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SOME NOTES ABOUT p-LIE ALGEBRAS

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Abstract

The aim of this paper is to present some results about the finite dimensional p-Lie algebras and some properties of p-c-supplemented subalgebras of p-Lie algebras. In addition some relations between p-c-supplemented Lie algebras and E-p-algebras are pointed out.

1 Preliminaries

We recall first some definitions in the theory of p-Lie algebras. The concept of p-Lie algebra has been introduced by Jacobson [4] and it is the following:

Definition 1.1 A p-Lie algebra is a Lie algebra, L, over a field, k, of characteristic $p > 0$, with a p-map $a \to a^{[p]}$ such that the following identities hold:

$$(\alpha a)^{[p]} = \alpha^p a^{[p]} \ for \ all \ \alpha \in K, \ a \in L,$$

$$[a, b^{[p]}] = a(adb^{[p]}) = a(adb)^p \ for \ all \ a, \ b \in L, \ and$$

$$(a+b)^{[p]} \ = \ a^{[p]} \ + \ b^{[p]} \ + \sum_{i=1}^{p-1} s_i(a,b) \ for \ all \ a, \ b \in L,$$

where $is_i(a,b)$ is the coefficient of X^{i-1} in the expansion of $a(ad(Xa + b))^{p-1}$,where X is an auxiliary indeterminate(see [4] pg. 187).

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In particular, we have $s_1(a,b) = [...[a,\underbrace{b]...b]}_{p-1}$. Note also that the sum $\sum s_i(a,b)$ belongs to the Lie algebra generated by a and b (with respect to the vector space operations and the multiplication given by $[x,y] = xy - yx$).

Definition 1.2 A subalgebra (respectively, ideal) of a Lie algebra L is a p-subalgebra (respectively, p-ideal) of L if it is close under the p-map.
Throughout this article, L will denote a finite-dimensional p-Lie algebra over a field K of characteristic $p > 0$.
We denote by $\Phi(L)$ the Frattini subalgebra of L. It is know that $\Phi(L)$ is the intersection of the maximal subalgebras of L.
We denote by $F(L)$ the Frattini ideal of L, that is the largest ideal of L which is contained in $\Phi(L)$. Analogously, $F_p(L)$ denotes the Frattini p-ideal of L i.e. the largest p-ideal of L which is contained in $\Phi_p(L)$, where $\Phi_p(L)$ is the Frattini p-subalgebra of L, that is the intersection of the maximal p-subalgebras of L.

Definition 1.3 If A is a p-subalgebra of L, the p-core of A is the largest ideal of L contained in A and we denote that by ${}_pA_L$.

We say that A is p-core-free in L if ${}_pA_L = 0$. A p-subalgebra A of L is p-c-suplemented in L if there is a p-subalgebra B of L such that $L = A + B$ and $A \cap B$ is a p-subalgebra of ${}_pA_L$. We say that L is p-c-supplemented if every p-subalgebra of L is p-c-supplemented in L.

Definition 1.4 A finite dimensional Lie algebra (respectively Lie p-algebra) L is called elementary (respectively p-elementary), if $F(A) = 0$ (respectively $F_p(A) = 0$) for every Lie subalgebra (respectively Lie p-subalgebra) A of L.
Following [3] we say that L is p-completely factorisable if for every p-subalgebra A of L there is a p-subalgebra B of L such that $L = A + B$ and $A \cap B = 0$

Definition 1.5 A finite dimensional Lie algebra (respectively Lie p-algebra), L, is an E-algebra (respectively, E-p-algebra) if for every Lie subalgebra (respectively, Lie p-subalgebra) U of L we have $F(U) \subseteq F(L)$ (respectively, $F_p(U) \subseteq F_p(L)$)

As usual, we denote by $[x,y]$ the product between x and y in L, $L^{(1)}$ the derived algebra of L, and we reconsider the subsets of L:

$$(A)_p = (\{x^{p^n} \mid x \in A, n \in N\}) \textit{ where } x^{p^n} = (x^{p^{n-1}})^p;$$

$$A^p = (\{x^p \mid x \in A\}) \textit{ where } A \textit{ is a subalgebra of } L;$$

$$L_1 := \bigcap_{i=1}^{\infty} L^{p^i};$$

$$L_0 := \{x \in L|\ x^{p^n} = 0,\ n \in N\};$$

$$C_L(M) := \{x \in L|\ \ [x, M] = 0\}\,,\ where\ M \subset L;$$

$$N_L(A) = \{x \in L|[x, A] \subseteq A\},\ where\ A \subset L.$$

Some properties of the p-ideals and p-subalgebras of an p-Lie algebra has been found by D.A. Towers [6]:

P.1 *Let A be a subalgebra of L. Then* $(A)_p^{(1)} \subseteq A^{(1)}$.

P.2 *If I is an ideal of L, then* $(I)_p \subseteq C_L(I)$ *. In particular,* $(I)_p$ *is an ideal of L.*

P.3 *If* $A \subseteq L$ *then* $N_L(A)$ *is p-closed.*

Now we present some results concerning $F_p(L)$ and $\Phi_p(L)$ inspired by the results for $F(L)$ and $\Phi(L)$ obtained by Stitzinger [5] for nilpotent Lie algebras and completed in Ciobanu [2].

Lemma 1.1 *For any p-Lie algebra , L, and any p-subalgebra of L, A, the following statements hold:*
(i) If $A + \Phi_p(L) = L$ *then* $A = L$.
(ii) If I is an ideal of L such that $I \subset \Phi_p(A)$*, then* $I \subset \Phi_p(L)$.

Lemma 1.2 *If I is a p-ideal of L, then the following asertions are true:*
(i) $(\Phi_p(L) + I)/I \subset \Phi_p(L/I)$.
(ii) $(F_p(L) + I)/I \subset F_p(L/I)$.
(iii) If $I \subset \Phi_p(L)$*, then (i) and (ii) are true with equality. Moreover, if* $F_p(L/I) = 0$*, then* $F_p(L) \subset I$.
(iv) If A is a minimal p-subalgebra of L such that $L = I + A$ *then* $I \cap A \subset F_p(A)$.
(v) If $I \cap F_p(L) = 0$*, then there is a p-subalgebra A of L such that* $L = I \dot{+} A$ *(where* $\dot{+}$ *denotes a vector space direct sum).*

Definition 1.6 (i) If L is a p-Lie algebra we denote by Sp(L) the sum of the minimal abelian p-ideals of L and we call it the abelian p-socle of L.
(ii) We say that L p-splits over an p-ideal I of p-Lie algebra L, if there is a p-subalgebra A of L such that $L = I \dot{+} A$, where $\dot{+}$ represents the direct sum of vector spaces.

Lemma 1.3 *The abelian p-socle Sp(L) is a p-ideal of L.*

2 F-free respectively F_p-free Lie algebras

In this section we present the relationship between $C_L(F(L))$ and $C_L(F_p(L))$.

Definition 2.1 A p-Lie algebra is called F-free (respectively, F_p-free) if $F(L) = 0$ (respectively, $F_p(L) = 0$).

Proposition 2.1 *If L is F_p-free, then L p-splits over its abelian p-socle.*

Proof. If L is F_p-free, then $F_p(L) = 0$. Let I=Sp(L) the abelian-p-socle of L which is an abelian p-ideal of L, then $I \cap F_p(L) = 0$ In accord with Lemma 2.5*(iv)*, there is a p-subalgebra A of L such that $L = I \dot{+} A$, so L p-splits over I.

Proposition 2.2 *For any p-Lie algebra, L, $F(L) \subset F_p(L)$.*

Proof. It is clear that $L/F_p(L)$ is $F_p(L)$-free and hence, by using Prop. 2.1, we get that $L/F_p(L)$ splits over $S(L/F_p(L))$. So $L/F_p(L)$ is F-free and it follows that $F(L) \subset F_p(L)$.

In the following we introduce two special subalgebras of L.

Definition 2.2 If L is a p-Lie algebra we note by T(L)-the intersection of all maximal subalgebras of L which are not ideals of L and correspondingly by $T_p(L)$ the intersection of all maximal p-subalgebras of L which are not p-ideals of L. We also define $\tau(L)$(respectively, $\tau_p(L)$) to be the largest ideal(respectively, p-ideal) of L that is contained in T(L)(respectively, $T_p(L)$). In these conditions the following statements occur :

Lemma 2.1 *If I is a p-ideal of L, then:*
(i) $(T_p(L) + I) \subset T_p(L/I)$.
(ii) $(\tau_p(L) + I)/I \subset \tau_p(L/I)$.
(iii) If $I \subset T_p(L)$,then statements (i) and (ii) occur with equality. More over, if $\tau_p(L/I) = 0$ then $\tau_p(L) \subset I$.

Lemma 2.2 *Let A be a maximal subalgebra of L, then:*
(i) if A is an ideal of L, $L^{(1)} \subset A$.
(ii) if A is not an ideal of L,then it is a p-subalgebra of L.

Proof. *(i)* Let $x \notin A$. Then $L = A + (x)_p$ and, according to P.1, it results that $L^{(1)} \subset A$.
(ii) We suppose that A is not p-close , then $(A)_p = L$, from which, according to P.1 we deduce that $L^{(1)} = (A)_p^{(1)} \subset A^{(1)} \subset A$, hence A is an ideal of L, contradicting the hypothesis, so any maximal subalgebra of L which is not an ideal of L is a p-subalgebra of L.

Proposition 2.3 *For any p-Lie algebra L the following statements hold:*
(i) $\tau_p(L) = C_L(F_p(L))$.
(ii) $\tau(L) = \tau_p(L)$.
(iii) If N is the nilradical of L, then $F_p(L) = \Phi_p(L) \cap N$.
(iv) If L is perfect(that is $L = L^{(1)}$), then $\Phi_p(L) = \Phi(L)$.

Proof.*(i)* According to Lemma 2.2*(i)* we have $[\tau_p(L), L] \subset L^{(1)} \cap \tau_p(L) \subset F_p(L)$, hence $\tau_p(L) \subset C_L(F_p(L))$. Now we suppose that $C_L(F_p(L)) \not\subseteq \tau_p(L)$. Then there is a maximal p-subalgebra A of L which is not an ideal of L such that $C_L(F_p(L)) \not\subseteq A$. Now it is easy to prove that $C_L(F_p(L))$ is p-closed , so $L = C_L(F_p(L)) + A$. Then $L^{(1)} \subset F_p(L) + A \subset A$ and A is an ideal of L, which is a contradiction. Therefore $C_L(F_p(L)) \subset \tau_p(L)$.

*(ii)*According to Lemma 3.2 *(ii)* immediatly results that $\tau_p(L) \subset \tau(L)$. Let now $x \in \tau(L)$. Then , according Prop. 2.2 $[x, L] \subset F(L) \subset F_p(L)$. Therefore according to Prop.2.3*(i)* $x \in C_L(F_p(L)) = \tau_p(L)$ and hence we deduce what must be demonstrated.

*(iii)*Let $A = \Phi_p(L) \cap N$. Then according to Prop.2.2 $N^{(1)} \subset F(L) \subset F_p(L)$ and so $N^{(1)} \subset A$. Let us assume that A is not an ideal of L. Then, since $AL \subset NL \subset N$, we have that $AL \not\subseteq \Phi_p(L)$. Hence, there is a maximal p-subalgebra M of L such that $AL \not\subseteq M$. Follows that $N \not\subseteq M$ and hence $L = N + M$. So, $AL = A(N + M) \subset N^{(1)} + M \subset M$, a contradiction. From these we can say that A is a p-ideal of L which is contained in $\Phi_p(L)$ and so $A \subset F_p(L)$. The reverse conclusion is immediate.

(iv) It is clear that any maximal p-subalgebra of L is a maximal subalgebra of L, so $\Phi_p(L) \subset \Phi(L)$ and according to Lemma 2.2 *(ii)* we have that $\Phi(L) \subset \Phi_p(L)$ hence $\Phi_p(L) = \Phi(L)$.

According with Prop. 2.2 and 2.3 we obtain that
$F(L) \subset F_p(L) \subset C_L(F(L)) = C_L(F_p(L))$.

3 $p - c -$ Supplemented Lie algebras and E$- \ p -$ algebras

In this section we present some results about p-c-supplemented subalgebras of p-Lie algebras considering also some similar results of Ballester-Bolinches, Wang and Xiuyun in [1].

Lemma 3.1 *Let L be p-Lie algebra and A be a p-subalgebra of L. The following statements are hold:*
(i) If B is a p-subalgebra of A and it is p-c-supplemented in L, then B is p-c-supplemented in A.
(ii) If I is a p-ideal of L and a p-subalgebra of A, then A is p-c-supplemented in L if and only if A/I is p-c-supplemented in L/I.

Proof.*(i)*Supose that A is a p-subalgebra of L and B is p-c-supplemented in L, then there is a p-subalgebra C of L such that $L = B + C$ and $B \cap C$ is a p-subalgebra of ${}_pB_L$. It follows that $A = (B + C) \cap A$ and $B \cap C \cap A$ is a p-subalgebra of ${}_pB_L \cap A$ which is a p-subalgebra of ${}_pB_A$, and so B is p-c-supplemented in A.

(ii) First we suppose that A/I is p-c-supplemented in L/I. Then there is a p-subalgebra B/I of L/I such that $L/I = A/I + B/I$ and $(A/I) \cap (B/I)$ is a p-subalgebra of ${}_p(A/I)_{L/I} =_p A_L/I$. It follows that $L = A + B$ and $A \cap B$ is a p-subalgebra of ${}_pA_L$, whence A is p-c-supplemented in L.

Now conversely we suppose that I is an p-ideal of L, more than that, I is a p-subalgebra of A and p-c-supplemented in L. In these circumstances there is a p-subalgebra B of L such that $L = A + B$ and $A \cap B$ is a p-c-subalgebra of ${}_pA_L$. Hence $L/I = A/I + (B+I)/I$ and $(A/I) \cap (B+I)/I = (A \cap (B+I))/I =$

$(I+A\cap B)/I$ but $(I+A\cap B)/I$ is a p-subalgebra of ${}_pA_L/I =_p (A/I)_{L/I}$, and so A/I is p-c-supplemented in L/I.

Lemma 3.2 *Let L be a p-Lie algebra and A, B be p-subalgebra of L such that A is a p-subalgebra of $F_p(B)$. If A is p-c-supplemented in L, then A is an ideal of L and A is a p-subalgebra of $F_p(L)$.*

Proof. First of all we suppose that $L = A+C$ and $A\cap C$ is a p-subalgebra of ${}_pA_L$. Then $B = B\cap L = B\cap(B+C) = A+B\cap C = B\cap C$ since A is a p-subalgebra of $F_p(B)$. From those we obtain that A is a p-subalgebra of B which is a p-subalgebra of C and $A = A\cap C$ which is a p-c subalgebra of A_L and A is an ideal of L. It then follows from Lemma 3.1 that A is a p-subalgebra of $F_p(L)$.

Corollary 3.1 *If L is p-c-supplemented, then L is an E-p-algebra.*

Proof. The result is immediately by taking $A = F_p(B)$

It is clear that if L is p-completely factorisable then it is p-c-supplemented.

Proposition 3.1 *Let L be a p-Lie algebra. Then the following statements are equivalent:*

(i) *L is p-c-supplemented.*

(ii) *$L/F_p(L)$ is p-completely factorisable and every subalgebra of $F_p(L)$ is an ideal of L.*

Proof. $(ii)\Rightarrow(i)$: We suppose first (ii) holds and let A be a p-subalgebra of L. Then there is a p-subalgebra $B/F_p(L)$ of $L/F_p(L)$ such that $L/F_p(L) = ((A+F_p(L))/F_p(L)) + (B/F_p(L))$ and $0 = ((A+F_p(L))/F_p)\cap(B/F_p(L)) = (A\cap B+F_p(L))/F_p(L)$. Hence $L = A+B$ and $A\cap B$ is a subalgebra of $F_p(L)$, so $A\cap B$ is an ideal of L and $A\cap B$ is a subalgebra of A_L; that is, L is p-c-supplemented.

$(i)\Rightarrow(ii)$: Suppose that L is F_p-free and p-c-supplemented, and consider A be a subalgebra of L. In these conditions there is a subalgebra B of L such that L= A+B. We choose C beeing a minimal subalgebra of L with respect to L=A+C. Now $A\cap C$ is a subalgebra of $F_p(C)$, by [7], whence $A\cap C = 0$ since L is p-elementary by Corollary 3.1. Hence L is p-completely factorisable, and *(ii)* follows from Lemma 3.1

The results obtained in this last section can be extended and we intend to do it in the future researches.

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