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GEOMETRIC METHODS IN STUDY OF THE STABILITY OF SOME DYNAMICAL SYSTEMS

Dumitru Bala

Abstract

In this paper we aim to analyse the stability of two dynamical systems given by differential equations or by systems of differential equations. The first model is a mechanical system which is described by a system of differential equations of the first degree. We study the stability of this system using the method of the Lyapunov function. The second studied model is the model of a vibrant tool machine described by a differential equation of second degree with two delay arguments. For the study of the stability of these models, we use the stage analysis of the differential equations systems with delayed arguments.

1 The Study of the Stability of a Dynamical System

Let us consider the dynamical system presented in Figure 1:

This mechanical system is described by the dynamical system of first degree

$$\begin{cases} \dot{y}_1 &= y_3 \\ \dot{y}_2 &= y_4 \\ \dot{y}_3 &= -\frac{c}{m}y_3 - \frac{k}{m}(y_1 - y_2) \\ \dot{y}_4 &= -\frac{c}{m}y_4 - \frac{k}{m}(y_2 - y_1) \end{cases} \quad . \tag{1}$$

The vector field determined by the system (1) is $X : \mathbb{R}^4 \to \mathbb{R}^4$,

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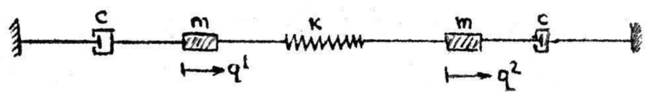

Figure 1: Dynamical system

$$X(y_1, y_2, y_3, y_4) = (y_3, y_4, -\frac{c}{m}y_3 - \frac{k}{m}(y_1 - y_2), -\frac{c}{m}y_4 - \frac{k}{m}(y_2 - y_1) \quad (2)$$

For the system (1), we find the prime integral

$$G(y_1, y_2, y_3, y_4) = [m(y_3 + y_4) + c(y_1 + y_2)] \quad (3)$$

and the Lyapunov function

$$V(y_1, y_2, y_3, y_4) = [m(y_3 + y_4) + c(y_1 + y_2)]^2, V(0, 0, 0, 0) = 0. \quad (4)$$

The set on which V is positively defined is $\mathbb{R}^4 \backslash M$ where

$M = \{(y_1, y_2, y_3, y_4) I (y_1, y_2, y_3, y_4) \in \mathbb{R}^4, (y_1, y_2, y_3, y_4) \neq (0, 0, 0, 0), m(y_3 + +y_4) + c(y_1 + y_2) = 0\}$.

We study the stability on the set $\mathbb{R}^4 \backslash M$. Applying the Lyapunov theorem for autonomous systems, it results that the system (1) is stable in $x_0 = = (0, 0, 0, 0)$. For the system (1), we have:

$$f = \frac{1}{2}(1 + \frac{c^2}{m^2})(y_3^2 + y_4^2) + \frac{k}{m^2}(y_1 - y_2)[k(y_1 - y_2) + c(y_3 - y_4)],$$

$$L = \frac{1}{2}\left[\left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2 + \left(\frac{dy_3}{dt}\right)^2 + \left(\frac{dy_4}{dt}\right)^2\right] -$$

$$-y_3\frac{dy_1}{dt} - y_4\frac{dy_2}{dt} + [\frac{c}{m}y_3 + \frac{k}{m}(y_1 - y_2)]\frac{dy_3}{dt} + [\frac{c}{m}y_4 + \frac{k}{m}(y_2 - y_1)]\frac{dy_4}{dt} +$$

$$+\frac{1}{2}(1 + \frac{c^2}{m^2})(y_3^2 + y_4^2) + \frac{k}{m^2}(y_1 - y_2)[k(y_1 - y_2) + c(y_3 - y_4)]$$

$$H = \frac{1}{2}\left[\left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2 + \left(\frac{dy_3}{dt}\right)^2 + \left(\frac{dy_4}{dt}\right)^2\right] -$$

$$-\frac{1}{2}(1 + \frac{c^2}{m^2})(y_3^2 + y_4^2) + \frac{k}{m^2}(y_1 - y_2)[k(y_1 - y_2) + c(y_3 - y_4)].$$

We calculate the Lagrangian L, the Hamiltonian H and the energy density f because these functions can be used to build the Lyapunov function. Also, there are theorems giving us the mathematical relations between L, H, f, prime integrals and the Lyapunov function.

2 Studying the stability of a regenerative vibrant machine tool with two delay arguments

The model of the regenerative machine tool with two delay arguments is given by the differential equation of second degree

$$\ddot{x}(t) + 2\delta_0 x(t) + \zeta_1 x(t-\tau_1) +$$
$$+\mu(x(t) - x(t-\tau_2)) - \varepsilon\sigma_2(x(t) - x(t-\tau_2))^2 - \varepsilon\sigma_3(x(t) - x(t-\tau_2))^3 = 0, \quad (5)$$

where $\delta_0, \sigma_2, \sigma_3$ are real parameters (δ_0 characterises the amortization of the tool machine), $0 \leq \varepsilon \ll 1, \mu$ is a real parameter, and τ_1, τ_2 are delay arguments with $\tau_1 < \tau_2$. We study the equation (5) by investigating the system of differential equations with two delay arguments, given by

$$\dot{x}_1(t) = x_2(t) \quad (6)$$

$$\dot{x}_2 = -2\delta_0 x_2(t) - \zeta_1 x_1(t-\tau_1) - \mu(x_1(t) - x_1(t-\tau_2)) + \varepsilon\sigma_2(x_1(t) - x_1(t-\tau_2))^2 + \varepsilon\sigma_3(x_1(t) - x_1(t-\tau_2))^3,$$

which is equivalent with the equation (5).

Let us consider $X(t) = (x_1(t), x_2(t))^T$ and the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -\mu & -2\delta_0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 \\ -\zeta_1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}. \quad (7)$$

Then we get the function

$$F(x_1(t), x_1(t-\tau_2)) =$$
$$= \varepsilon \begin{pmatrix} 0 \\ \sigma_2(x_1(t) - x_1(t-\tau_2))^2 + \sigma_3(x_1(t) - x_1(t-\tau_2))^3 \end{pmatrix}. \quad (8)$$

From (6), (7), (8) , it results the matrix system

$$\dot{X}(t) = AX(t) + B_1 X(t-\tau_1) + B_2 X(t-\tau_2) + F(x_1(t), x(t-\tau_2)), \quad (9)$$

which is a nonlinear differential equation system with two delay arguments τ_1, τ_2. From the fact that, for the system (9), the right member is described by continuous functions that satisfy the Lipschitz condition, the system (9) has unique solution for an initial condition given by $X(\theta) = \Phi(\theta), \theta \in \in [-\tau_2, 0]$. From (9), it results that the point $0 = (0,0)^T$ is an equilibrium point. This equilibrium point represents the stationary solution of the system (9). The investigation of the system (9) is done in the neighborhood of this equilibrium point, using the stage analysis of the equation systems with delayed arguments from [2].

The linearized system in the equilibrium point $O = (0,0)^T$ is

$$\dot{Y}(t) = AY(t) + B_1 Y(t-\tau_1) + B_2 Y(t-\tau_2). \quad (10)$$

The characteristic equation of (10) is

$$D(\lambda) = \det(\lambda I - A - e^{-\lambda\tau_1}B_1 - e^{-\lambda\tau_2}B_2), \tag{11}$$

where I is the identity matrix of the type 2×2. From (11), we get a transcendent equation

$$D(\lambda) = \lambda^2 + 2\delta_0\lambda + \xi_1 e^{-\lambda\tau_1} + \mu(1 - e^{-\lambda\tau_2}) = 0. \tag{12}$$

Using Theorem 3.22 [3], we obtain the following:

Proposition 1.1. *If* $\delta_0 > \dfrac{|\xi_1| + |\mu|}{2|\xi_2|}$, *then the roots of the equation (12) have negative real parts, for any* $\tau_1, \tau_2 \in [0, \infty)$.

From Proposition 1.1. and the variant of the Hartman - Coarbman theorem for our system of equations with delayed arguments, we obtain

Proposition 1.2. *If* $\delta_0 > \dfrac{|\xi_1| + |\mu|}{2|\xi_2|}$, *then the stationary solution* $X(t) = 0$, *for all* t, *of the system (9), and also of the equation (5), is asimptotically stable for any* $\tau_1, \tau_2 \in [0, \infty)$.

We determine the D-curves that describe the boundaries of the stability regions in terms of the parameters ξ_1, μ,for a fixed δ_0. These curves are obtained by setting the condition that the equation (12) admits purely imaginary roots, that depends upon the parameters ξ_1, μ. Let us take $\lambda = \pm i\varpi$, where, $\varpi = \varpi(\xi_1, \mu) > 0$ root of (12). By replacing this in (12) and cancelling the real part and the imaginary part of the obtained relation, it results the system of equations:

$$\begin{aligned} &\xi_1 \cos\varpi\tau_1 + \mu(1 - \cos\varpi\tau_2) - \varpi^2 = 0, \\ &\xi_1 \sin\varpi\tau_1 + \mu\sin\varpi\tau_2 - 2\delta_0\varpi = 0. \end{aligned} \tag{13}$$

From (13) we obtain

$$\begin{aligned} \xi_1 &= \frac{2\delta_0\varpi(1 - \cos\varpi\tau_2) + \varpi^2\sin\varpi\tau_2}{\cos\varpi\tau_1\sin\varpi\tau_2 + (1 - \cos\varpi\tau_2)\sin\varpi\tau_1}, \\ \mu &= \frac{\varpi^2\sin\varpi\tau_1 - 2\delta_0\varpi\cos\varpi\tau_1}{\cos\varpi\tau_1\sin\varpi\tau_2 + (1 - \cos\varpi\tau_2)\sin\varpi\tau_1}. \end{aligned} \tag{14}$$

By fixing τ_1, τ_2, and taking variable, the formulae (14) give the coordinates of a point in the plan $(\xi_1\mu)$ that describes the curves of the equation (12).

For the values $\delta_0 = 0,085, \quad 0,1035 \leq \tau_1 \leq 0,1045$, and $\tau_2 = 1,03\tau_1$, the stability domain is given by the black region in the Figure 2. The coordinates of the points on the D-curves in Figure 2 represent values of μ for which the equation (12) has the roots $i\varpi$. Further on, we will consider a fixed value $\varpi = \varpi_0$ for which, from (14), it results the values $\xi_1 = \xi_1^0$, respectively $\mu = \mu_0$.

We analyse the system (10), for δ_0, $\xi_1 = \xi_1^0, \tau_1, \tau_2$ fixed and $\mu = \mu_0 + \varepsilon\mu$, where μ is a paramether $|\mu| < 1$. The coefficient μ is a parameter that intervenes in the stationary solution of the nonlinear system (9).

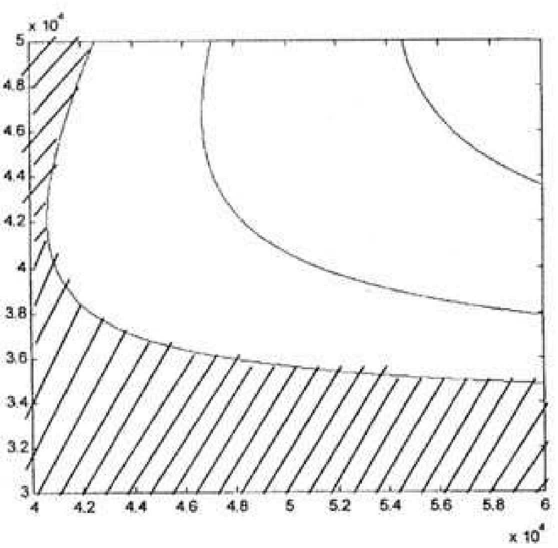

Figure 2: The stability domain–the black region

It is called a *Hopf bifurcation* with respect to the parameter μ, a value of the parameter μ_0, for which

$$Re\ \lambda(\mu_c) = 0\ Re \left(\frac{dx(\mu)}{d\mu} \right) |_{\mu_E=\mu_c} \neq 0. \tag{15}$$

The orbit $(t, x(t_1))$ of the differential equation (5) in the neighborhood of the stationary solution $x(t) = 0$ for all t is

$$x(t) = 2x_1(t) + r_{20}^{(0)}(x_1(t)^2 - y_1(t)^2) + r_{11}^{(0)}(x_1(t)^2 - y_1(t)^2) + 2i_{20}^{(0)} x_1(t) y_1(t), \tag{16}$$

where $(x_1(t), y_1(t))$ is a solution of the differential system with the initial conditions

$x_1(0) = Re(\bar{\Psi}^*(s), \varphi(\theta))$,

$y_1(0) = I_m(\bar{\Psi}^*(s), \varphi(\theta))$.

Here $x(\theta) = \varphi(\theta), \theta \in [-\tau_2, 0]$ is the initial condition of the equation (5), and $r_y = Re(W_y(0)), y(0) = I_m(W_{ij}, 0)$.

The orbit $(t, x(t - \tau_1))$ of the differential equation (5) in the neighborhood of the stationary solution $x(t) = 0$, for all t, is

$$\begin{aligned} x(t-\tau_1) &= 2x_1(t)\cos\varpi_0\tau_1 + 2y_1^{'}(t)\sin\varpi_0\tau_1 + r_{20}(-\tau_1)(x_1(t)^2 - y_1(t)^2) + \\ &+ r_{11}(-\tau_1)(x_1(t)^2 - y_1(t)^2) - 2i_{20}(-\tau_1)x_1(t) - y_1(t), \end{aligned} \tag{17}$$

where $r_{ij}(-\tau) = Re(W_{ij}(-\tau_1)),\ i_{ij}(-\tau) = I_m(W_{ij}(-\tau_1))$.

The orbit $(t, x(t, t-\tau_2))$ of the differential equation (5) in the neighborhood of the stationary solution $x(t) = 0$, for all t, is given by the next formula:

$$x(t-\tau_2) = 2x_1(t)\cos\varpi_0\tau_2 + 2y_1(t)\sin\varpi_0\tau_2 + r_{20}(-\tau_2)(x_1(t)^2 - y_1(t)^2) + \\ + r_{11}(-\tau_2)(x_1(t)^2 - y_1(t)^2) - 2i_{20}(-\tau_2)x_1(t) - y_1(t), \tag{18}$$

where $xr_{ij}(-\tau) = Re(W_{ij}(-\tau_2))$, $i_{ij}(-\tau) = I_m(W_{ij}(-\tau_2))$.

The Hopf bifurcation given by $\mu = \mu_0$ is called *supercritical (subcritical)* if, for $\mu > \mu_0(\mu < \mu_0)$, the equation (5) has periodical solutions. The Hopf bifurcation given by $\mu = \mu_0$ it is called *orbitally stable (unstable)* if the orbit $(t, x(t))$of the equation (5) is stable (unstable).

Following the theory of the normal forms, the characterization of the Hopf bifurcation is done by using the coefficients μ_2, β_2, T_2 given by

$$\mu_2 = -\frac{Re(C_1)}{Re(M)},\ T_2 = \frac{Im(C_1) + \mu_2 Im(M)}{\varpi_0},\ \beta_2 = Re(C_1), \tag{19}$$

where

$$C_1 = \frac{i}{2\varpi_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{1}{2}g_{21}$$

$$M = \frac{e^{-i\varpi_0\tau_2} - 1}{2i\varpi_0 + 2\delta_0 - \mu_0\tau_2 e^{-i\varpi_0\tau_2} + \xi\tau_2 e^{-i\varpi_0\tau_1}}.$$

Proposition 2.1. [2] *The following statements are true:*

i) *If* $\mu_2 > 0(< 0)$, *then the Hopf bifurcation is supercritical (subcritical) and there are periodical solutions of the bifurcation for* $\mu > \mu_0(\mu < \mu_0)$.

ii) *If* $\beta_2 < 0(> 0)$, *then the orbits of the bifurcation are orbitally stable (unstable).*

iii) *If* $T_2 > 0(< 0)$, *then the periods of the orbit of the bifurcation are increasing (decreasing).*

For $\mu = \mu_0 + \varepsilon\mu$, the equation (5) is written as

$$\dot{x}\,(t) + 2\delta_0 x(t) + \xi_1 x(t-\tau_1) + \mu_0(x(t) - x(t-\tau_2)) - \varepsilon\sigma_2(x(t) - x(t-\tau_2))^2 - \\ - \varepsilon\sigma_3(x(t) - x(t-\tau_2))^3 + \varepsilon\tilde{\mu}(x(t-\tau_2)) = 0. \tag{20}$$

The matriceal system associated to the equation (20) is given by

$$\dot{X}\,(t) = AX(t) + B_1 X(t-\tau_1) + B_2 X(t-\tau_2) + \tilde{F}(x_1(t), x_1(t-\tau_2)), \tag{21}$$

where $A, B_1 B_2$ are given by (7) and $\tilde{F}$ is defined by

$$\tilde{F}(x_1(t), x_1(t-\tau_2)) = F(x_1(t), x_1(t-\tau_2)) - \varepsilon\tilde{\mu}\begin{pmatrix} 0 \\ x_1(t) - x_1(t-\tau_2) \end{pmatrix}. \tag{22}$$

The normal form of the system (21) on the central variety $W^C(\mu_0)$ is

$$z'(t) = (i\varpi_0 - \varepsilon\tilde{\mu}(1 - e^{-i\varpi_0\tau_2}))z(t) + \frac{1}{2}\tilde{g}_{20}z(t)^2 + \tilde{g}_{11}z(t)\bar{z}(t)^2 + \\ + \frac{1}{2}\tilde{g}_{02}\bar{z}(t)^2 + \frac{1}{2}\tilde{g}_{11}z(t)^2\bar{z}(t), \tag{23}$$

where $z(t) = x_1(t) + iy_1(t)$.

The equation (23) is written under the form

$$x_1^{'}(t) = -\varepsilon\tilde{\mu}(1-\cos\varpi_0\tau_2)x_1(t) - (\varpi_0 - \varepsilon\tilde{\mu}\sin\varpi_0\tau_2)y_1(t) + \frac{1}{2}(\tilde{R}_{20} + 2\tilde{R}_{11} + \tilde{R}_{02})x_1(t)^2 - \frac{1}{2}(\tilde{R}_{20} + 2\tilde{R}_{11} + \tilde{R}_{02})y_1(t)^2 + (\tilde{I}_{02} - \tilde{I}_{20})x_2(t)y_1(t) + \frac{1}{2}\tilde{R}_{21}x_1(t)(x_1(t)^2 + y_1(t)^2) - \frac{1}{2}\tilde{I}_{21}y_1(t)(x_1(t)^2 + y_1(t)^2) \tag{24}$$

$$y_1^{'}(t) = (\varpi_0 - \varepsilon\tilde{\mu}\sin\varpi_0\tau_2)x_1(t) - \varepsilon\tilde{\mu}(1-\cos\varpi_0\tau_2)y_1(t) + \frac{1}{2}(\tilde{I}_{20} + 2\tilde{I}_{11} + \tilde{I}_{02})y_1(t) - \frac{1}{2}(\tilde{I}_{20} + 2\tilde{I}_{11} + \tilde{I}_{02})x_1(t)^2 + (\tilde{R}_{02} - \tilde{R}_{20})x_2(t)y_1(t) + \frac{1}{2}\tilde{R}_{21}y_1(t)(x_1(t)^2 + y_1(t)^2) - \frac{1}{2}\tilde{I}_{21}y_1(t)(x_1(t)^2 + y_1(t)^2), \tag{25}$$

where

$R_{ij} = Re(\tilde{g}_{ij}),\quad I_{ij} = I_m(\tilde{g}_{ij})$.

The orbit $(t, x(t))$ of the differential equation (20) is given by (16), $x_1(t), y_1(t_1)$ being a solution of the ordinary differential system (24).

The orbit $(t, x(t-\tau_1)), (t, x(t-\tau_2))$ of the differential equation (20) in the variants of the stationary solutions $x(t) = 0, \forall t$, is given by (17), respectively (18), with $(x_1(t), y_1(t))$ a solution of (24).

Using the program realised with the soft Maple 9, we obtained the figures presented below.

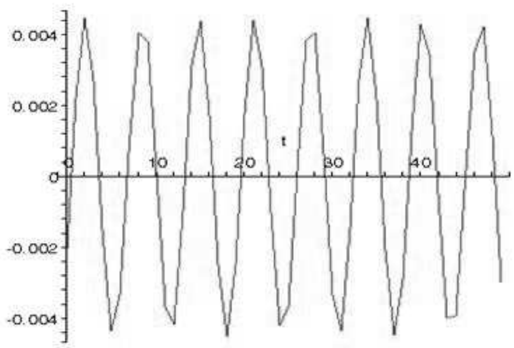

Figure 3: Orbit $(t,\, x(t))$

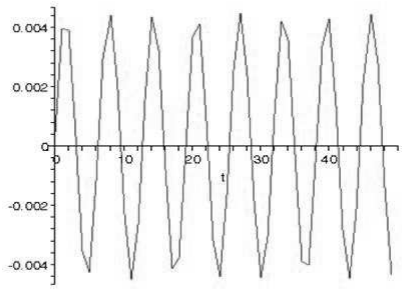

Figure 4: Orbit $(t,\, x(t-\tau_1))$

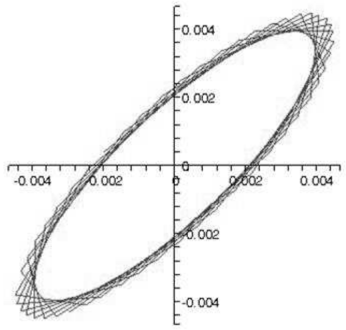

Figure 5: Orbit $(x(t),\, x(t-\tau_1))$

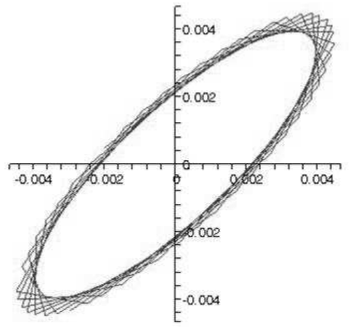

Figure 6: Orbit $(x(t),\, x(t-\tau_2))$

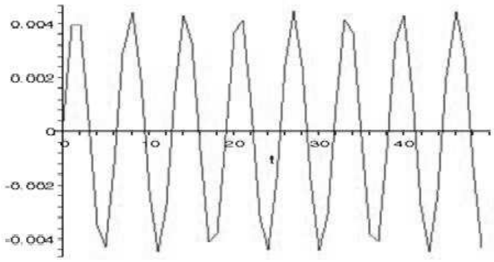

Figure 7: Orbit $(t,\, x(t-\tau_2))$

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Dumitru Bălă
University of Craiova, Section Tg. Jiu
E-mail: dumitru_bala@yahoo.com