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ITERATIVE ALGORITHM FOR A CONVEX FEASIBILITY PROBLEM

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Abstract

The purpose of this paper is to study convex feasibility problems in the setting of a real Hilbert space. The approximation of common elements of solution set of variational inequality problems and fixed point set of nonexpansive mappings is considered. Strong convergence theorems are established in the framework of Hilbert spaces.

1 Introduction and Preliminaries

Recently, many authors studied the following convex feasibility problem (CFP):

$$\text{finding a } p \in \bigcap_{i=1}^{r} C_i, \tag{1.1}$$

where $r \geq 1$ is an integer and each C_i is a nonempty closed and convex subset of a real Hilbert space H. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [9,12], computer tomography [19] and radiation therapy treatment planning [10].

In this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed

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and convex subset of H. Recall that a mapping A is said to be α-inverse-strongly monotone if there exists a real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Recall that the classical variational inequality problem, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

Given $z \in H, u \in C$, the following inequality holds

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that the projection P_C is firmly nonexpansive. That is,

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \quad \forall x, y \in H.$$

One can see that the variational inequality problem (1.2) is equivalent to a fixed point problem. It is easy to see that an element $u \in C$ is a solution of the variational inequality (1.2) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping, that is,

$$u \in VI(C, A) \Longleftrightarrow u = P_C(I - \lambda A)u.$$

In [11], Iiduka and Takahashi showed that if A is α-inverse-strongly monotone and $\lambda \leq 2\alpha$, then the mapping $I - \lambda A$ is nonexpansive. This implies that $P_C(I - \lambda A)$ is also nonexpansive. In [1], Browder showed that if C is a bounded closed and convex subset of H, then nonexpansive mapping on C has a unique fixed point. Moreover, the fixe point set if closed and convex, see also [2] and [13].

In this paper, we shall consider the case that C_i is the set of solutions of the variational inequality problem (1.2). That is, $C_i = VI(C, A_i)$ for each $1 \leq i \leq r$. Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to stand for the set of fixed points of the mapping S. Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Recently, iterative algorithms for the classical variational inequality (1.2) and fixed point problem of nonexpansive mappings have received rapid development, see, for example, [5-8,11-17,20,21,23] and the references therein. Recently, Iiduka and Takahashi [11] constructed an iterative algorithm to study the problem of finding a common element of the set of solution of a variational inequality for an inverse-strongly monotone mapping and of the set of fixed points of a nonexpansive mapping. To be more precise, they proved the following theorem:

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \tag{1.3}$$

for every $n = 1, 2, \dots,$ where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

Recently, Y. Yao and J.C. Yao [23] further studied the approximation common elements of solution set of the variational inequality (1.1) and of the fixed point set of a nonexpansive mapping by considering the following iterative algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$, A is an α-inverse-strongly monotone mapping of C into H and S is a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. They proved that the sequence $\{x_n\}$ generated by the algorithm (1.4) converges strongly to $x^* = P_{F(S)\cap VI(C,A)}u$.

Quite recently, Ceng, Wang and Yao [5] considered the problem for a pair of inverse-strongly monotone mappings by the following iterative algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$, A and B are two inverse-strongly monotone mappings and S

is a nonexpansive mapping. They also obtained a strong convergence theorem of the iterative algorithm (1.5).

In this paper, motivated by Ceng et al. [5], Iiduka and Takahashi [11], Y. Yao and J.C. Yao [23], we study the convex feasibility problem (1.1) by considering a family of inverse-strongly monotone mappings and a single nonexpansive mapping. The results presented in this paper improve and extend the corresponding results announced by many others.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 (Suzuki [18]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.2 (Bruck [4]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_i : 1 \le i \le r\}$ be a sequence of nonexpansive mappings on C. Suppose $\cap_{i=1}^r F(T_i)$ is nonempty. Let $\{\mu_i\}$ be a sequence of positive numbers with $\sum_{i=1}^r \mu_i = 1$. Then a mapping S on C defined by*

$$Sx = \sum_{i=1}^r \mu_i T_i x$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \cap_{i=1}^\infty F(T_i)$ holds.

Recall that a mapping $S : C \to C$ is closed at zero if $\{x_n\}$ is a sequence in C converging strongly to $x \in C$ and Sx_n converges strongly to zero, then $Sx = 0$.

Recall that a mapping $S : C \to C$ is demiclosed at zero if $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and Sx_n converges strongly to zero, then $Sx = 0$.

Lemma 1.3 (Browder [3]). *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

Lemma 1.4 (Xu [22]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2 Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a μ_i-inverse-strongly monotone mapping for each $1 \le i \le r$, where r is some positive integer. Let $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} := \cap_{i=1}^{r} VI(C, A_i) \cap F(S) \ne \emptyset$. Let $\{x_n\}$ be a sequence defined by the following manner:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S \sum_{i=1}^{r} \eta_i P_C(x_n - \lambda_i A_i x_n), \quad n \ge 1, \tag{2.1}$$

where $u \in C$ is a fixed point, $\lambda_1, \lambda_2, \ldots$ and λ_r are real numbers such that $\lambda_i \in (0, 2\mu_i)$ for each $1 \le i \le r$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that the above control sequences satisfies the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^{r} \eta_i = 1$, $\forall n \ge 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ generated in the iterative algorithm (2.1) converges strongly to $p = P_{\mathcal{F}} u$.

Proof. The proof is split into five steps.

Step 1. Show that the sequence $\{x_n\}$ is bounded.

Note that the mapping $I - \lambda_i A_i$ is nonexpansive for each i. Indeed, for any $x, y \in C$, we see that

$$\begin{aligned}
&\|(I - \lambda_i A_i)x - (I - \lambda_i A_i)y\|^2 \\
&= \|x - y\|^2 - 2\lambda_i \langle A_i x - A_i y, x - y\rangle + \lambda_i^2 \|A_i x - A_i y\|^2 \\
&\le \|x - y\|^2 - \lambda_i(2\mu_i - \lambda_i)\|A_i x - A_i y\|^2.
\end{aligned}$$

Since, for each $1 \le i \le r$, $\lambda_i \in (0, 2\mu_i)$, we see that $I - \lambda_i A_i$ is nonexpansive.

Put $y_n = \sum_{i=1}^r \eta_i P_C(x_n - \lambda_i A_i x_n)$ for each $n \geq 1$. For any $x^* \in \mathcal{F}$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n S y_n - x^*\| \\
&\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|S y_n - x^*\| \\
&\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \sum_{i=1}^r \eta_i \|P_C(x_n - \lambda_i A_i x_n) - x^*\| \\
&\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\
&= \alpha_n \|u - x^*\| + (1 - \alpha_n) \|x_n - x^*\|.
\end{aligned}$$

By mathematical inductions, we can obtain that

$$\|x_n - x^*\| \leq \{\|x_1 - x^*\|, \|u - x^*\|\}, \quad \forall n \geq 1.$$

This shows that the sequence $\{x_n\}$ is bounded. since $I - \lambda_i A_i$ is nonexpansive for each i, we obtain that

$$\begin{aligned}
\|y_n - x^*\| &= \|\sum_{i=1}^r \eta_i P_C(x_n - \lambda_i A_i x_n) - \sum_{i=1}^r \eta_i x^*\| \\
&\leq \sum_{i=1}^r \eta_i \|P_C(x_n - \lambda_i A_i x_n) - x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned}$$

This shows that $\{y_n\}$ is also bounded.

Step 2. Show that $x_{n+1} - x_n \to 0$ as $n \to \infty$.

Note that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\sum_{i=1}^r \eta_i P_C(x_{n+1} - \lambda_i A_i x_{n+1}) - \sum_{i=1}^r \eta_i P_C(x_n - \lambda_i A_i x_n)\| \\
&\leq \|x_{n+1} - x_n\|.
\end{aligned} \tag{2.2}$$

Put $l_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$, for all $n \geq 1$. That is,

$$x_{n+1} = (1 - \beta_n) l_n + \beta_n x_n, \quad \forall n \geq 1. \tag{2.3}$$

Note that

$$\begin{aligned}
& l_{n+1} - l_n \\
&= \frac{\alpha_{n+1} u + \gamma_{n+1} S y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n S y_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} u + \frac{1 - \beta_{n+1} - \alpha_{n+1}}{1 - \beta_{n+1}} S y_{n+1} - \frac{\alpha_n}{1 - \beta_n} u - \frac{1 - \beta_n - \alpha_n}{1 - \beta_n} S y_n \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \big(u - S y_{n+1}\big) + \frac{\alpha_n}{1 - \beta_n} \big(S y_n - u\big) + S y_{n+1} - S y_n.
\end{aligned}$$

It follows that

$$\begin{aligned}\|l_{n+1}-l_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-u\|+\|Sy_{n+1}-Sy_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-u\|+\|y_{n+1}-y_n\|.\end{aligned}$$

By virtue of (2.2), we arrive at

$$\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-u\|.$$

It follows from the conditions (ii) and (iii) that

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_{n+1}\|)\le 0.$$

Thanks to Lemma 1.1, we obtain that

$$\lim_{n\to\infty}\|l_n-x_n\|=0.$$

In view of (2.3), we have

$$x_{n+1}-x_n=(1-\beta_n)(l_n-x_n).$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.4}$$

Step 3. Show that $Sx_n-x_n\to 0$ as $n\to\infty$.

Note that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n u+\beta_n x_n+\gamma_n Sy_n-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|S\sum_{i=1}^r \eta_i P_C(x_n-\lambda_i A_i x_n)-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|\sum_{i=1}^r \eta_i P_C(x_n-\lambda_i A_i x_n)-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^r \eta_i\|P_C(x_n-\lambda_i A_i x_n)-x^*\|^2.\end{aligned} \tag{2.5}$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \sum_{i=1}^{r} \eta_i \|x_n - x^* - \lambda_i (A_i x_n - A_i x^*)\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \sum_{i=1}^{r} \eta_i (\|x_n - x^*\|^2 \\
&\quad - 2\lambda_i \langle A_i x_n - A_i x^*, x_n - x^* \rangle + \lambda_i^2 \|A_i x_n - A_i x^*\|^2) \\
&\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n \sum_{i=1}^{r} \eta_i \lambda_i (2\mu_i - \lambda_i) \|A_i x_n - A_i x^*\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\gamma_n \sum_{i=1}^{r} \eta_i \lambda_i (2\mu_i - \lambda_i) \|A_i x_n - A_i x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_n - x_{n+1}\|.
\end{aligned}$$

Thanks to conditions (ii) and (iii), one obtains that

$$\lim_{n\to\infty} \|A_i x_n - A_i x^*\| = 0, \quad \forall 1 \le i \le r. \tag{2.6}$$

On the other hand, one has

$$\begin{aligned}
&\|P_C(I - \lambda_i A_i) x_n - x^*\|^2 \\
&= \|P_C(I - \lambda_i A_i) x_n - P_C(I - \lambda_i A_i) x^*\|^2 \\
&\le \langle (I - \lambda_i A_i) x_n - (I - \lambda_i A_i) x^*, P_C(I - \lambda_i A_i) x_n - x^* \rangle \\
&= \frac{1}{2} \big(\|(I - \lambda_i A_i) x_n - (I - \lambda_i A_i) x^*\|^2 + \|P_C(I - \lambda_i A_i) x_n - x^*\|^2 \\
&\quad - \|(I - \lambda_i A_i) x_n - (I - \lambda_i A_i) x^* - (P_C(I - \lambda_i A_i) x_n - x^*)\|^2 \big) \\
&\le \frac{1}{2} \big(\|x_n - x^*\|^2 + \|P_C(I - \lambda_i A_i) x_n - x^*\|^2 \\
&\quad - \|x_n - P_C(I - \lambda_i A_i) x_n - \lambda_i (A_i x_n - A_i x^*)\|^2 \big) \\
&= \frac{1}{2} \big(\|x_n - x^*\|^2 + \|P_C(I - \lambda_i A_i) x_n - x^*\|^2 - \|x_n - P_C(I - \lambda_i A_i) x_n\|^2 \\
&\quad + 2\lambda_i \langle A_i x_n - A_i x^*, x_n - P_C(I - \lambda_i A_i) x_n \rangle - \lambda_i^2 \|A_i x_n - A_i x^*\|^2 \big).
\end{aligned}$$

It follows that

$$\|P_C(I-\lambda_i A_i)x_n-x^*\|^2 \le \|x_n-x^*\|^2-\|x_n-P_C(I-\lambda_i A_i)x_n\|^2+M_i\|A_ix_n-A_ix^*\|, \tag{2.7}$$

where M_i is given by

$$M_i = \sup\{2\lambda_i\|x_n - P_C(I-\lambda_i A_i)x_n\| : \forall n \ge 1\}.$$

On the other hand, we have

$$\|y_n - x_n\|^2 = \|\sum_{i=1}^{r}\eta_i P_C(I-\lambda_i A_i)x_n - x_n\|^2 \le \sum_{i=1}^{r}\eta_i\|P_C(I-\lambda_i A_i)x_n - x_n\|^2,$$

which combines with (2.7) yields that

$$\sum_{i=1}^{r}\eta_i\|P_C(I-\lambda_i A_i)x_n-x^*\|^2 \le \|x_n-x^*\|^2-\|y_n-x_n\|^2+\sum_{i=1}^{r}\eta_i M_i\|A_ix_n-A_ix^*\|.$$

From (2.5), we see that

$$\|x_{n+1}-x^*\|^2 \le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^{r}\eta_i M_i\|A_ix_n-A_ix^*\|-\gamma_n\|y_n-x_n\|^2,$$

from which it follows that

$$\begin{aligned}\gamma_n\|y_n - x_n\|^2 \le{}& \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \\ &+ \gamma_n\sum_{i=1}^{r}\eta_i M_i\|A_ix_n - A_ix^*\| \\ \le{}& \alpha_n\|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ &+ \gamma_n\sum_{i=1}^{r}\eta_i M_i\|A_ix_n - A_ix^*\|.\end{aligned}$$

It follows from (2.4), (2.6) and the conditions (ii) and (iii) that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.8}$$

Note that

$$Sy_n - x_n = \frac{(x_{n+1} - x_n) - \alpha_n(u - x_n)}{\gamma_n}.$$

Combining this with the condition (ii) and (iii) gives that

$$\lim_{n\to\infty}\|Sy_n - x_n\| = 0. \tag{2.9}$$

Observe that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\le \|x_n - Sy_n\| + \|y_n - x_n\|.\end{aligned}$$

It follows from (2.8) and (2.9) that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{2.10}$$

Step 4. Show that

$$\limsup_{n\to\infty} \langle u - p, x_n - p\rangle \le 0,$$

where $p = P_{\mathcal{F}}u$.

To show it, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - p, x_n - p\rangle = \lim_{i\to\infty} \langle u - p, x_{n_i} - p\rangle. \tag{2.11}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to f. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup f$. Define a mapping $W : C \to C$ by

$$Wx = \sum_{i=1}^{r} \eta_i P_C(I - \lambda_i A_i)x, \quad \forall x \in C.$$

From Lemma 1.2, we see that W is nonexpansive such that

$$F(W) = \cap_{i=1}^{r} F(P_C(I - \lambda_i A_i)) = \cap_{i=1}^{r} VI(C, A_i).$$

From (2.8), we see that

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{2.12}$$

From Lemma 1.3, we can obtain that $f \in F(W)$. In view of (2.10) and Lemma 1.3, we see that $f \in F(S)$. This proves that

$$f \in F(W) \cap F(S) = \cap_{i=1}^{r} VI(C, A_i) \cap F(S).$$

It follows from (2.11) that

$$\limsup_{n\to\infty} \langle u - p, x_n - p\rangle \le 0.$$

Step 5. Show that $x_n \to p$ as $n \to \infty$.

Note that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&= \langle \alpha_n u + \beta_n x_n + \gamma_n S y_n - p, x_{n+1}-p\rangle \\
&= \alpha_n\langle u-p, x_{n+1}-p\rangle + \beta_n\langle x_n-p, x_{n+1}-p\rangle \\
&\quad + \gamma_n\langle S y_n - p, x_{n+1}-p\rangle \\
&\le \alpha_n\langle u-p, x_{n+1}-p\rangle + \beta_n\|x_n-p\|\|x_{n+1}-p\| + \gamma_n\|S y_n-p\|\|x_{n+1}-p\| \\
&\le \alpha_n\langle u-p, x_{n+1}-p\rangle + \beta_n\|x_n-p\|\|x_{n+1}-p\| + \gamma_n\|y_n-p\|\|x_{n+1}-p\| \\
&\le \alpha_n\langle u-p, x_{n+1}-p\rangle + (1-\alpha_n)\|x_n-p\|\|x_{n+1}-p| \\
&\le \alpha_n\langle u-p, x_{n+1}-p\rangle + \frac{1-\alpha_n}{2}\|x_n-p\|^2 + \frac{1}{2}\|x_{n+1}-p\|^2,
\end{aligned}
\tag{2.13}
$$

which implies that

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2 + 2\alpha_n\langle u-p, x_{n+1}-p\rangle.$$

Applying Lemma 1.4 to (2.13), we obtain that

$$\lim_{n\to\infty}\|x_n-p\| = 0.$$

This completes the proof.

Putting $S=I$, the identity mapping, we have the following result.

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a μ_i-inverse-strongly monotone mapping for each $1 \le i \le r$, where r is some positive integer. Assume that $\mathcal{F} := \cap_{i=1}^r VI(C, A_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by the following manner:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{r} \eta_i P_C(x_n - \lambda_i A_i x_n), \quad n \ge 1,$$

where $u \in C$ is a fixed point, $\lambda_1, \lambda_2, \ldots$ and λ_r are real numbers such that $\lambda_i \in (0, 2\mu_i)$ for each $1 \le i \le r$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that the above control sequences satisfies the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^r \eta_i = 1$, $\forall n \ge 1$;

(ii) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}u$.

Next, we give a special case of Theorem 2.1 on a pair of inverse-strongly monotone mappings.

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a μ_1-inverse-strongly monotone mapping and $B : C \to H$ a μ_2-inverse-strongly monotone mapping, respectively. Let $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} := VI(C, A) \cap VI(C, B) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by the following manner:*

$$\begin{cases} x_1 \in C, \ \textit{chosen arbitrarily} \\ y_n = \eta P_C(x_n - \lambda A x_n) + (1-\eta) P_C(x_n - \rho B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S y_n, \quad n \geq 1, \end{cases}$$

where $u \in C$ is a fixed point, η is a real number in $(0,1)$, λ and ρ are real numbers such that $\lambda \in (0, 2\mu_1)$ and $\rho \in (0, 2\mu_2)$, respectively, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that the above control sequences satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}} u$.

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References

[1] F.E. Browder, Nonexpansive nonlinear operators in Banach spaces, Proc. Natl. Acad. Sci. USA, 54 (1965), 1041–1044.

[2] F.E. Browder and W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl., 20 (1967), 197–228.

[3] F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proc. Symp. Pure. Math., (18) (1976), 78–81.

[4] R.E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Tras. Amer. Math. Soc., 179 (1973) 251–262.

[5] L.C. Ceng, C.Y. Wang and J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Meth. Oper. Res., 67 (2008), 375–390.

[6] L.C. Ceng and J.C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput., 190 (2007), 205–215.

[7] J. Chen, L. Zhang and T. Fan, Viscosity approximation methods for nonexpansive mappings and monotone mappings, J. Math. Anal. Appl., 334 (2007), 1450–1461.

[8] Y.J. Cho and X. Qin, Systems of generalized nonlinear variational inequalities and its projection methods, Nonlinear Anal., 69 (2008), 4443–4451.

[9] P.L. Combettes, The convex feasibility problem: in image recovery, Advances in Imaging and Electron Physics, P. Hawkes, Ed., vol. 95, pp. 155–270, Academic Press, Orlando, Fla, USA, 1996.

[10] Y. Censor and S.A. Zenios, Parallel Optimization. Theory, Algorithms, and Applications, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, NY, USA, 1997.

[11] H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal., 61 (2005), 341–350.

[12] T. Kotzer, N. Cohen and J. Shamir, Images to ration by a novel method of parallel projection onto constraint sets, Opt. Lett., 20 (1995), 1172–1174.

[13] G. Marino and H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl., 329 (2007), 336–346.

[14] X. Qin and Y. Su, Approximation of a zero point of accretive operator in Banach spaces, J. Math. Anal. Appl., 329 (2007), 415–424

[15] X. Qin, M. Shang and Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Math. Comput. Model., 48 (2008), 1033–1046.

[16] X. Qin, Y.J. Cho and S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math., 225 (2009), 20–30.

[17] X. Qin, Y.J. Cho, J.I. Kang and S.M. Kang, Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces, J. Comput. Appl. Math., 230 (2009), 121-127.

[18] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochne integrals, J. Math. Anal. Appl., 305 (2005), 227–239.

[19] M.I. Sezan and H. Stark, Application of convex projection theory to image recovery in tomograph and related areas, in Image Recovery: Theory and Application, H. Stark, Ed., pp. 155–270 Academic Press, Orlando, Fla, USA, 1987.

[20] W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003), 417–428.

[21] M. Shang, Y. Su and X. Qin, Three-step iterations for nonexpansive mappings and inverse-strongly monotone mappings, J. Syst. Sci. Complex., 22 (2009), 333–343.

[22] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc., 66 (2002), 240–256.

[23] Y. Yao and J.C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, Appl. Math. Comput., 186 (2007), 1551–1158.

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