An. Şt. Univ. Ovidius Constanţa Vol. 17(3), 2009, 69–77

ON MONOTONE SOLUTIONS FOR A NONCONVEX SECOND-ORDER FUNCTIONAL DIFFERENTIAL INCLUSION

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Abstract

The existence of monotone solutions for a second-order functional differential inclusion is obtained in the case when the multifunction that define the inclusion is upper semicontinuous compact valued and contained in the Fréchet subdifferential of a ϕ-convex function of order two.

1 Introduction

Functional differential inclusions, known also as differential inclusions with memory, express the fact that the velocity of the system depends not only on the state of the system at a given instant but depends upon the history of the trajectory until this instant. The class of differential inclusions with memory encompasses a large variety of differential inclusions and control systems. In particular, this class covers the differential inclusions, the differential inclusions with delay and the Volterra inclusions. A detailed discussion on this topic may be found in [1].

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space with the norm $||.||$ and the scalar product $\langle .,. \rangle$. Let $\sigma > 0$ and $\mathcal{C}_\sigma := \mathcal{C}([-\sigma, 0], \mathbf{R}^n)$ the Banach space of continuous functions from $[-\sigma, 0]$ into $\mathbf{R}^n$ with the norm given by $||x(.)||_\sigma := \sup\{||x(t)||; t \in [-\sigma, 0]\}$. For each $t \in [0, \tau]$, we define the operator $T(t) : \mathcal{C}([-\sigma, \tau], \mathbf{R}^n) \to \mathcal{C}_\sigma$ as follows: $(T(t)x)(s) := x(t+s)$, $s \in [-\sigma, 0]$. $T(t)x$ represents the history of the state from the time $t - \sigma$ to the present time t.

Key Words: monotone solutions, ϕ- monotone operators, differential inclusion.
Mathematics Subject Classification: 334A60.
Received: April 2009
Accepted: October 2009

Let $K \subset \mathbf{R}^n$ be a closed set, $\Omega \subset \mathbf{R}^n$ an open set and P a lower semicontinuous multifunction from K into the family of all nonempty subsets of K with closed graph satisfying the following two conditions

$$\forall\, x \in K, \quad x \in P(x),$$

$$\forall\, x, y \in K, y \in P(x) \quad \Rightarrow \quad P(y) \subseteq P(x).$$

Under these conditions, a preorder (reflexive and transitive relation) on K is defined by $x \preceq y$ iff $y \in P(x)$.

Let $K_0 := \{\varphi \in \mathcal{C}_\sigma; \varphi(0) \in K\}$, let F be a multifunction defined from $K_0 \times \Omega$ into the family of nonempty compact subsets of $\mathbf{R}^n$ and $(\varphi_0, y_0) \in K_0 \times \Omega$ be given that define the second-order functional differential inclusion

$$\begin{array}{l} x'' \in F(T(t)x, x') \quad a.e.\ ([0,\tau]) \\ x(t) = \varphi_0(t) \quad \forall t \in [-\sigma, 0], \quad x'(0) = y_0, \\ x(t) \in P(x(t)) \subset K \quad \forall t \in [0,\tau], \quad x(s) \preceq x(t) \quad \forall\, 0 \leq s \leq t \leq \tau. \end{array} \tag{1.1}$$

Existence of solutions of problem (1.1) has been studied my many authors, mainly in the case when the multifunction is convex valued, $P(x) \equiv K$ and $T(t) = I$ ([2,5,7,10,11] etc.). Recently in [9], the situation when the multifunction is not convex valued is considered. More exactly, in [9] it is proved the existence of solutions of problem (1.1) when F is an upper semicontinuous multifunction contained in the subdifferential of a proper convex function.

The aim of the present paper is to relax the convexity assumption on the function $V(.)$ that appear in [9], in the sense that we assume that $F(.)$ is contained in the Fréchet subdifferential of a ϕ-convex function of order two. Since the class of proper convex functions is strictly contained into the class of ϕ- convex functions of order two, our result generalizes the one in [9].

On the other hand, the result in the present paper is an extension of the result in [5] obtained for differential inclusions. At the same time, our result may be considered as an extension of the result in [6] obtained for first order functional differential inclusions to second-order functional differential inclusions of form (1.1). The proof follows the general ideas in [5] and [9].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

We denote by $\mathcal{P}(\mathbf{R}^n)$ the set of all subsets of $\mathbf{R}^n$, by $cl(A)$ we denote the closure of the set $A \subset R^n$ and by $co(A)$ we denote the convex hull of A. For $x \in \mathbf{R}^n$ and $r > 0$ let $B(x, r) := \{y \in \mathbf{R}^n; ||y - x|| < r\}$ be the open ball centered in x with radius r, and let $\overline{B}(x,r)$ be its closure. For $\varphi \in \mathcal{C}_\sigma$ let $B_\sigma(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; ||\psi - \varphi||_\sigma < r\}$ and $\overline{B}_\sigma(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; ||\psi - \varphi||_\sigma \leq r\}$.

Let $\Omega \subset \mathbf{R}^n$ be an open set and let $V : \Omega \to \mathbf{R} \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in \mathbf{R}^n;\ V(x) < +\infty\}$.

Definition 2.1. The multifunction $\partial_F V : \Omega \to \mathcal{P}(\mathbf{R}^n)$, defined as

$$\partial_F V(x) = \{\alpha \in \mathbf{R}^n, \liminf_{y \to x} \frac{V(y) - V(x) - < \alpha, y - x >}{||y - x||} \geq 0\} \text{ if } V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$ is called the *Fréchet subdifferential* of V.

We also put $D(\partial_F V) = \{x \in \mathbf{R}^n; \partial_F V(x) \neq \emptyset\}$.

According to [8] the values of $\partial_F V(.)$ are closed and convex.

Definition 2.2. Let $V : \Omega \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous function. We say that V is a *ϕ-convex of order 2* if there exists a continuous map $\phi_V : (D(V))^2 \times \mathbf{R}^2 \to \mathbf{R}_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$V(y) \geq V(x) + < \alpha, x - y > -\phi_V(x, y, V(x), V(y))(1 + ||\alpha||^2)||x - y||^2.$$

In [4], [8] there are several examples and properties of such maps. For example, according to [4], if $K \subset \mathbf{R}^2$ is a closed and bounded domain, whose boundary is a C^2 regular Jordan curve, the indicator function of K

$$V(x) = I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K \\ +\infty, & \text{otherwise} \end{cases}$$

is ϕ- convex of order 2.

The second-order contingent set of a closed subset $C \subset \mathbf{R}^n$ at $(x, y) \in C \times \mathbf{R}^n$ is defined by:

$$T_C^2(x, y) = \{v \in \mathbf{R}^n; \quad \liminf_{h \to 0+} \frac{d(x + hy + \frac{h^2}{2}v, M)}{h^2/2} = 0\}.$$

For properties of second-order contingent set see, for example, [2].

A multifunction $F : K_0 \to \mathcal{P}(\mathbf{R}^n)$ is upper semicontinuous at $(\varphi, y) \in K_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(\psi, z) \subset F(\varphi, y) + B(0, \varepsilon), \quad \forall (\psi, z) \in B_\sigma(\varphi, \delta) \times B(y, \delta).$$

We recall that a continuous function $x(.) : [-\sigma, \tau] \to \mathbf{R}^n$ is said to be a solution of (1.1) if $x(.)$ is absolutely continuous on $[0, \tau]$ with absolutely continuous derivative $x'(.)$, $T(t)x \in K_0, \forall t \in [0, \tau]$, $x'(t) \in \Omega$ a.e. $[0, \tau]$ and (1.1) is satisfied.

The next technical result is proved in [9].

Lemma 2.3 ([9]). *Let $K \subset \mathbf{R}^n$ be a closed set, $\Omega \subset \mathbf{R}^n$ be an open set and $P : K \to \mathcal{P}(K)$ a lower semicontinuous multifunction with closed values, $K_0 := \{\varphi \in \mathcal{C}_\sigma; \varphi(0) \in K\}$, $F : K_0 \times \Omega \to \mathcal{P}(\mathbf{R}^n)$ upper semicontinuous with nonempty compact values and $(\varphi_0, y_0) \in K_0 \times \Omega$.*

Assume also that $\forall x \in K$, $x \in P(x)$; there exist $r, M \geq 0$ such that $\sup\{||z||; z \in F(\psi, y)\} \leq M$, $\forall(\psi, y) \in (K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B}(y_0, r)$; $F(\varphi, y) \subset T^2_{P(\varphi(0))}(\varphi(0), y)$, $\forall(\varphi, y) \in K_0 \times \Omega$.

Then there exists $\tau > 0$ such that for any $m \in \mathbf{N}$ there exist $l_m \in \mathbf{N}$, a set of points $\{t_0^m = 0 < t_1^m < ... < t_{l_m-1}^m \leq \tau < t_{l_m}^m\}$; the points $x_p^m, y_p^m, z_p^m \in \mathbf{R}^n$, $p = 0, 1, ..., l_m - 1$ with $x_0^m = \varphi_0(0)$ and $y_0^m = y_0$; a continuous function $x_m(.) : [-\sigma, \tau] \to \mathbf{R}^n$ with $x_m(t) = \varphi_0(t)$ $\forall t \in [-\sigma, 0]$ and with the following properties for $p = 0, 1, ..., l_m - 1$

(i) $h_{p+1}^m := t_{p+1}^m - t_p^m < \frac{1}{m}$,

(ii) $z_p^m = u_p^m + w_p^m$, *with* $u_p^m \in F(T(t_p^m)x_m, y_p^m)$ *and* $w_p^m \in B(0, \frac{1}{m})$,

(iii) $x_m(t) = x_p^m + (t - t_p^m)y_p^m + \frac{1}{2}(t - t_p^m)^2 z_p^m$, $t \in [t_p^m, t_{p+1}^m]$,

(iv) $x_{p+1}^m = x_p^m + h_{p+1}^m y_p^m + \frac{1}{2}(h_{p+1}^m)^2 z_p^m = x_m(t_{p+1}^m)$,

(v) $x_{p+1}^m \in P(x_p^m) \cap B(\varphi_0(0), r) \subset K$, $y_{p+1}^m = y_p^m + h_{p+1}^m z_p^m \in \overline{B}(y_0, r) \subset \Omega$,

(vi) $x_m(t) \in B(\varphi_0(0), r)$, $\forall t \in [t_p^m, t_{p+1}^m]$,

(vii) $T(t_{p+1}^m)x_m \in B_\sigma(\varphi_0, r) \cap K_0$.

3 The main result

We are now able to prove our main result.

Theorem 3.1. *Let K, Ω and $P(.)$ as in Lemma 2.3. In addition, assume that K_0 is locally compact, $P(.)$ has closed graph and $\forall x \in K$, $y \in P(x)$ it follows $P(y) \subseteq P(x)$.*

Consider $F : K_0 \times \Omega \to \mathcal{P}(\mathbf{R}^n)$ an upper semicontinuous multifunction with nonempty compact values such that $F(\varphi, y) \subset T^2_{P(\varphi(0))}(\varphi(0), y)$ $\forall(\varphi, y) \in K_0 \times \Omega$ and there exists a proper lower semicontinuous function of order two $V : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ with $F(\varphi, y) \subseteq \partial_F V(y)$ $\forall(\varphi, y) \in K_0 \times \Omega$.

Then for any $(\varphi_0, y_0) \in K_0 \times \Omega$ there exists $\tau > 0$ and $x(.) : [0, \tau] \to K$ a solution to problem (1.1)

Proof. Let $(\varphi_0, y_0) \in K_0 \times \Omega$. Since K_0 is locally compact there exists $r > 0$ such that $K_0 \cap B_\sigma(\varphi_0, r)$ is compact and $\overline{B}(y_0, r) \subset \Omega$. Using the fact that $F(., .)$ is upper semicontinuous with compact values, by Proposition 1.1.3 in [1] $F((K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B}(y_0, r))$ is compact. Take $M := \sup\{||z||; z \in F(\psi, y); (\psi, y) \in (K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B}(y_0, r)\}$.

Let ϕ_V the continuous function appearing in Definition 2.2. Since $V(.)$ is continuous on $D(V)$ (e.g. [8]), by possibly decreasing r one can assume that for all $y \in B_r(y_0) \cap D(V)$, $|V(y) - V(y_0)| \leq 1$. Set $S := \sup\{\phi_v(y_1, y_2, z_1, z_2); y_i \in \overline{B}_r(y_0), z_i \in [V(y_0) - 1, V(y_0) + 1], i = 1, 2\}$.

One may apply Lemma 2.3 and according to the definition of x_m for all $m \geq 1$, all $p = 0, 1, ..., l_m - 1$ and all $t \in [t_p^m, t_{p+1}^m]$ we have

$$x'_m(t) = y_p^m + (t - t_p^m)z_p^m, \quad x''_m(t) = z_p^m \in F(T(t_p^m)x_m, y_p^m) + B(0, \frac{1}{m}).$$

From (ii) and (v) of Lemma 2.3 one has

$$||x'_m(t)|| \leq ||y_p^m|| + h_{p+1}^m ||z_p^m|| \leq ||y_0|| + \frac{5r}{4} \quad \forall t \in [0, \tau], \tag{3.1}$$

$$||x''_m(t)|| \leq M + \frac{1}{m} \quad \forall t \in [0, \tau]. \tag{3.2}$$

Then the sequences $\{x_m\}$ and $\{x'_m\}$ are echicontinuous in $C([0, \tau], \mathbf{R}^n)$. Applying Arzela-Ascoli theorem, there exists a subsequence (again denoted) $\{x_m(.)\}$ and an absolutely continuous function $x(.) : [0, \tau] \to \mathbf{R}^n$ with absolutely continuous derivative $x'(.)$ such that $x_m(.)$ converges uniformly to $x(.)$ on $[0, \tau]$, $x'_m(.)$ converges uniformly to $x'(.)$ on $[0, \tau]$ and $x''_m(.)$ converges weakly to $x''(.)$ in $L^2([0, \tau], \mathbf{R}^n)$. Furthermore, since all the functions $x_m(.)$ are equal with $\varphi_0(.)$ on $[-\sigma, 0]$, then $x_m(.)$ converges uniformly to $x(.)$ on $[-\sigma, \tau]$, where $x_m = \varphi_0)$ on $[-\sigma, 0]$.

For each $t \in [0, \tau]$ and each $m \geq 1$ let $\delta_m(t) = t_p^m$, $\theta_m(t) = t_{p+1}^m$ if $t \in (t_p^m, t_{p+1}^m]$ and $\delta_m(0) = \theta_m(0) = 0$. If $t \in (t_p^m, t_{p+1}^m]$ we get

$$x''_m(t) = z_p^m \in F(T(t_p^m)x_m, y_p^m) + B(0, \frac{1}{m})$$

and for all $m \geq 1$ and a.e. on $[0, \tau]$

$$x''_m(t) \in F(T(\delta_m(t))x_m, x'_m(\delta_m(t))) + B(0, \frac{1}{m}).$$

Also for all $m \geq 1$ and a.e. on $[0, \tau]$ $T(\theta_m(t))x_m \in B_\sigma(\varphi_0, r) \cap K_0$, $x_m(t) \in B(\varphi_0(0), r)$, $x_m(\theta_m(t)) \in P(x_m(\delta_m(t)) \subset K$.

Note that $\forall t \in [0, \tau]$, $\lim_{m\to\infty} T(\theta_m(t))x_m = T(t)x$ in $\mathcal{C}_\sigma$ and $\lim_{m\to\infty} x'_m(\delta_m(t)) = x'(t)$ (e.g., [9]).

Taking into account the upper semicontinuity of $F(.,.)$, Theorem 1.4.1 in [1] and (3.1) one deduces

$$x''(t) \in \mathrm{co}F(T(t)x, x'(t)) \subset \partial_F V(x'(t)) \quad a.e. \ ([0, \tau]). \tag{3.3}$$

The next step of the proof shows that $x''_m(.)$ has a subsequences that converges pointwise to $x''(.)$. From property (ii) of Lemma 2.3

$$z_p^m - w_p^m \in F(T(t_p^m)x_m, y_p^m) \subset \partial_F V(y_p^m) = \partial_F V(x'_m(t_p^m))$$

for $p = 0, 1, 2, ..., l_m - 2$.

From the definition of the Fréchet subdifferential for $p = 0, 1, 2, ..., l_m - 2$ one has

$$\begin{aligned} &V(x'_m(t_{p+1}^m)) - V(x'_m(t_p^m)) \geq < z_p^m - w_p^m, x'_m(t_{p+1}^m) - x'_m(t_p^m) > - \\ &\phi_V(x'_m(t_{p+1}^m), x'_m(t_p^m), V(x'_m(t_{p+1}^m)), V(x'_m(t_p^m)))(1 + ||z_p^m - w_p^m||^2). \\ &.||x'_m(t_{p+1}^m) - x'_m(t_p^m)||^2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &V(x'_m(\tau)) - V(x'_m(t_{l_m-1}^m)) \geq < z_{l_m-1}^m - w_{l_m-1}^m, x'_m(\tau) - x'_m(t_{l_m-1}^m) > \\ &-\phi_V(x'_m(\tau)), x'_m(t_{l_m-1}^m), V(x'_m(\tau)), V(x'_m(t_{l_m-1}^m)))(1 + ||z_{l_m-1}^m - w_{l_m-1}^m||^2). \\ &.||x'_m(\tau) - x'_m(t_{l_m-1}^m)||^2 \end{aligned} \tag{3.5}$$

By adding the $l_m - 1$ inequalities from (3.4) and the inequality from (3.5), one has

$$V(x'_m(\tau)) - V(x'_m(0)) \geq \int_0^\tau ||x''_m(t)||^2 dt + \alpha(m) + \beta(m),$$

where

$$\alpha(m) = -\sum_{p=0}^{l_m-2} < w_p^m, \int_{t_p^m}^{t_{p+1}^m} x''_m(t)dt > - < w_{l_m-1}^m, \int_{t_{l_m-1}^m}^{\tau} x''_m(t)dt >,$$

$$\begin{aligned} &\beta(m) = -\textstyle\sum_{p=0}^{l_m-2} \phi_V(x'_m(t_{p+1}^m), x'_m(t_p^m), V(x'_m(t_{p+1}^m)), V(x'_m(t_p^m)))(1+ \\ &||z_p^m - w_p^m||^2)||x'_m(t_{p+1}^m) - x_k(t_p^m)||^2 - \phi_V(x'_m(\tau), x'_m(t_{l_m-1}^m), V(x'_m(\tau)), \\ &V(x'_m(t_{l_m-1}^m)))(1 + ||z_{l_m-1}^m - w_{l_m-1}^m||^2)||x'_m(\tau) - x'_m(t_{l_m-1}^m)||^2. \end{aligned}$$

One may write

$$\begin{aligned} &|\alpha(m)| \leq \\ &(M+1)[\textstyle\sum_{p=0}^{l_m-2} ||w_p^m||(t_{p+1}^m - t_p^m) + ||w_{l_m-1}^m||(\tau - t_{l_m-1}^m)] \leq \frac{\tau(M+1)}{m}, \end{aligned}$$

$$\begin{aligned} &|\beta(m)| \leq S(1+M^2)[\textstyle\sum_{p=0}^{l_m-2} || \int_{t_p^m}^{t_{p+1}^m} x''_m(t)dt||^2 + || \int_{t_{l_m-1}^m}^{\tau} x''_m(t)dt||^2] \\ &\leq S(1+M^2)[\textstyle\sum_{p=0}^{l_m-2} \frac{1}{m} \int_{t_p^m}^{t_{p+1}^m} ||x''_m(t)||^2 dt + \frac{1}{m} \int_{t_{l_m-1}^m}^{\tau} ||x''_m(t)||^2 dt] \\ &\leq \frac{1}{m} S(1+M^2) \int_0^\tau ||x''_m(t)||^2 dt \leq \frac{1}{m} S(1+M^2)\tau(M+1)^2. \end{aligned}$$

Therefore, $\lim_{m\to\infty}\alpha(m)=\lim_{m\to\infty}\beta(m)=0$ and thus

$$V(x'_m(\tau))-V(y_0)\geq \limsup_{m\to\infty}\int_0^\tau ||x''_m(t)||^2dt. \tag{3.6}$$

From (3.3) and Theorem 2.2 in [4] we deduce that there exists $\tau_1>0$ such that the mapping $t\to V(x'(t))$ is absolutely continuous on $[0,\min\{\tau,\tau_1\}]$ and

$$(V(x'(t)))' = < x''(t), x''(t) > \quad a.e.\ ([0,\min\{\tau,\tau_1\}]).$$

Without loss of generality we may assume that $\tau=\min\{\tau,\tau_1\}$. Hence, $V(x'(\tau))$
$-V(x'(0))=\int_0^\tau ||x''(t)||^2dt$; therefore from (3.2) one has

$$\int_0^\tau ||x''(t)||^2dt\geq \limsup_{m\to\infty}\int_0^\tau ||x''_m(t)||^2dt$$

and, since $x''_m(.)$ converges weakly in $L^2([0,\tau],\mathbf{R}^m)$ to $x''(.)$, by the lower semicontinuity of the norm in $L^2([0,\tau],\mathbf{R}^n)$ (e.g., Proposition III 30 in [3]), we obtain that $x''_m(.)$ converges strongly in $L^2([0,\tau],\mathbf{R}^m)$ to $x''(.)$, hence a subsequence (again denote by) $x''_m(.)$ converges pointwise a.e. to $x''(.)$.

On the other hand, since $F(.,.)$ is upper semicontinuous with close values, then $graph(F(.,.)$ is closed (e.g., Proposition 1.1.2 in [1]) and by the facts that $T(t)x_m$ converges uniformly to $T(t)x$, x'_m converges uniformly to x' and x''_m converges pointwise o x'' it follows that $x''(t)\in F(T(t)x,x'(t))$ a.e. $[0,\tau]$.

It remains to prove that

$$(x(t),x'(t))\in K\times\Omega,\quad \forall t\in[0,\tau],$$

$$x(s)\in P(x(t))\quad \forall t,s\in[0,\tau],\quad t\leq s.$$

First, from property (iii) of Lemma 2.3 it follows that $x_m(\delta_m(t))\in$ $\overline{B}(\varphi_0(0),r)$ and $x'_m(\delta_m(t))\in\overline{B}(y_0,r)\cap\Omega$. Since $\lim_{m\to\infty}x_m(\delta_m(t))=x(t)$ and $\lim_{m\to\infty}x'_m(\delta_m(t))=x'(t)$ then $x(t)\in\overline{B}(\varphi_0(0),r)$ and $x'(t)\in\overline{B}(y_0,r)\cap\Omega$.

Secondly, let $t,s\in[0,\tau],\quad t\leq s$. For m large enough we can find $p,q\in\{0,1,2,...,l_m-2\}$ such that $p>q$, $t\in[t^m_q,t^m_{q+1}]$, $s\in[t^m_p,t^m_{p+1}]$. If $j=p-q$, then property (v) of Lemma 2.3 gives

$$P(x_m(t^m_p))\subseteq P(x_m(t^m_{p-1}))\subseteq P(x_m(t^m_{p-2}))\subseteq ...\subseteq P(x_m(t^m_q)).$$

This implies $P(x_m(\delta_m(s)))\subseteq P(x_m(\delta_m(t)))$ and since $x_m(\delta_m(s))\in$ $P(x_m(\delta_m(s)))$ it follows $x_m(\delta_m(s))\in P(x_m(\delta_m(t)))$ which completes the proof.

Remark 3.2. If $V(.) : \mathbf{R}^n \to \mathbf{R}$ is a proper lower semicontinuous convex function then (e.g. [8]) $\partial_F V(x) = \partial V(x)$, where $\partial V(.)$ is the subdifferential in the sense of convex analysis of $V(.)$, and Theorem 3.1 yields the main result in [9]. On the other hand, if $P(x) \equiv K$ and $T(t) = I$ then Theorem 3.1 yields the main result in [5], namely Theorem 3.2.

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