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SOMETHING ABOUT FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

Camelia Ciobanu

Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

In this paper we consider $\mathbb{Q}$ the class of solvable Lie algebras $\mathcal{L}$ with the following property: if $\mathcal{A}$ is a subalgebra of $\mathcal{L}$, then $\Phi(\mathcal{A}) \subseteq \Phi(\mathcal{L})$ (where $\Phi(\mathcal{L})$ denotes the Frattini subalgebra of $\mathcal{L}$;that is $\Phi(\mathcal{L})$ is the intersection of all maximal subalgebras of $\mathcal{L}$). The class $\mathbb{Q}$ is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal $\mathcal{I}$of $\mathcal{L} \in \mathbb{Q}$ be the Frattini subalgebra of $\mathcal{L}$. We considered here only solvable Lie algebras of finite dimension.

1. Introduction

In this section, we recall some notions and properties necessary in the paper.

We shall define by $\Phi(\mathcal{L})$ the Frattini subalgebra of $\mathcal{L}$, to be the intersection of all maximal subalgebras of $\mathcal{L}$.

We let $\mathcal{N}(\mathcal{L})$ be the nilradical of $\mathcal{L}$ and $\mathcal{S}(\mathcal{L})$ be the socle of $\mathcal{L}$; that is $\mathcal{S}(\mathcal{L})$ is the union of all minimal ideals of $\mathcal{L}$. If $\mathcal{A}$ and $\mathcal{B}$ are subalgebras of $\mathcal{L}$, let $Z_{\mathcal{B}}(\mathcal{A})$ be the centralizer of $\mathcal{A}$ in $\mathcal{B}$.

The center of A will be denoted by $Z(\mathcal{A})$. If $[\mathcal{B}, \mathcal{A}] \subseteq \mathcal{A}$, we let

$$Ad_{\mathcal{A}}(\mathcal{B}) = \{adb \text{ restricted to } \mathcal{A};\ \text{for all } b \in \mathcal{B}\}.$$

$\mathcal{L}^{'}$ will be the derived algebra of $\mathcal{L}$ and $\mathcal{L}^{''} = (\mathcal{L}^{'})^{'}$.

Key Words: p-Lie algebra, p-ideal, p-subalgebra, Cartan subalgebra

2.Lie algebras $\mathcal{L}$ with $\mathcal{L}^{'}$ nilpotent

Proposition 2.1. *Let $\mathcal{L}$ be a Lie algebra such that $\mathcal{L}^{'}$ is nilpotent. Then the following are equivalent.*
(i) $\Phi(\mathcal{L}) = 0$.
(ii) $\mathcal{N}(\mathcal{L}) = \mathcal{S}(\mathcal{L})$ *and* $\mathcal{N}(\mathcal{L})$ *is complemented by a subalgebra.*
(iii) $\mathcal{L}^{'}$ *is abelian, is a semi-simple* $\mathcal{L}$ *-module and is complemented by a subalgebra.*
Under these conditions, Cartan subalgebras of $\mathcal{L}$ *are exactly those subalgebras complementary to* $\mathcal{L}'$.

Proof. Assume (i) holds. Nilpotency of $\mathcal{L}^{'}$ implies $\Phi(\mathcal{L}) \supseteq \mathcal{L}^{''}$, so $\mathcal{L}^{'}$ is abelian and may be regarded as an $\mathcal{L}/\mathcal{L}'$-module. We may assume $\mathcal{L}^{'} = \sum \oplus \mathcal{V}_\rho$, $\mathcal{V}_\rho$ indecomposable $\mathcal{L}/\mathcal{L}'$-submodules. If $\mathcal{M}$ is a maximal subalgebra of $\mathcal{L}$ and if $\mathcal{V}_\rho \nsubseteq \mathcal{M}$, then $\mathcal{M} \cap \mathcal{V}_\rho$ is an ideal of $\mathcal{L}$. If $\mathcal{J}$ is an $\mathcal{L}/\mathcal{L}'$-submodule of V_ρ properly contained between $\mathcal{M} \cap V_\rho$ and V_ρ, then $\mathcal{M} + \mathcal{J}$ is a subalgebra of $\mathcal{L}$ properly contained between $\mathcal{M}$ and $\mathcal{L}$, contradicting the maximality of $\mathcal{M}$. Therefore $\mathcal{M}$ contains all maximal submodules of V_ρ for each $\mathcal{S}$. Then $\Phi(\mathcal{L}) = 0$ implies the interections of all maximal submodules of V_ρ is zero for each ρ. If $V_1, V_2, ..., V_s$ are maximal submodul of V_ρ with $V_1 \cap ... V_s = 0$ and are minimal with respect to this property, we have $V = V_2 \cap ... V_s \neq 0$ and $V \cap V_1 = 0$ so that $V \oplus V_1 = V_\rho$, contradicting indecomposability. Therefore each V_ρ is irreducible and $\mathcal{L}'$ is a completely reducible $\mathcal{L}/\mathcal{L}'$- module and is also a completely reducible $\mathcal{L}$- module. Since $\mathcal{L}$ is solvable it contains Cartan subalgebras by Theorem 3 of [1]. Let $\mathcal{C}$ be a Cartan subalgebra of $\mathcal{L}$ and let $\mathcal{L}_0$ and $\mathcal{L}_1$, be the Fitting null and one component of $\mathcal{L}$ with respect to $\mathcal{C}$. Then $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \equiv \mathcal{C} + \mathcal{L}_1 \subseteq \mathcal{C} + \mathcal{L}'$ shows $\mathcal{L} = \mathcal{C} + \mathcal{L}'$. We claim that $\mathcal{C} \cap \mathcal{L}^{'} = 0$. If $\mathcal{C} \cap \mathcal{L}^{'} \neq 0$, then, since $\mathcal{L}'$ is abelian, $\mathcal{C}$ is nilpotent and $\mathcal{L}'$ is a completely reducible $\mathcal{L}$-module, $\mathcal{L}'$ is a sum of irreducible $\mathcal{C}$-modules, $\mathcal{U}_1, ..., \mathcal{U}_q$, such that for each $\mathcal{U}$ $\underbrace{{}_i[..., [\mathcal{U}_i, \mathcal{C}], ...\mathcal{C}]}_{k} = 0$ for some k, hence $[\mathcal{U}_i, \mathcal{C}] = 0$. Thus $[\mathcal{C}, \mathcal{L}^{'} \cap \mathcal{C}] = 0$. One sees that each $\mathcal{U}_i$ is a central minimal ideal of $\mathcal{L}$, and since $\Phi(\mathcal{L}) = 0$, U_i is complemented by a maximal subalgebra $\mathcal{M}$. Therefore $\mathcal{U}_i$ is a one-dimensional direct summand of $\mathcal{L}$, contradicting $\mathcal{U}_i \subseteq \mathcal{L}^{'}$. Hence $\mathcal{L}^{'} \cap \mathcal{C} = 0$ and $\mathcal{C}$ is a complement to $\mathcal{L}^{'}$ in $\mathcal{L}$. Since $[\mathcal{C}, \mathcal{C}] \subseteq \mathcal{C} \cap \mathcal{L}^{'} = 0$, $\mathcal{C}$ is abelian. Any minimal ideal not in $\mathcal{L}^{'}$ satisfies $[\mathcal{L}, \mathcal{A}] \subseteq \mathcal{A} \cap \mathcal{L}^{'} = 0$, so is central.

Therefore $\mathcal{S}(\mathcal{L}) = \mathcal{L}^{'} + Z(\mathcal{L})$ and, since $\mathcal{C}$ is a Cartan subalgebra, $Z(\mathcal{L}) \subseteq \mathcal{C}$. Let $\mathcal{C}_0$ be a complementary subspace to $Z(\mathcal{L})$ in $\mathcal{C}$.

One sees that

$$\mathcal{N}(\mathcal{L}) = \mathcal{L}^{'} + Z(\mathcal{L}) + (\mathcal{N}(\mathcal{L}) \cap \mathcal{C}_0) = \mathcal{S}(\mathcal{L}) + (\mathcal{N}(\mathcal{L}) \cap \mathcal{C}_0).$$

If c is a nonzero element in $\mathcal{N}(\mathcal{L}) \cap \mathcal{C}_0$, and c is nilpotent but not zero with implies $[V_\rho, c] = V_\rho$ for some $V_\rho \subseteq \mathcal{L}^{'}$ and $[...[V_\rho, \underbrace{c]...c]}_{k} = 0$ for some k, a contradicting. This $\mathcal{S}(\mathcal{L}) = \mathcal{N}(\mathcal{L})$ and $\mathcal{C}_0$ is a complement. Consecuently (i) implies (ii).

Assume *(ii)* holds and proceed by induction on the dimension of $\mathcal{L}$. Since $\mathcal{L}' \subseteq \mathcal{N}(\mathcal{L}) = S(L)$ and minimal ideals are abelian, $\mathcal{L}^{'}$ is abelian. If every minimal ideal of $\mathcal{L}$ is contained in $\mathcal{L}^{'}$, then $\mathcal{S}(\mathcal{L}) = \mathcal{L}^{'}$ and (iii) follows. Therefore let $\mathcal{J}$ be the minimal ideal of $\mathcal{L}$ such that $\mathcal{J} \nsubseteq \mathcal{L}^{'}$.

Hence $\mathcal{J} \nsubseteq \Phi(\mathcal{L})$and there exists a maximal subalgebra $\mathcal{M}$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{M} + \mathcal{J}$.

Since $[\mathcal{L}, \mathcal{J}] \subseteq \mathcal{J} \cap \mathcal{L}^{'}, \mathcal{J}$ is central, hence one-dimensional. It follows that $\mathcal{L}$ is the Lie algebra direct sum of $\mathcal{M}$ and $\mathcal{J}$. Since $\mathcal{M}$ inherits the condition (ii), $\mathcal{M}$ satisfies (iii) by induction. It now follows that $\mathcal{L}$ also satisfies (iii). Assume (iii) holds. Then $\mathcal{L}^{'}$ is a sum of minimal ideals of $\mathcal{L}$, which we denote by $\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_k$ and $\mathcal{L} = \mathcal{L}^{'} + \mathcal{D}$, $\mathcal{D}$ a subalgebra of $\mathcal{L}$. Since $\mathcal{D}^{'} \subseteq \mathcal{D} \cap \mathcal{L}^{'} = 0, \mathcal{D}$ is abelian. One sees that $\mathcal{L}^{'} = [\mathcal{L}^{'}, \mathcal{D}]$ and, consequently, $\mathcal{J}_i = [\mathcal{J}_i, \mathcal{D}]$ for all i. Since $Z_{\mathcal{J}}(\mathcal{D})$ is central in $\mathcal{L}$, $Z_{\mathcal{J}_i}(\mathcal{D})$ is an ideal in $\mathcal{L}$ contained in $\mathcal{J}_i$. Since $Z_{\mathcal{J}_i}(\mathcal{D}) \neq \mathcal{J}_i$, $Z_{\mathcal{J}_i}(\mathcal{D}) = 0$. It follows that $\mathcal{D}$ is its own normalizer, hence is a Cartan subalgebra of $\mathcal{L}$.

Now $\mathcal{D} + \mathcal{J}_i + ... + \widehat{\mathcal{J}_i} + ... + \mathcal{J}_k$ is a maximal subalgebra of $\mathcal{L}$ since any containing algebra has a nonzero projection on $\mathcal{J}_i$ which is *ad* $\mathcal{D}$ stable, hence equal to $\mathcal{J}_i$. Therefore $\Phi(\mathcal{L}) \subseteq \mathcal{D}$ and $\Phi(\mathcal{L}) \subseteq \mathcal{D} \cap \mathcal{L}^{'} = 0$. Hence (i) holds.

The fact that complements to $\mathcal{L}^{'}$ are Cartan subalgebras is shown in (iii) implies (i). The fact that Cartan subalgebras are complements to $\mathcal{L}^{'}$ is shown in (i) implies (ii). This completes the proof of Proposition 2.1. □

Theorem 2.1. *Let $\mathcal{L}$ be a Lie algebra such that $\mathcal{L}'$ is nilpotent and* $\Phi(\mathcal{L})$=0. *Then, for any subalgebra $\mathcal{D}$ of $\mathcal{L}$, $\Phi(\mathcal{D}) = 0$.*

Proof. Suppose $\mathcal{L}^{'} \subseteq \mathcal{D}$. Let ξ be a complement to $\mathcal{L}^{'}$ in $\mathcal{L}$, so $\xi \cap \mathcal{D}$ is a complement to $\mathcal{L}^{'}$ in $\mathcal{D}$. Since $\mathcal{L}$ acts completely reducibly on $\mathcal{L}^{'}$ and $\mathcal{L}^{'}$ is abelian, ξ acts completely reducible on $\mathcal{L}^{'}$. Then, since ξ is abelian, $\mathcal{D} \cap \xi$ acts completely reducibly on $\mathcal{L}^{'}$, hence so does $\mathcal{D}$.

Therefore $\mathcal{L}^{'} = \mathcal{D}^{'} \oplus \mathcal{J}$ for some ideal $\mathcal{J}$ in $\mathcal{D}$ when $\mathcal{D}$ acts completely reducibly on $\mathcal{D}^{'}$ and $\mathcal{J} + (\xi \cap \mathcal{D})$ is a complementary subalgebra of $\mathcal{D}^{'}$in $\mathcal{D}$. Thus, by Proposition 2.1, $\Phi(\mathcal{D}) = 0$.

Suppose $\mathcal{L}^{'} \nsubseteq \mathcal{D}$. Since $\mathcal{D} + \mathcal{L}^{'}$ falls in the preceding case, we may assume $\mathcal{D} + \mathcal{L}^{'} = \mathcal{L}$. Since $\mathcal{L}^{'}$ is abelian, $\mathcal{L}^{'} \cap \mathcal{D}$ is an ideal in $\mathcal{L}$, $\mathcal{D}/(\mathcal{L}^{'} \cap \mathcal{D})$ complements $\mathcal{L}^{'}/(\mathcal{L}^{'} \cap \mathcal{D}) = (\mathcal{L}/\mathcal{L}^{'} \cap \mathcal{D})^{'}$in $\mathcal{L}/(\mathcal{L}^{'} \cap \mathcal{D})$ and $\mathcal{D}/(\mathcal{L}^{'} \cap \mathcal{D})$ acts completely reducibly in $\mathcal{L}^{'}/(\mathcal{L}^{'} \cap \mathcal{D})$, $\mathcal{D}/(\mathcal{L}^{'} \cap \mathcal{D})$ is a Cartan subalgebra of $\mathcal{L}/(\mathcal{L}^{'} \cap \mathcal{D})$.

Let ζ be a Cartan subalgebra of $\mathcal{D}$, so it is a Cartan subalgebra of $\mathcal{L}$. This ζ is a complement to $\mathcal{L}^{'}$ and $\zeta + (\mathcal{L}^{'} \cap \mathcal{D}) = \mathcal{D}$ since $\subseteq \mathcal{D}$. Hence ζ is a complement to $\mathcal{L}^{'} \cap \mathcal{D}$ in $\mathcal{D}$. Since $\mathcal{D}$ acts completely reducibly on $\mathcal{L}^{'} \cap \mathcal{D}$ and $\mathcal{D}^{'} \subseteq \mathcal{L}^{'} \cap \mathcal{D}, \mathcal{D}$ acts completely reducibly on $\mathcal{D}^{'}$,

$$\mathcal{L}^{'} \cap \mathcal{D} = \mathcal{D}^{'} \oplus (\mathcal{L}^{'} \cap Z(\mathcal{D}))$$

and, since $Z(\mathcal{D}) \subseteq \zeta$,

$$Z(\mathcal{D}) \cap \mathcal{L}^{'} \subseteq \zeta \cap \mathcal{L} = 0.$$

Therefore $\zeta = \mathcal{D}^{'} = \mathcal{D}$ and $\zeta \cap \mathcal{D}^{'} = 0$. Now $\mathcal{D}$ satisfies part (iii) of Proposition 2.1 hence, $\Phi(\mathcal{D}) = 0$. $\square$

If $\mathcal{L}$ is a solvable Lie algebra it has been shown that $\Phi(\mathcal{L})$ is an ideal of $\mathcal{L}$. We look for a condition on the subalgebras of $\mathcal{L}/\Phi(\mathcal{L})$ with are necessary and sufficient that $\mathcal{L} \in \mathbb{Q}$. In order to do this, the following concept is introduced.

Definition 2.1. We shall say that a Lie algebra $\mathcal{L}$ is *the reduced partial sum of an ideal* $\mathcal{J}$ *and a subalgebra* $\mathcal{A}$ if $\mathcal{L} = \mathcal{A} + \mathcal{J}$ and for any subalgebra $\mathcal{D}$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{J} + \mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{A}$ then $\mathcal{D} = \mathcal{A}$.

It is noted that if $\mathcal{J} \nsubseteq \Phi(\mathcal{L})$,then there exists a subalgebra $\mathcal{A} \neq \mathcal{L}$ such that $\mathcal{L}$ is the reduced partial sum of $\mathcal{J}$ and $\mathcal{A}$. On the other hand, if $\mathcal{J} \subseteq \Phi(\mathcal{L})$and $\mathcal{L}$ is the reduced partial sum of $\mathcal{J}$ and $\mathcal{A}$, then $\mathcal{A} = \mathcal{L}$.

Lemma 2.1. *Let* $\mathcal{L}$ *be the reduced partial sum of* $\mathcal{J}$ *and* $\mathcal{A}$.
Then $\mathcal{J} \cap \mathcal{A} \subseteq \Phi(\mathcal{A})$.

Proof. Suppose $\mathcal{B} = \mathcal{J} \cap \mathcal{A} \nsubseteq \Phi(\mathcal{A})$.Then $\mathcal{A}$ contains a subalgebra $\mathcal{D}$ such that $\mathcal{B} + \mathcal{D} = \mathcal{A}$.Then $\mathcal{L} = \mathcal{J} + \mathcal{A} = \mathcal{J} + \mathcal{B} + \mathcal{D} = \mathcal{J} + \mathcal{D}$.This contradicts the minimality of $\mathcal{A}$, then $\mathcal{J} \cap \mathcal{A} \subseteq \Phi(\mathcal{A})$. $\square$

Lemma 2.2. *Let* $\mathcal{L}$ *be the reduced partial sum of* $\mathcal{J}$ *and* $\mathcal{A}$.
Then $\Phi(\mathcal{L}/\mathcal{J}) \cong \mathcal{J} + \Phi(\mathcal{A})/\mathcal{J}$.

Proof. Since $\mathcal{J} \cap \mathcal{A} \subseteq \Phi(\mathcal{A})$, $\mathcal{J} \cap \Phi(\mathcal{A}) = \mathcal{J} \cap \mathcal{A}$. Since

$$\mathcal{L}/\mathcal{J} \cong \mathcal{A} + \mathcal{A}/\mathcal{J} \cong \mathcal{A}/\mathcal{J} \cap \mathcal{A},$$

$\Phi(\mathcal{L}/\mathcal{J}) \cong \Phi(\mathcal{A}/\mathcal{J} \cap \mathcal{A}) \cong \Phi(\mathcal{A})/\mathcal{J} \cap \mathcal{A} = \Phi(\mathcal{A})/\mathcal{J} \cap \Phi(\mathcal{A}) \cong \mathcal{J} + \Phi(\mathcal{A})/\mathcal{J}$. $\square$

Proposition 2.2. *For a Lie algebra* $\mathcal{L}$*, the following statements are equivalent:*
(ii) $\mathcal{L} \in \mathbb{Q}$.

(ii) For any subalgebra $\mathcal{A}$ of $\mathcal{L}/\Phi(\mathcal{L})$, $\Phi(\mathcal{A}) = 0$.

Proof. Let $\mathcal{L}$ satisfy (i) and let $\pi : 1 \to \mathcal{L}/\Phi(\mathcal{L})$ be the natural homomorphism. Then $\Phi(\pi(\mathcal{L})) = \pi(\Phi(\mathcal{L})) = 0$.

Let $\overline{\mathcal{B}}$ be a subalgebra of $\mathcal{L}/\Phi(\mathcal{L})$ and let $\mathcal{B}$ be the subalgebra of $\mathcal{L}$ which contains $\Phi(\mathcal{L})$ and corresponds to $\overline{\mathcal{B}}$. Since $\mathcal{L}$ satisfies (i), $\Phi(\mathcal{B}) \subseteq \Phi(\mathcal{L})$. If $\Phi(\mathcal{B}) =\Phi(\mathcal{L})$, then

$$\Phi\left(\pi(\mathcal{B})\right) = \pi\left(\Phi(\mathcal{B})\right) = \pi\left(\Phi(\mathcal{L})\right) = 0.$$

Suppose then that $\Phi(\mathcal{B}) \subset \Phi(\mathcal{L})$.Then $\mathcal{B}$ can be represented as a reduced partial sum $\mathcal{B} = \Phi(\mathcal{L}) + \mathcal{P}$. Let $\mathcal{R}$ be a subalgebra of $\mathcal{B}$ such that $\mathcal{R}/\Phi(\mathcal{L}) \cong \Phi\left(\mathcal{B}/\Phi(\mathcal{L})\right)$.If $\mathcal{R}/\Phi(L) \neq 0$,then

$$\mathcal{R} = \mathcal{R} \cap (\Phi(\mathcal{L}) + \mathcal{P}) == (\mathcal{R} \cap \Phi(\mathcal{L})) + (\mathcal{R} \cap \mathcal{P}) = \Phi(\mathcal{L}) + (\mathcal{R} \cap \mathcal{P}).$$

Consequently there exists an $x \in \mathcal{R} \cap \mathcal{P}$, $x \notin \Phi(\mathcal{L})$. Since $\Phi(\mathcal{P}) \subseteqq \Phi(\mathcal{L})$, $x \notin \Phi(\mathcal{P})$ and there exists a maximum subalgebra $\mathcal{M}$ of $\mathcal{P}$ such that $x \notin M$. We claim that either $\Phi(\mathcal{L}) + \mathcal{M} = \mathcal{B}$ or $\Phi(\mathcal{L}) + \mathcal{M}$ is maximal in $\mathcal{B}$.

Suppose $\Phi(\mathcal{L}) + \mathcal{M} \neq \mathcal{B}$ and let J be a subalgebra of $\mathcal{B}$ which contains $\Phi(\mathcal{L}) + \mathcal{M}$. Then $\mathcal{M} \subseteq J \cap \mathcal{P}$, so, by the maximality of $\mathcal{M}$, either $J{\cap}\mathcal{P} = \mathcal{M}$ or $J{\cap}\mathcal{P} = \mathcal{P}$. If $J{\cap}\mathcal{P} = \mathcal{M}$ then

$$\Phi(\mathcal{L}) + \mathcal{M} = \Phi(\mathcal{L}) + (J \cap \mathcal{P}) = J \cap (\Phi(\mathcal{L}) + \mathcal{P}) = J \cap \mathcal{B} = J.$$

If $J \cap \mathcal{P}{=}\mathcal{P}$, then $J \supseteqq \mathcal{P}$ and, since $J \supseteqq \Phi(\mathcal{L})$, $J \supseteqq \Phi(\mathcal{L}){+}\mathcal{P}{=}\mathcal{B}$, hence $J = \mathcal{B}$. Consequently, there exists no subalgebras of $\mathcal{B}$ properly contained between $\Phi(\mathcal{L}){+}\mathcal{M}$ and $\mathcal{B}$, hence either $\Phi(\mathcal{L}) + \mathcal{M} = \mathcal{B}$ or $\Phi(\mathcal{L}){+}\mathcal{M}$ is maximal in $\mathcal{B}$. If $\Phi(\mathcal{L}) + \mathcal{M} = \mathcal{B}$, then $\Phi(\mathcal{L}) + \mathcal{P}$ is not a reduced partial sum which is a contradiction. If $\Phi(\mathcal{L}) + \mathcal{M}$ is maximal in $\mathcal{B}$, then

$$\Phi(\mathcal{L}) + \mathcal{M}/\Phi(\mathcal{L}) \supseteqq \Phi\left(\mathcal{B}/\Phi(\mathcal{L})\right) \cong \left(\mathcal{R} \cap \Phi(\mathcal{L})\right).$$

Hence $R \subseteqq \Phi(\mathcal{L}) + \mathcal{M}$. Since $\mathcal{M} \subseteqq \Phi(\mathcal{L}) + \mathcal{M}$ and

$$x \in \mathcal{R} \cap \mathcal{P} \subset \mathcal{R} \subseteqq \Phi(\mathcal{L}) + \mathcal{M},$$

$$\mathcal{P} = \{\mathcal{M}, x\} \subseteqq \Phi(\mathcal{L}) + \mathcal{M}.$$

Then $\mathcal{B} =\Phi(\mathcal{L}) + \mathcal{P} \subseteqq \Phi(L) + \mathcal{M} \subseteq \mathcal{B}$ implies $\Phi(\mathcal{L}) + \mathcal{P}$ is not a reduced partial sum, a contradiction.

Hence $\Phi(\overline{\mathcal{B}}) = \mathcal{R}/\Phi(\mathcal{L}) = 0$ and (ii) is satisfied.

If $\mathcal{L}/\Phi(\mathcal{L})$ satisfies (ii), then $\pi\left(\Phi(\mathcal{A})\right) \subseteqq \Phi\left(\pi(\mathcal{A})\right) = 0$ for every subalgebra $\mathcal{A}$ of $\mathcal{L}$. Then $\Phi(\mathcal{A}) \subseteq \Phi(\mathcal{L})$for every subalgebra $\mathcal{A}$ of $\mathcal{L}$. □

(Combining Proposition 2.2 and Theorem 2.1 we have:

Theorem 2.2. *Let $\mathcal{L}$ be a Lie algebra such that $\mathcal{L}^{'}$ is nilpotent. Then $\mathcal{L} \in \mathbb{Q}$.* $\square$

Theorem 2.3. *Let $\mathcal{L} \in \mathbb{Q}$ and let φ be a Lie homomorphism of $\mathcal{L}$. Then $\varphi(\Phi(\mathcal{L})) = \Phi(\varphi(\mathcal{L}))$.*

Proof. $\varphi(\Phi(\mathcal{L}))$ is always contained in $\Phi(\varphi(\mathcal{L}))$. If k=*Kernel* $\varphi \subseteqq \Phi(\mathcal{L})$, then equality holds. Suppose $\mathcal{P} \nsubseteqq \Phi(\mathcal{L})$.

Let $\mathcal{L} = \mathcal{P} + k$ be a reduced partial sum. Using Lemma 2.2,

$$\Phi(\varphi(\mathcal{L})) = \Phi(\mathcal{L}/\mathcal{P}) \cong \mathcal{P} + \Phi(k)/\mathcal{P} = \varphi(\Phi(k)).$$

Since $\varphi(\mathcal{P}+\Phi(\mathcal{L})) = \varphi(\Phi(\mathcal{L})) \subseteqq \Phi(\varphi(\mathcal{L})) = \Phi(\mathcal{L}/\mathcal{P}) = \varphi(\mathcal{P} + \Phi(k))$,

$$\mathcal{P}+\Phi(\mathcal{L}) \subseteq \mathcal{P}+\Phi(k) \subseteq \mathcal{P} + \Phi(\mathcal{L}).$$

Hence $\mathcal{P}+\Phi(\mathcal{L}) = \mathcal{P}+\Phi(k)$ and $\Phi(\varphi(\mathcal{L})) = \varphi(\Phi(k)) = \varphi(\Phi(\mathcal{L}))$. $\square$

Theorem 2.4. Let $\mathcal{L} \in \mathbb{Q}$. *Necessary conditions that an ideal $\mathcal{J}$ of $\mathcal{L}$ be the Frattini subalgebra of $\mathcal{L}$ are the following:*
(*i*) $\Phi(Ad_{\mathcal{J}}(\mathcal{L}) = Ad_{\mathcal{J}}(\Phi(\mathcal{L}))$.
(*ii*) *There exists a subalgebra $\mathcal{A}$ of $\mathcal{L}$ such that* $\mathcal{A}/\mathcal{J} \cong Ad_{\mathcal{J}}(\mathcal{L})/Ad_{\mathcal{J}}(\Phi(\mathcal{L}))$.

Proof.(*i*) Let φ be the mapping from $\mathcal{L}$ into the derivation algebra of $\mathcal{J}$ defined by $\varphi(x) = adx$ restricted to $\mathcal{J}$ for all $x \in \mathcal{L}$. Then

$$\varphi(\Phi(\mathcal{L})) = Ad_{\mathcal{J}}(\Phi(\mathcal{L})) = \Phi(\varphi(\mathcal{L})) = \Phi(Ad_{\mathcal{J}}(\mathcal{L})).$$

(*ii*) Let $\mathcal{B} = Z_{\mathcal{L}}(\Phi(\mathcal{L}))$. Suppose that $\mathcal{B} \nsubseteqq \Phi(\mathcal{L})$ and let $\xi = \mathcal{L}/\Phi(\mathcal{L})$ and $\mathcal{R} = (\mathcal{B} + \Phi(\mathcal{L}))/\Phi(\mathcal{L})$. Since $Ad_{\Phi(\mathcal{L})}(\mathcal{L}) \cong \mathcal{L}/\mathcal{B}$ and

$$Ad_{\Phi(\mathcal{L})}(\Phi(\mathcal{L})) \cong \Phi(\mathcal{L})/Z(\Phi(\mathcal{L})) = \Phi(\mathcal{L})/\mathcal{B} \cap \Phi(\mathcal{L}) = (\mathcal{B} + \Phi(\mathcal{L}))/\mathcal{B},$$

$$\xi/\mathcal{R} \cong (\mathcal{L}/\Phi(\mathcal{L}))/(\mathcal{B} + \Phi(\mathcal{L})/\Phi(\mathcal{L})) \cong \mathcal{L}/(\mathcal{B} + \Phi(\mathcal{L})) \cong$$

$$\cong (\mathcal{L}/\mathcal{B})/((\mathcal{B} + \Phi(\mathcal{L}))/\mathcal{B}) \cong Ad_{\Phi(\mathcal{L})}(\mathcal{L})/Ad_{\Phi(\mathcal{L})}(\Phi(\mathcal{L})).$$

Since $\Phi(\xi) = 0$, there exists a subalgebra $\mathcal{D}$ in ξ such that ξ is the reduced partial sum of $\mathcal{R}$ and $\mathcal{D}$.

Using Proposition 2.2 and Lemma 2.1, $\mathcal{R} \cap \mathcal{D} \subseteqq \Phi(\mathcal{D}) = 0$, hence $\mathcal{R} \cap \mathcal{D} = 0$.

Let $\mathcal{H}$ be the subalgebra of $\mathcal{L}$ which contains $\Phi(\mathcal{L})$ and corresponds to $\mathcal{D}$. Then

$$\mathcal{H}/\Phi(\mathcal{L}) \cong \mathcal{D} \cong \xi/\mathcal{R} \cong Ad_{\Phi(\mathcal{L})}(\mathcal{L})/Ad_{\Phi(\mathcal{L})}(\Phi(\mathcal{L})). \ \square$$

If $\mathcal{B} \subseteq \Phi(\mathcal{L})$,then

$$Ad_{\Phi(\mathcal{L})}(\mathcal{L})/Ad_{\Phi(\mathcal{L})}\left(\Phi(\mathcal{L})\right) \cong (\mathcal{L}/\mathcal{B})/\left(\Phi(\mathcal{L})/Z\left(\Phi(\mathcal{L})\right)\right) =$$

$$= (\mathcal{L}/\mathcal{B})/\left(\Phi(\mathcal{L})/\mathcal{B} \cap \Phi(\mathcal{L})\right) = (\mathcal{L}/\mathcal{B})/\left(\Phi(L)/B\right) \cong \mathcal{L}/\Phi(\mathcal{L}).$$

Related to part (i) of Theorem 2.4, there are the following results.

Theorem 2.5. *Let $\mathcal{L} \in \mathbb{Q}$ and let $\mathcal{J}$ be an ideal of $\mathcal{L}$ containing $\Phi(\mathcal{L})$. Then, $\Phi\left(Ad_{\mathcal{J}}(\mathcal{L})\right) \cong Ad_{\mathcal{J}}(\mathcal{J})$ if and only if $\mathcal{J} = \Phi(\mathcal{L}) + Z(\mathcal{J})$.* □

Theorem 2.6. *Let $\mathcal{L} \in \mathbb{Q}$ and let $\mathcal{J}$ be an ideal of $\mathcal{L}$ contained in $\Phi(\mathcal{L})$. Then, $\Phi\left(Ad_{\mathcal{J}}(\mathcal{L})\right) \cong Ad_{\mathcal{J}}(\mathcal{J})$ if and only if $\Phi(\mathcal{L}) = \mathcal{J} + Z_{\Phi(\mathcal{L})}(\mathcal{J})$.* □

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Department of Mathematics and Informatics,
"Mircea cel Bătrân" Naval Academy,
Fulgerului 1
8700 - Constantza
Romania