational reduction procedures.

Présimplifiable rings were introduced by Bouvier in [5, 6] and later studied by D. D. Anderson and S. Valdes-Leon in [3]. Domainlike rings have been studied by M. Axtell, S. F. Forman and J. Stickles in [2]. We are motivated in this regard by the recent success of studies of the notions of "présimplifiable", "strongly associate" and "domainlike" commutative rings in such articles as [1, 3] and [2] because they are useful rings for study "factorization in commutative rings with zero-divisors". The factorization of nonunits into atoms is a central theme in algebra. Classically the theory has concentrated on integral domains. Much of this theory generalizes to the case of rings with zero-divisors, but important differences remain (see [1,3]). The authors study ways in which factorization in a ring with zero-divisors differ markedly for the domain case.

In the present article we introduce a new class of semirings, called partitioning semirings (see Definition 2.2), and we study it in details from the "zero-sum" problem point of view. We focus on a class of semirings, properly containing the class of semidomains, that is representable by the property that every zero-divisor of the semiring is nilpotent. However, we demonstrate here that some results from the papers listed in [1, 2] can be generalized by means of certain ideal-theoretic properties connected with the set of zero-divisors of a semiring. In Section 2, we begin by presenting several examples and fundamental results concerning partitioning semirings that prove useful throughtout this article. In Section 3, a number of basic results concerning semidomainlike semirings are given. We also establish a connection between the s-présimplifiable semirings, semidomainlike semirings and strongly associate semirings (see Sections 2 and Section 3).

We now give some definitions that are used frequently throughout this article. A commutative semiring R is defined as an algebraic system $(R,+,.)$ such that $(R,+)$ and $(R,.)$ are commutative semigroups, connected by $a(b+c)=ab+ac$ for all $a,b,c\in R$, and there exists $0\in R$ such that $r+0=r$ and $r0=0r=0$ for each $r\in R$. In this article all semirings considered will be assumed to be commutative semirings with $1\neq 0$. A nonempty subset I of R is called an ideal of R if $a+b\in I$ and $ra\in R$ whenever $a,b\in I$ and $r\in R$. A prime ideal of R is a proper ideal I of R in which $x\in I$ or $y\in I$ whenever $xy\in I$. The radical of the ideal I is given by $\mathrm{rad}(I)=\{a\in R:$ there exists a positive integer n such that $a^n\in I\}$. A proper ideal I of R is called a primary ideal of R if whenever $a,b\in R$ such that $ab\in I$ and $a\notin I$, it must be the case that there exists a positive integer n for which $b^n\in I$. In this case, $\mathrm{rad}(I)$ is the smallest prime ideal of R containing I. The ideal I of R is called subtractive (simply k-ideal) if $a,b\in R$ such that both $a+b\in I$ and $b\in I$

imply that $a \in I$ (so $\{0\}$ is a k-ideal of R). Given a nonempty subset X of R, the ideal generated by X is $(X) = \{r_1x_1 + r_2x_2 + ... + r_nx_n : r_i \in R, x_i \in X\}$. In particular, for $x \in R$, the set $(x) = Rx = \{rx : r \in R\}$ is an ideal of R called the principal ideal of R generated by x.

Following [12, 13], an element a in a semiring R is called a zero-sum of R if there exists an element $b \in R$ such that $a + b = 0$. In such a case, the element b is unique (we use $S(R)$ to denote the set of all zero-sum elements of R). A semiring R is called zerosumfree if 0 is the only zero-sum of R. An element a of of R is called zero-divisor of R if there exists $0 \neq b \in R$ such that $ab = 0$ (note here that we include 0 in the set of zero-divisors of a semiring). The collection of all zero-divisors of R will be denoted by $Z(R)$. Furthermore, the subset $\{a \in R :$ there exists a positive integer n such that $a^n = 0\}$ of $Z(R)$ consisting of the nilpotent elements of R will be denoted by $\mathrm{nil}(R)$, the nilradical of R. A semiring R is said to be a semidomain if $ab = 0$ $(a, b \in R)$ implies either $a = 0$ or $b = 0$. It is clear that R is semidomain if and only if (0) is a prime ideal of R. Finally, a semiring R is called cancellative whenever $ac = ab$ for some elements a, b and c of R with $a \neq 0$, then $b = c$.

Let R be a semiring. A non-zero element a of R is said to be a semi-unit in R if there exist $r, s \in R$ such that $1 + ra = sa$. The set of all semi-units of R will be denoted by $U(R)$. A semiring R is said to be a local semiring if and only if R has a unique maximal k-ideal. Moreover, a is a semi-unit of R if and only if a lies outside each maximal k-ideal of R (see [8, Lemma 4]).

2 Properties of partitioning semirings

An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that $R = \cup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q-ideal of a semiring R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$, and $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is the unique element such that $q_1q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is the zero element of R/I. Clearly, if R is commutative, then so is R/I, see [4], [7], and [8] for a background of the quotient semirings and their properties.

Recall that if I is a proper Q-ideal of R, then there exists a maximal k-ideal P of R with $I \subseteq P$ (see [8, Theorem 3]). Also, we define the Jacobson radical of R, denoted by $\mathrm{Jac}(R)$, to be the intersection of all the maximal k-ideals of

R. Then by [8, Lemma 2], the Jacobson radical of R always exists and by [7, Lemma 2.12], it is a k-ideal of R. Our starting point is the following lemma.

Lemma 2.1. *A semiring R is a ring if and only if some semi-unite of R is a zero-sum of R.*

Proof. The necessity is clear. For the sufficiency, Let $x \in R$, and choose a semi-unit u of R that is a zero-sum of R. Let v be such that $u + v = 0$. There are elements $r, s \in R$ such that $1+ru = su$, $asu+asv = 0$ and $a+aru+asv = 0$, whence a is a zero-sum of R. Thus R is a ring. □

Definition 2.2. *(1) A semiring R is called a partitioning semiring, if every proper principal ideal of R is a partitioning ideal.*

(2) A semiring R is called a strongly zero-sum semiring, if every nonsemi-unit element of R is a zero-sum element.

(4) A proper ideal I of R is said to be a strongly zero-sum ideal, if every element of I is a zero-sum element.

Example 2.3. *(1) If I is a partitioning ideal of a semiring R, then I is a k-ideal (see [12, Corollary 8.23]), but the converse is not true. To see this, let Z_+ denote the set of all nonnegative integers and consider the semiring $R = (Z_+, \text{gcd}, \text{lcd})$. Then the ideal $2Z_+$ of R is a k-ideal but not partitioning. Thus R is not a partitioning semiring.*

(2) Clearly, every commutative ring with non-zero identity is a strongly zero-sum semiring and partitioning semiring.

(3) There exist infinite semidomains with nontrivial zero-sums that are not rings; for example, the polynomial semiring $XZ[X] + N$, where Z is the ring of integers and N is the semiring of nonnegative integers.

(4) Let Z_+ denote the semiring of nonnegative integers with the usual operrations of addition and multiplication. An inspection will show that if $n \in Z_+ - \{0\}$, then the ideal $Rn = \{kn : k \in Z_+\}$ is a Q-ideal when $Q = \{0, 1, ..., n-1\}$ and if $n = 0$, the ideal Rn is a Q-ideal when $Q = Z_+$. Thus R is a partitioning semiring. Moreover, A simple argument will show that the ideal $Z_+ - \{1\}$ can not be a Q-ideal. Also, it is clear that R is not a strongly zero-sum semiring.

(5) Let $R = \{0, 1, 2, ..., 20\}$, and define $a + b = \max\{a, b\}$, $a.b = \min\{a, b\}$ for each $a, b \in R$. Then $(R, +, .)$ is easily checked to be a commutative semiring with 20 as identity. Let J_4 denote the ring integer modulo 4. Let $J_4 \oplus R = \{(a, b) : a \in J_4, b \in R\}$ denote the direct sum of semirings J_4 and R. Then $J_4 \oplus R$ is a commutative semiring. An inspection will show that $I_0 = \{(a, 0) : a \in J_4\}$ is a proper strongly zero-sum ideal in $J_4 \oplus R$ and $I_{10} = \{(0, n) : n \leq 10\}$ is a proper ideal of $J_4 \oplus R$, which is not a strongly zero-sum ideal. Moreover, R is not a partitioning and strongly zero-sum semiring.

Remark 2.4. *Let R be a semiring.*

(1) Since $1+0.1=1.1$, we have 1 is a semi-unit of R. Moreover, if $rs=1$ (resp. $rs \in U(R)$) for some $r,s \in R$, then r and s are semi-units of R since $1+0.r=rs$ and $1+0.s=rs$ (resp. $1+rsu=rsw$ for some $u,w \in R$).

(2) Let $0 \neq a \in R$ be such that $ab=0$ for some non-zero element b of R. If $1+ra=sa$ for some $r,s \in R$, then $b=0$, which is a contradiction. So every zero-divisor of R is not a semi-unit.

Proposition 2.5. *Let R be a partitioning semiring that is not a ring.*

(1) If a is a zero-sum of R, then a is a zero-divisor of R.

(2) If R is a strongly zero-sum semiring, then R is a local semiring with maximal ideal $Z(R)$.

Proof. (1) By assumption, $a+b=0$ for some $b \in R$, so $b \in Ra$ since Ra is a k-ideal. Choose $r \in R$ such that $b=ra$. Then $(1+r)a=0$. It then follows from Lemma 2.1 that $1+r \neq 0$. Thus a is a zero-divisor of R.

(2) By Remark 2.4 (2) and (1), $S(R)=Z(R)$ is the set of nonsemi-units of R that is a k-ideal. Now R is a local semiring by [8, Theorem 5]. □

Theorem 2.6. *If R is a partitioning semidomain such that there exists a nonzero zero-sum a of R, then R is a domain. In particular, every strongly zero-sum partitioning semidomain is a domain.*

Proof. By a similar argument like that Proposition 2.5 (1), there exists $r \in R$ such that $(1+r)a=0$. Then 1 is a zero-sum element of R since R is semidomain. Now the assertion follows from Lemma 2.1 and Remar 2.4 (1). □

Theorem 2.7. *Let I be a proper Q-ideal of a partitioning semiring R. If R is a strongly zero-sum semiring, then so is R/I.*

Proof. Suppose that q'_0 is the unique element of Q such that $1=q'_0+a$ for some $a \in I$; we show that q'_0+I is the identity element in R/I. Let $q+I \in R/I$. Then $(q'_0+I) \odot (q+I) = q'+I$, where $q' \in Q$ is the unique element such that $q'_0q+I \subseteq q'+I$, whence there exist $a',b' \in I$ such that $q'_0q+a'=q'+b'$. As $q+a'=q'_0q+qa+a'=q'+b'+qa \in (q+I)\cap(q'+I)$, we have $q=q'$. So q'_0+I is the identity element of R/I.

Assume that q_1+I is a nonsemi-unit of R/I ($q_1 \in Q$); we show that q_1 is not a semi-unit of R. If not, then $1+rq_1=sq_1$ for some $r,s \in R$, so $1 \in Rq_1$, whence $Rq_1=R$ by [12, Lemma 1]. Thus $rq_1=1$ for some $r \in R$. There exist $q_2 \in Q$ and $c \in I$ such that $1=q'_0+a=q_1q_2+cq_1$. Then there is a unique element $q_3 \in Q$ such that $(q_1+I)\odot(q_2+I)=q_3+I$, where $q_1q_2+I \subseteq q_3+I$, so $q_1q_2+e=q_3+f$ for some $e,f \in I$. It follows that $q'_0+a+e=q_3+f+cq_1$; hence $q_3=q'_0$. Thus $(q_1+I)\odot(q_2+I)=q'_0+I$. It then follows from Remark

2.4 that $q_1 + I$ is a semi-unit of R/I, which is a contradiction. So q_1 is not a semi-unit of R.

Let q_0 be the unique element in Q such that $q_0 + P$ is the zero in R/P. By assumption, there are elements $q_4 \in Q$ and $d \in I$ such that $q_1 + q_4 + d = 0$. So there is a unique element $q_5 \in Q$ such that $(q_1 + I) \oplus (q_4 + I) = q_5 + I$, where $q_1 + q_4 + I \subseteq q_5 + I$, so $q_1 + q_4 + e' = q_5 + f'$ for some $e', f' \in I$. It follows that $0 + e' = q_1 + q_4 + e' + d = q_5 + f' + d$, so $0 + I \subseteq (q_0 + I) \cap (q_5 + I)$, whence $q_0 = q_5$, as needed. □

Proposition 2.8. *Let R be a partitioning semiring, and let $r \in S(R)$. Then $r \in J(R)$ if and only if, for every $a \in R$, the element $1 + ra$ is a semi-unit of R.*

Proof. By [10, Lemma 3.4], it suffices to show that if for each $a \in R$, the element $1+ra$ is a semi-unit of R, then $r \in J(R)$. Let P be a maximal k-ideal of R: we show that $r \in P$. Suppose not. There is an element $r' \in R$ such that $r + r' = 0$, so $r' \notin P$ since P is a k-ideal. Then we should have $M + Rr' = R$, so that there exist $p \in P$ and $a \in R$ with $p + r'a = 1$; hence $1 + ra = p \in P$, which is a contradiction by [8, Lemma 1]. Thus $r \in J(R)$. □

Let R be a partitioning semiring and $a, b \in R$. We say a and b are associate, denoted $a \sim b$, if $a|b$ and $b|a$, or equivalently, if $Ra = Rb$. We say a and b are strongly associate, denoted $a \approx b$, if there exists a $u \in U(R)$ such that $a = ub$. We say a and b are very strongly associate, denoted $a \cong b$, if $a \sim b$ and further when $a \neq 0$, $a = ub$ ($u \in R$) implies $u \in U(R)$. A semiring R is called s-présimplifiable if $ab = a$ for $a, b \in R$ implies that either $a = 0$ or $b \in U(R)$. A semiring R is a strongly associate semiring if $a \sim b$ implies $a \approx b$. These definitions are the same as that introduced in [1].

Lemma 2.9. *Suppose that R is a partitioning semiring and let $\approx$ and $\sim$ be both equivalence relations on R. Then the relation $\cong$ is an equivalence relation on R if and only if R is s-présimplifiable.*

Proof. The proof is straightforward. □

Lemma 2.10. *Let R be a partitioning semiring and $a, b \in R$. Then the following hold:*

(1) If $a \approx b$, then $a \sim b$.

(2) If $a \cong b$, then $a \approx b$.

(3) If $a \sim b \Rightarrow a \cong b$ For all $a, b \in R$, then $a \approx b \Rightarrow a \cong b$. In particular, R is a s-présimplifiable semiring (so R is strongly associate).

Proof. (1) There exists a $u \in U(R)$ such that $a = ub$, so $Ra \subseteq Rb$. There are elements $r, s \in R$ such that $1 + ru = su$, which implies $b + ra = sa$; thus $b \in Ra$ since Ra is a k-ideal, and so we have $Ra = Rb$. Thus $a \sim b$. (2) Is clear.

(3) The first part follows from (1) and (2). To see that R is s-présimplifiable, assume that $c = cd$. Now $c \cong c$. Hence $c = 0$ or $d \in U(R)$. □

The following theorem shows how the property of s-présimplifiable is related to the types of associate elements defined above.

Theorem 2.11. *For a strongly zero-sum partitioning semiring R that is not a ring the following are equivalent.*

(1) For all $a, b \in R$, $a \sim b \Rightarrow a \cong b$.

(2) For all $a, b \in R$, $a \approx b \Rightarrow a \cong b$.

(3) R is s-présimplifiable.

(4) $Z(R) \subseteq J(R)$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follows from Lemma 2.10. For (3) $\Rightarrow$ (4), assume that $x \in Z(R)$. Then $xy = 0$ for some $0 \neq y \in R$, so $y(1 + x) = y$, whence $1+x \in U(R)$. For $r \in R$, $rx \in Z(R) = S(R)$ and hence $1+rx \in U(R)$. Thus $x \in J(R)$ by Proposition 2.8, and so we have $Z(R) \subseteq J(R)$. (4) $\Rightarrow$ (1) Suppose that $a = rb$. Then $Ra = rRb = rRa$ and there exists $a' \in R$ such that $a + a' = 0$. Now $rRa = rRa'$. Thus $a = rsa'$ for some $s \in R$, so $a(1+rs) = 0$; hence $1+rs \in Z(R) \subseteq J(R)$. Then rs is a semi-unit of R (otherwise, $1 \in J(R)$ since R is local), and so r itself is a semi-unit by Remark 2.4. Hence $r \in U(R)$. So $a \cong b$. □

Corollary 2.12. *If R is a strongly zero-sum partitioning semiring that is not a ring, then R is a s-présimplifiable semiring (so R is strongly associate).*

Proof. Apply Theorem 2.11 and Proposition 2.5. □

3 Properties of semidomainlike semirings

We begin the key definition of this section.

Definition 3.1. *A semiring R is called semidomainlike semiring, if $Z(R) \subseteq$ nil(R) [1].*

Proposition 3.2, which is well-known in the ring-theoretic context (see [1, 2]), provides characterizations in terms of the notions of "primary ideal" and "prime ideal" of when certain equalities amongst these algebraic structures occur.

Proposition 3.2. *Let R be a semiring.*

(1) (0) *is a primary ideal of R if and only if $Z(R) = \text{nil}(R)$. In particular, if* (0) *is a primary ideal of R, then $Z(R)$ is an ideal of R.*

(2) (0) *is primary if and only if R is semidomainlike.*

(3) Every semidomailike semiring is s-présimplifiable.

(4) If R is semidomainlike, Then $Z(R)$ is the unique minimal prime ideal of R.

Proof. (1) Let (0) is a primary ideal of R. It is straightforward to see that $\text{nil}(R) = \text{rad}(0) \subseteq Z(R)$. It suffices to show that $Z(R) \subseteq \text{nil}(R)$. Let $x \in Z(R)$. Then $xy = 0$ for some $0 \neq y \in R$. Since (0) is a primary ideal of R, there exists a positive integer n such that $x^n = 0$, whence $a \in \text{rad}(0)$, and so we have equality. Conversely, suppose that $Z(R) = \text{nil}(R)$, and let $a, b \in R$ such that $ab = 0$ but $b \neq 0$. Then $b \in Z(R) = \text{rad}(0)$. So (0) must be a primary ideal of R. (2) and (3) follows from (1) (note that $\text{nil}(R) \subseteq J(R)$). To see that (4), as $\text{rad}(Z(R)) = Z(R) = \text{rad}(0)$ by (1), we must have $Z(R)$ is a prime ideal of R since (0) is primary. Now if P is a prime ideal, then $Z(R) = \text{nil}(R) \subseteq P$, as desired. □

Let R be an Artinian cancellative semiring. Then every element of R is either a semi-unit or a nilpotent element, and $\text{J}(R) = \text{nil}(R)$ (see [10, Proposition 3.7] and [8, Theorem 12]). Then R is a semidomainlike semiring (so R is a s-pr*é*simplifiable semiring by Proposition 3.2).

Let R be a given semiring, and let S be the set of all multiplicatively cancellable elements of R (so $1 \in S$). Clearly, the set S is multiplicatively closed. Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denot the equivalence class of $\sim$ which contains (a, s) by a/s, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S can be given the structure of a commutative semiring under oprerations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = (ab)/st$ for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the semiring of fractions of R with respect to S; its zero element is $0/1$, its multiplicative identity element is $1/1$ and each element of S has a multiplicative inverse in R_S (see [14]). The next theorem is a classical result stated in terms of the semidomainlike property.

Theorem 3.3. *Let R be a semiring. Then R is semidomainlike if and only if R_S is semidomainlike.*

Proof. Assume that R is a semidomainlike semiring and let $0/1 \neq a/s \in Z(R_S)$. Then $(a/s)(a'/s') = 0/1$ for some $0/1 \neq a'/s' \in R_S$, whence $aa' = 0$.

If $a \neq 0$, then $a'^n = 0$ for some n since (0) is primary in R by Proposition 3.2. Hence $(a'/s')^n = 0/1$, and so $(0/1)$ is primary in R_S. Thus R_S is a semidomainlike semiring. Conversely, let $r, s \in R$ with $r \neq 0$ (so $r/1 \neq 0/1$) and assume $rs = 0$. Then $(r/1)(s/1) = 0/1$, so $(s/1)^m = 0/1$; hence $s^m = 0$, and so (0) is a primary ideal of R. Therefore R is a semidomainlike semiring by Proposition 3.2. □

Lemma 3.4. *Let P be a proper Q-ideal of a semiring R, and let q_0 be the unique element in Q such that $q_0 + P$ is the zero in R/P. If $a \in P$, then $a + P = q_0 + P$.*

Proof. There are elements $q_1 \in Q$ and $p \in P$ such that $a = q_1 + p$, so $q_1 \in P$ since every Q-ideal is a k-ideal. By [10, Lemma 2.3], $q_1 = q_0$, and so we have equality. □

A classical result of commutative semiring theory is that a Q-ideal P is prime if and only if R/P is a semidomain (see [7, Theorem 2.6]). The following theorem is a parallel result for semidomainlike semirings.

Theorem 3.5. *Let P be a proper Q-ideal of a semiring R. Then P is primary if and only if R/P is semidomainlike.*

Proof. Let q_0 be the unique element in Q such that $q_0 + P$ is the zero in R/P. Let P be a primary ideal of R and let $q_1 + P \in Z(R/P)$. Then there exists a non-zero element $q_2 + P$ of R/P such that $(q_1 + P) \odot (q_2 + P) = q_0 + P$, where $q_1, q_2 \in Q$ and $q_1q_2 + P \subseteq q_0 + P = P$. So $q_1q_2 + e = f$ for some $e, f \in P$; hence $q_1q_2 \in P$ since P is a k-ideal of R. If $q_2 \in P$, then $q_2 \in (q_0 + P) \cap (q_2 + P)$; hence $q_2 + P = q_0 + p$, which is a contradiction. Then P primary gives $(q_1 + P)^n = q_1^n + P = q_0 + P$ for some n by Lemma 3.4. Thus $q_1 + P \in \text{nil}(R)$ and so R/P is a semidomainlike semiring. Conversely, let $ab \in P$, where $a, b \in R$. There are elements $q_1, q_2 \in Q$ such that $a \in q_1 + P$ and $b \in q_2 + P$, so $a = q_1 + c$ and $b = q_2 + d$ for some $c, d \in P$. As $ab = q_1q_2 + q_1d + cq_2 + cd$, we must have $q_1q_2 \in P$ since P is a k-ideal of R. Let $(q_1 + P) \odot (q_2 + P) = q_4 + P$, where $q_4 \in Q$ is the unique element such that $q_1q_2 + P \subseteq q_4 + P$. It follows that $q_4 \in P \cap Q$; hence $q_4 + P = q_0 + P$ by Lemma 3.4 (so $q_4 = q_0$). If $q_1 + P = q_0 + P$, then $a \in P$. Similarly, for $q_2 + P = q_0 + P$. So we may assume that $q_1 + P \neq q_0 + P$ and $q_2 + P \neq q_0 + P$. Therefore, by assumption, $(q_1 + P)^m = q_1^m + P = q_0 + P$ for some m. Then $a^m = q_1^m + e \in P$ for some $e \in P$. Thus P is primary. □

Example 3.6. *Let $R = Z_+$ denote the semiring of nonnegative integers with the usual operrations of addition and multiplication. Then by example 2.3, R*

is a partitioning semiring. Let $R6$ *denote the ideal generated by* 6*. It is clear that* $R6$ *is not a prime* k*-ideal. The only prime ideals in* R *that contain* $R6$ *are* $R2$*,* $R3$ *and* $\{0\} \cup \{x \in R : x > 1\}$*. Since* $R2 \cap R3 \cap \{x \in R : x > 1\}$ *is equal to* $R6$*, it follows that* $\mathrm{rad}(R6) = R6$ *is a partitioning ideal of* R*.*

Theorem 3.7. *Let* I *be an ideal of a partitioning semiring* R *such that* $\mathrm{rad}(I)$ *is a* Q*-ideal of* R*. Then* $R/\mathrm{rad}(I)$ *is semidomainlike if and only if* $R/\mathrm{rad}(R)$ *is a semidomain.*

Proof. Let q_0 be the unique element in Q such that $q_0 + \mathrm{rad}(I)$ is the zero in R/rad (I). Let $R/\mathrm{rad}(I)$ be semidomainlike, and let $(q_1 + \mathrm{rad}(I)) \odot (q_2 + \mathrm{rad}(I)) = q_0 + \mathrm{rad}(I)$ in $R/\mathrm{rad}(I)$ with $q_1 + \mathrm{rad}(I) \neq q_0 + \mathrm{rad}(I)$, where $q_1, q_2 \in Q$ and $q_1q_2 + \mathrm{rad}(I) \subseteq q_0 + \mathrm{rad}(I)$. Then $q_1q_2 \in \mathrm{rad}(I)$, but $q_1 \notin \mathrm{rad}(I)$. Since $R/\mathrm{rad}(I)$ is semidomainlike, $\mathrm{rad}(I)$ is primary by Theorem 3.5. Thus $q_2^n \in \mathrm{rad}(I)$ for some positive integer m, whence $q_2 \in \mathrm{rad}(I)$, $q_2 + \mathrm{rad}(I) = q_0 + \mathrm{rad}(I)$ by [14, Lemma 2.3], and $R/\mathrm{rad}(I)$ is a semidomain. The other implication is similar. □

Corollary 3.8. *Let* R *be a partitioning semiring such that* $\mathrm{nil}(R)$ *is a* Q*-ideal of* R*. Then* $R/\mathrm{nil}(R)$ *is semidomainlike if and only if* $R/\mathrm{nil}(R)$ *is a semidomain.*

Proof. Apply Theorem 3.7. □

Theorem 3.9. *Let* R *be a partitioning semiring such that* $\mathrm{nil}(R)$ *is a* Q*-ideal of* R*. If* R *is a semidomainlike semiring, then* $R/\mathrm{nil}(I)$ *is a semidomainlike semiring.*

Proof. By [7, Theorem 2.6] and Corollary 3.8, it is enough to show that $\mathrm{nil}(R)$ is a prime ideal of R. Let $a, b \in R$ such that $ab \in \mathrm{nil}(R)$ with $a \notin \mathrm{nil}(R)$. Then $a^nb^n = 0$ and $a^n \neq 0$ for some n. We may assume that $b^n \neq 0$. Therefore $b^n \in Z(R) \subseteq \mathrm{nil}(R)$ since R is semidomainlike, whence $b \in \mathrm{nil}(R)$, as required. □

Proposition 3.10. *Let* I *be a* Q*-ideal of a semiring* R *and let* $u = q + a \in U(R)$ *for some* $q \in Q$*. Then* $q + I \in U(R/I)$*.*

Proof. Let q_0 (resp. q_0') be the unique element in Q such that $q_0 + I$ (resp. $q_0' + I$) is the zero in R/I (resp. $q_0' + I$ is the identity in R/I). Since u is a semi-unit, $1 + ru = su$ for some $r, s \in R$. There are elements q_1, q_2 and q_3 of Q, and a_1, b_1 of I such that $r = q_2 + b_1$ and $s = q_3 + c_1$, so $1 + qq_2 + l = qq_3 + l'$ for some $l, l' \in I$. It follows that

$$q_0' + qq_2q_0' + q_0'l = qq_3q_0' + q_0'l'. \quad (1)$$

By (1), since $q'_0 + I$ is the identity element of R/I, we must have

$$q'_0 + qq_2 + I \subseteq qq_3 + I. \tag{2}$$

We show that $(q'_0 + I) \oplus ((q + I) \odot (q_2 + I) = (q + I) \odot (q_3 + I))$ (in fact, $1_{R/I} + \bar{q}_2\bar{q} = \bar{q}_3\bar{q}$, where $\bar{q} = q + I, \bar{q}_2 = q_2 + I$ and $\bar{q}_3 = q_3 + I$). There are unique elements q_4, q_5 and q'_4 of Q such that $qq_2 + I \subseteq q_4 + I$, $q'_0 + q_4 + I \subseteq q_5 + I$ and $qq_3 + I \subseteq q'_4 + I$. It suffices to show that $q_5 = q'_4$. By assumption, there exist elements a_1, a_2, a_3, a_4, a_5 and a_6 of I such that $qq_2 + a_1 = q_4 + a_2$, $q'_0 + q_4 + a_3 = q_5 + a_4$ and $qq_3 + a_5 = q'_4 + a_6$. Therefore, $q'_0 + q_4 + a_3 + a_2 = q_5 + a_4 + a_2 = q'_0 + qq_2 + a_1 + a_3$. It follows from (2) that $q'_0 + qq_2 \in (q_5 + I) \cap (qq_3 + I) \subseteq (q_5 + I) \cap (q'_4 + I)$; hence $q_5 = q'_4$, as desired. □

Theorem 3.11. *Let R be a strongly zero-sum partitioning semiring that R is not a ring. If $\mathrm{nil}(R)$ is a Q-ideal of R, then $R/\mathrm{nil}(R)$ is a s-présimplifiable.*

Proof. Let q_0 be the unique element in Q such that $q_0 + \mathrm{nil}(R)$ is the zero in $R/\mathrm{nil}(R)$. Suppose $(e + \mathrm{nil}(R)) \odot (f + \mathrm{nil}(R)) = e + \mathrm{nil}(R)$ and $e + \mathrm{nil}(R) \neq q_0 + \mathrm{nil}(R)$ (so $e \notin \mathrm{nil}(R)$), where $e, f \in Q$. By Theorem 2.7, there is an element f' of Q such that

$$(f + \mathrm{nil}(R)) \oplus (f' + \mathrm{nil}(R)) = q_0 + \mathrm{nil}(R),$$

whence $(e + \mathrm{nil}(R)) \oplus ((e + \mathrm{nil}(R)) \odot (f' + \mathrm{nil}(R))) = q_0 + \mathrm{nil}(R)$. Thus $e(1 + f') \in \mathrm{nil}(R)$. This implies $e^n(1 + f')^n = 0$ for some n. Since $e \notin \mathrm{nil}(R)$, we have $e^n \neq 0$, and so $(1 + f')^n \in Z(R) \subseteq J(R)$ by Theorem 2.11 and Corollary 2.12. Since by Proposition 2.5, R is local, we get $1 + f' \in J(R)$; hence $f' \in U(R)$ and so $f' + \mathrm{nil}(R) \in U(R/\mathrm{nil}(R))$ by Proposition 3.10. Now a semi-unit of $R/\mathrm{nil}(R)$ is a zero-sum of $R/\mathrm{nil}(R)$, so $R/\mathrm{nil}(R)$ is a ring by Lemma 2.1. Hence $f + \mathrm{nil}(R) \in U(R/\mathrm{nil}(R))$, as required. □

Let R be a non-trivial semiring. An expression $P_0 \subset P_1 \subset ... \subset P_n$ (note the strict inclusions) in which $P_0, ..., P_n$ are prime k-ideals of R, is called a chain of prime k-ideals of R; the length of such a chain is the number of "links", that is, 1 less than the number of prime k-ideals present. The dimension of R, denoted by $\dim(R)$, is defined to be $\sup\{n \in Z_+ :$ there exists a chain of prime k-ideals of R of length $n\}$ if this supremum exists, and ∞ otherwise (Z_+ is the set of non-negative integers).

Proposition 3.12. *Assume that R is a strongly zero-sum partitioning semiring that R is not a ring. If $\mathrm{nil}(R)$ is a Q-ideal of R, then the following hold:*

(1) If (0) is not a primary ideal of R, then $\dim(R) \geq 1$.

(2) If $\dim(R) = 0$*, then* (0) *is primary. In particular,* R *is a semidomain-like semiring.*

Proof. (1) Since (0) is not primary, there are elements $a, b \in R$ such that $ab = 0$, where $b \neq 0$ and $a^n \neq 0$ for all n. So $a \in Z(R) \subseteq J(R)$ by Theorem 2.11 and Corollary 2.12, and hence a is contained in every maximal k-ideal of R. Set $T = \{a^n : n \text{ is a positive integer}\}$. Then T is a multiplicatively closed set with $\text{nil}(R) \cap T = \emptyset$. We denote by T_R the set of all Q-ideals of R. Since $\text{nil}(R)$ is a proper Q-ideal of R, the set $\Delta = \{J \in T_R : \text{nil}(R) \subseteq J \text{ and } J \cap T = \emptyset\}$ is not empty and the set Δ of Q-ideals of R (partially ordered by inclusion) has at least one maximal element P, and any such maximal element of Δ is a prime Q-ideal of R with $P \cap T = \emptyset$. Then $a \notin P$ and so P is not a maximal k-ideal. Thus $P \subsetneqq M$ for some maximal k-ideal M of R by [8, Theorem 3], which implies $\dim(R) \geq 1$. (2) follows from (1) and Proposition 3.2. □

Theorem 3.13. *Let* R *be a strongly zero-sum partitioning semiring that* R *is not a ring. If* $\text{nil}(R)$ *is a* Q*-ideal of* R*, then the following are equivalent.*

(1) $\dim(R) = 0$.

(2) R *has a unique prime* k*-ideal.*

(3) R *is semidomainlike.*

Proof. (1) $\Rightarrow$ (2) Since $\dim(R) = 0$, all prime k-ideals are maximal k-ideals. By Propositions 3.12 (2) and 3.2, we must have $Z(R) = \text{nil}(R)$ is a prime k-ideal, and hence a maximal k-ideal. Since $\text{nil}(R)$ is a maximal k-ideal and is the intersection of all prime ideals of R, we have $\text{nil}(R)$ as the only prime ideal in R (note that R is s-pr*é*simplifiable by Corollary 2.12).

(2) $\Rightarrow$ (3) Since R is a strongly zero-sum semiring, $Z(R) = \text{nil}(R)$ by Propositin 2.5 and so R is a semidomainlike semiring.

(3) $\Rightarrow$ (1) By proposition 3.2, we have $Z(R)$ is the unique minimal prime ideal of R. Since all nonsemi-units are zero-divisors, we have that $Z(R)$ is the unique prime k-ideal of R. □

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