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SPATIAL DISCRETIZATION OF AN IMPULSIVE COHEN-GROSSBERG NEURAL NETWORK WITH TIME - VARYING AND DISTRIBUTED DELAYS AND REACTION - DIFFUSION TERMS

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Abstract

An impulsive Cohen-Grossberg neural network with time-varying and distributed delays and reaction-diffusion terms is considered.

The reaction-diffusion terms are approximated by divided differences. For simplicity of notation the spatial domain Ω is assumed to be a finite closed interval $[a, b]$. Under suitable conditions in terms of $M-$matrices it is proved that the system obtained has a unique equilibrium point which is globally exponentially stable

1 Introduction

Since Cohen-Grossberg neural networks were proposed by Cohen and Grossberg [2] in 1983, extensive work has been done on this subject due to their numerous applications in classification of patterns, associative memories, image processing, quadratic optimization, and other areas. In implementation of neural networks, however, time delays inevitably occur due to the finite switching speed of neurons and amplifiers.

Most widely studied and used neural networks can be classified as either continuous or discrete. Recently, there has been a somewhat new category of

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neural networks which are neither purely continuous-time nor purely discrete-time. This third category of neural networks called impulsive neural networks displays a combination of characteristics of both the continuous and discrete systems [4].

It is well known that diffusion effect cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields [8], so the activations must be considered to vary in space as well as in time. The papers [6, 7] are devoted to the exponential stability of impulsive Cohen-Grossberg neural networks with, respectively, time-varying and distributed delays and reaction-diffusion terms.

In the present paper, we consider an impulsive Cohen-Grossberg neural network with both time-varying and distributed delays and reaction-diffusion terms as in [9] which are of a form more general than in [6, 7], and zero Neumann boundary conditions. For simplicity of notation, the spatial domain Ω is assumed to be a finite closed interval $[a, b]$. Under suitable conditions in terms of $M-$matrices, it is proved that the system obtained has a unique equilibrium point which is globally exponentially stable.

2 Model description and preliminaries

We consider the following system of impulsive Cohen-Grossberg neural networks with time-varying and distributed delays and reaction-diffusion terms, and zero Neumann boundary conditions:

$$\begin{aligned} \frac{\partial u_i(t,x)}{\partial t} &= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(D_{ik}(t,x)\frac{\partial u_i(t,x)}{\partial x_k}\right) - \alpha_i(u_i(t,x))\Bigg[\beta_i(u_i(t,x)) \\ &- \sum_{j=1}^{m} a_{ij} f_j(u_j(t,x)) - \sum_{j=1}^{m} b_{ij} g_j(u_j(t-\tau_{ij}(t),x)) \qquad (1) \\ &- \sum_{j=1}^{m} c_{ij}\int_0^{+\infty} K_{ij}(s) h_j(u_j(t-s,x))\,ds + J_i\Bigg], \\ & t>0, \quad t\neq t_k, \qquad i=\overline{1,m}, \qquad x\in\Omega\subset\mathbb{R}^n, \end{aligned}$$

$$\Delta u_i(t_k,x) \equiv u_i(t_k^+,x) - u_i(t_k^-,x) = I_{ik}(u_i(t_k,x)), \quad i=\overline{1,m}, \quad x\in\Omega, \quad k\in\mathbb{N},$$

$$\left.\frac{\partial u_i}{\partial\nu}\right|_{\partial\Omega} = 0, \quad i=\overline{1,m}, \qquad u_i(s,x)=\phi_i(s,x), \quad s\le 0, \quad x\in\Omega, \quad i=\overline{1,m},$$

where $m \geq 2$ is the number of neurons in the network; Ω is a bounded compact set with smooth boundary $\partial\Omega$ and $\text{mes}\,\Omega > 0$; $\partial/\partial\nu$ is the outward normal derivative; $D_{ik}(t,x) > 0$ are smooth functions corresponding to the transmission diffusion operator along the $i-$th neuron; $\alpha_i(u_i)$ represent amplification functions; $\beta_i(u_i)$ are appropriately behaving functions which support the stabilizing feedback term $-\alpha_i(u_i)\beta_i(u_i)$ of the $i-$th neuron; a_{ij}, b_{ij}, c_{ij} denote the connection weights (or strengths) of the synaptic connections between the $j-$th neuron and the $i-$th neuron; $f_j(u_j)$, $g_j(u_j)$, $h_j(u_j)$ denote the activation functions of the $j-$th neuron; J_i denotes external input to the $i-$th neuron; $\tau_{ij}(t)$ correspond to the transmission delays and satisfy $0 \le \tau_{ij}(t) \le \tau_{ij}$; the delay kernels $K_{ij} : [0,+\infty) \to [0,+\infty)$ are real-valued continuous functions satisfying

$$\int_0^{+\infty} e^{\lambda s} K_{ij}(s)\, ds = k_{ij}(\lambda),$$

where $k_{ij}(\lambda)$ are continuous functions on $[0,\delta)$ for some $\delta > 0$ and $k_{ij}(0) = 1$, $i,j = \overline{1,m}$; the moments (instants) of impulse effect t_k satisfy $0 < t_1 < t_2 < \cdots < t_k \to +\infty$ as $k \to +\infty$; $u_i(t_k^-,x)$ and $u_i(t_k^+,x)$ denote respectively the left-hand and right-hand limit at t_k; I_{ik} is the impulsive perturbation of the $i-$th neuron at time t_k.

As usual in the theory of impulsive differential equations (and unlike [6, 7]), at the points of discontinuity t_k of the solution $t \mapsto u_i(t,x)$, we assume that $u_i(t_k,x) \equiv u_i(t_k^-,x)$. It is clear that, in general, the derivatives $\frac{\partial u}{\partial t}(t_k,x)$ do not exist. On the other hand, according to the first equality of (1), there do exist the limits $\frac{\partial u}{\partial t}(t_k^{\mp},x)$. According to the above convention, we assume $\frac{\partial u}{\partial t}(t_k,x) \equiv \frac{\partial u}{\partial t}(t_k^-,x)$.

Throughout the paper we assume that:

A1 The amplification functions $\alpha_i : \mathbb{R} \to (0,+\infty)$ are continuous and bounded in the sense that $0 < \underline{\alpha}_i \le \alpha_i(u) \le \overline{\alpha}_i$ for $u \in \mathbb{R}$, $i = \overline{1,m}$.

A2 The stabilizing functions $\beta_i : \mathbb{R} \to \mathbb{R}$ are continuous and monotone increasing, namely, $0 < \beta_i \le \dfrac{\beta_i(u) - \beta_i(v)}{u - v}$ for $u, v \in \mathbb{R}$, $u \ne v$, $i = \overline{1,m}$.

A3 For the activation functions $f_i(u), g_i(u), h_i(u)$ there exist positive constants F_i, G_i, H_i such that $F_i = \sup\limits_{u\ne v}\left|\dfrac{f_i(u) - f_i(v)}{u - v}\right|$, $G_i = \sup\limits_{u\ne v}\left|\dfrac{g_i(u) - g_i(v)}{u - v}\right|$, $H_i = \sup\limits_{u\ne v}\left|\dfrac{h_i(u) - h_i(v)}{u - v}\right|$ for all $u, v \in \mathbb{R},\ u \ne v,\ i = \overline{1,m}$.

A4 For the impulse functions $\mathcal{I}_{ik} : \mathbb{R} \to \mathbb{R}$, $\mathcal{I}_{ik}(u) = u + I_{ik}(u)$ there exist positive constants γ_{ik} such that $\gamma_{ik} = \sup\limits_{u\ne v}\left|\dfrac{\mathcal{I}_{ik}(u) - \mathcal{I}_{ik}(v)}{u - v}\right|$ for all $u, v \in \mathbb{R},\ u \ne v,\ i = \overline{1,m},\ k \in \mathbb{N}$.

Next the reaction-diffusion terms are approximated by divided differences. For simplicity of notation the spatial domain Ω is assumed to be a finite closed interval $[a, b]$. Then for a sufficiently large positive integer N we choose a discretization step $h = (b-a)/N$, denote $x_\ell \equiv a + \ell h$ ($\ell = \overline{-1, N+1}$) and for $i = \overline{1, m}$ we write

$$\frac{\partial}{\partial x}\left(D_i(t,x)\frac{\partial u_i(t,x)}{\partial x}\right)\bigg|_{x=x_\ell} \approx$$

$$\frac{D_i(t,x_{\ell+1})u_i(t,x_{\ell+1}) - \big(D_i(t,x_{\ell+1}) + D_i(t,x_\ell)\big)u_i(t,x_\ell) + D_i(t,x_\ell)u_i(t,x_{\ell-1})}{h^2}.$$

Further we denote for brevity $u_i(t,\ell) \equiv u_i(t,x_\ell)$ ($\ell = \overline{-1, N+1}$), $D_i(t,\ell) \equiv D_i(t,x_\ell)$ ($\ell = \overline{0, N+1}$), $D_i(t,\ell+1/2) \equiv \big(D_i(t,x_{\ell+1}) + D_i(t,x_\ell)\big)/2$ ($\ell = \overline{0, N}$). Finally we approximate the zero Neumann boundary conditions by

$$u_i(t,-1) = u_i(t,0), \quad u_i(t,N+1) = u_i(t,N), \quad t > 0, \quad i = \overline{1, m}.$$

Thus we obtain the following spatial discretization of system (1)

$$\begin{gathered}
\frac{\partial u_i(t,\ell)}{\partial t} = \\
\frac{D_i(t,\ell+1)u_i(t,\ell+1) - 2D_i(t,\ell+1/2)u_i(t,\ell) + D_i(t,\ell)u_i(t,\ell-1)}{h^2} \\
-\alpha_i(u_i(t,\ell))\Bigg[\beta_i(u_i(t,\ell)) - \sum_{j=1}^{m} a_{ij} f_j(u_j(t,\ell)) \qquad (2) \\
- \sum_{j=1}^{m} b_{ij} g_j(u_j(t-\tau_{ij}(t),\ell)) - \sum_{j=1}^{m} c_{ij}\int_0^{+\infty} K_{ij}(s) h_j(u_j(t-s,\ell))\,ds + J_i\Bigg], \\
t > 0, \quad t \neq t_k, \qquad i = \overline{1, m}, \qquad \ell = \overline{0, N}, \\
u_i(t,-1) = u_i(t,0), \quad u_i(t,N+1) = u_i(t,N), \qquad t > 0, \quad i = \overline{1, m}, \\
\Delta u_i(t_k,\ell) = I_{ik}(u_i(t_k,\ell)), \quad i = \overline{1, m}, \quad \ell = \overline{0, N}, \quad k \in \mathbb{N}, \\
u_i(s,\ell) = \phi_i(s,\ell), \quad s \le 0, \quad \ell = \overline{0, N}, \quad i = \overline{1, m},
\end{gathered}$$

which can be regarded as a neural network with $m(N+1)$ neurons.

The components of an equilibrium point $u^* = (u_1^*, \ldots, u_m^*)$ of system (2) (or (1)) are governed by the algebraic sytem

$$\beta_i(u_i^*) - \sum_{j=1}^{m}\big(a_{ij} f_j(u_j^*) + b_{ij} g_j(u_j^*) + c_{ij} h_j(u_j^*)\big) + J_i = 0, \qquad i = \overline{1, m}. \quad (3)$$

and satisfy the equalities

$$I_{ik}(u_i^*) = 0, \qquad i = \overline{1,m}, \quad k \in \mathbb{N}. \tag{4}$$

We assume that

A5 The impulse functions I_{ik} satisfy the equalities (4) for any solution u^* of system (3).

Denote

$$\|u_i(t,\cdot)\| = \left(\sum_{\ell=0}^{N} \big(u_i(t,\ell)\big)^2 h\right)^{1/2}.$$

Definition 1 An equilibrium point $u^* = (u_1^*, \ldots, u_m^*)$ of system (2) is said to be *globally exponentially stable* if there exist constants $\lambda > 0$ and $M \geq 1$ such that for any solution $u(t,\ell) = (u_1(t,\ell), \ldots, u_m(t,\ell))^T$ of system (2) we have

$$\sum_{i=1}^{m} \|u_i(t,\cdot) - u^*\| \leq M \sup_{s\leq 0} \sum_{i=1}^{m} \|\phi_i(s,\cdot) - u^*\| e^{-\lambda t} \quad \text{for all} \;\; t \geq 0.$$

Definition 2 [1] A real matrix $A = (a_{ij})_{m\times m}$ is said to be an $M-$matrix if $a_{ij} \leq 0$ for $i,j = \overline{1,m}$, $i \neq j$ and all successive principle minors of A are positive.

Lemma 1 [1] *Let $A = (a_{ij})_{m\times m}$ be a real matrix with non-positive off-diagonal elements. Then A is an $M-$matrix if and only if one of the following conditions holds:*

(1) *There exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ with $\xi_i > 0$ such that every component of $\xi^T A$ is positive — that is, $\sum\limits_{i=1}^{m} \xi_i a_{ij} > 0$, $j = \overline{1,m}$.*

(2) *There exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ with $\xi_i > 0$ such that every component of $A\xi$ is positive — that is, $\sum\limits_{j=1}^{m} a_{ij}\xi_j > 0$, $i = \overline{1,m}$.*

For more details about $M-$matrices the reader is referred to [3, 5].

Now let us introduce the following matrices:

$$\begin{array}{lll} \underline{\alpha} = \mathrm{diag}(\underline{\alpha}_1, \ldots, \underline{\alpha}_m), & \overline{\alpha} = \mathrm{diag}(\overline{\alpha}_1, \ldots, \overline{\alpha}_m), & \beta = \mathrm{diag}(\beta_1, \ldots, \beta_m), \\ F = \mathrm{diag}(F_1, \ldots, F_m), & G = \mathrm{diag}(G_1, \ldots, G_m), & H = \mathrm{diag}(H_1, \ldots, H_m), \\ |A| = (|a_{ij}|)_{m\times m}, & |B| = (|b_{ij}|)_{m\times m}, & |C| = (|c_{ij}|)_{m\times m}. \end{array}$$

Lemma 2 [6, 7] *Let the assumptions* **A1**–**A3** *and* **A5** *hold and suppose that $\mathcal{A} = \underline{\alpha}\beta - \overline{\alpha}\big(|A|F + |B|G + |C|H\big)$ is an $M-$matrix. Then system (2) has a unique equilibrium point.*

Proof. Since $\mathcal{A}$ is an $M-$matrix, and $\overline{\alpha}^{-1}\underline{\alpha} \leq E$ (E is the identity matrix), then $\beta - \big(|A|F + |B|G + |C|H\big)$ is also an $M-$matrix. From [10, Corollary 2] it is easy to deduce that system (2) without impulses ($I_{ik} \equiv 0$) has a unique equilibrium point. By **A5** it is also an equilibrium point of system (2). $\square$

3 Main results

Theorem 1 *Let the system* (2) *satisfy assumptions* **A1**–**A5**. *If the following conditions hold:*

(*a*) *There exists a vector* $\xi = (\xi_1, \dots, \xi_m)^T > 0$ *and a number* $\lambda > 0$ *such that*

$$\sum_{j=1}^{m} \left\{(\lambda - \underline{\alpha}_i)\delta_{ij} + \overline{\alpha}_i \left[|a_{ij}|F_j + |b_{ij}|G_j e^{\lambda\tau_{ij}} + |c_{ij}|H_j k_{ij}(\lambda)\right]\right\}\xi_j < 0, \quad i = \overline{1,m}, \tag{5}$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$ *for* $j \neq i$;

(*b*) $\eta = \sup_{k\in\mathbb{N}} \left\{\dfrac{\ln \eta_k}{t_k - t_{k-1}}\right\} < \lambda$, *where* $\eta_k = \max_{i=\overline{1,m}} \{1, \gamma_{ik}\}$, $k \in \mathbb{N}$, $t_0 = 0$,

then the system (2) *has a unique equilibrium point which is globally exponentially stable with convergence rate* $\lambda - \eta$.

Proof. First let us note that the condition (5) holds if and only if $\mathcal{A}$ is an $M-$matrix. In fact, if $\mathcal{A}$ is an $M-$matrix, from Lemma 1 there exists a vector $\xi > 0$ such that $\left[-\underline{\alpha}\beta + \overline{\alpha}(|A|F + |B|G + |C|H)\right]\xi < 0$. By continuity, there exists $\lambda > 0$ such that (5) holds.

Conversely, if (5) holds for some $\lambda_0 > 0$, then it still holds for all $\lambda \in [0, \lambda_0]$. For $\lambda = 0$, from Lemma 1, we deduce that $\mathcal{A}$ is an $M-$matrix.

Thus the condition (5), by virtue of Lemma 2, ensures the existence of a unique equilibrium point $u^* = (u_1^*, \dots, u_m^*)^T$ for system (2). For any other solution $u(t,\ell) = (u_1(t,\ell), \dots, u_m(t,\ell))^T$ of the system (2), denote

$$w_i(t,\ell) = u_i(t,\ell) - u_i^*, \qquad i = \overline{1,m}, \quad \ell = \overline{-1, N+1}.$$

Thus the system (2) is transformed into

$$\frac{\partial w_i(t,\ell)}{\partial t} =$$

$$= \frac{D_i(t,\ell+1)w_i(t,\ell+1) - 2D_i(t,\ell+1/2)w_i(t,\ell) + D_i(t,\ell)w_i(t,\ell-1)}{h^2} -$$

$$-\tilde{\alpha}_i(w_i(t,\ell))\left[\tilde{\beta}_i(w_i(t,\ell)) - \sum_{j=1}^{m} a_{ij}\tilde{f}_j(w_j(t,\ell)) - \right. \tag{6}$$

$$\left. - \sum_{j=1}^{m} b_{ij}\tilde{g}_j(w_j(t-\tau_{ij}(t),\ell)) - \sum_{j=1}^{m} c_{ij}\int_0^{+\infty} K_{ij}(s)\tilde{h}_j(w_j(t-s,\ell))\,ds\right],$$

$$w_i(t,-1) = w_i(t,0), \qquad w_i(t,N+1) = w_i(t,N).$$

$$\Delta w_i(t_k,\ell) = \tilde{I}_{ik}(w_i(t_k,\ell)),$$

where

$$\tilde{\alpha}_i(w_i) = \alpha_i(w_i + u_i^*), \qquad \tilde{\beta}_i(w_i) = \beta_i(w_i + u_i^*) - \beta_i(u_i^*),$$

$$\tilde{f}_j(w_j) = f_j(w_j + u_j^*) - f_j(u_j^*), \qquad \tilde{g}_j(w_j) = g_j(w_j + u_j^*) - g_j(u_j^*),$$

$$\tilde{h}_j(w_j) = h_j(w_j + u_j^*) - h_j(u_j^*), \qquad \tilde{I}_{ik}(w_i) = I_{ik}(w_i + u_i^*).$$

We multiply the $i-$th differential equation in (6) by $w_i(t,\ell)\,h$ and sum up for $\ell = \overline{0,N}$:

$$\frac{1}{2}\frac{d}{dt}\sum_{\ell=0}^{N}\big(w_i(t,\ell)\big)^2 h =$$

$$\sum_{\ell=0}^{N}\big\{D_i(t,\ell+1)w_i(t,\ell+1) - 2D_i(t,\ell+1/2)w_i(t,\ell) + D_i(t,\ell)w_i(t,\ell-1)\big\}w_i(t,\ell)\,h^{-1} -$$

$$-\sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))\tilde{\beta}_i(w_i(t,\ell))w_i(t,\ell)\,h + \sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))w_i(t,\ell)\sum_{j=1}^{m} a_{ij}\tilde{f}_j(w_j(t,\ell))\,h +$$

$$+\sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))w_i(t,\ell)\sum_{j=1}^{m} b_{ij}\tilde{g}_j(w_j(t-\tau_{ij}(t),\ell))\,h +$$

$$+\sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))w_i(t,\ell)\sum_{j=1}^{m} c_{ij}\int_0^{+\infty} K_{ij}(s)\tilde{h}_j(w_j(t-s,\ell))\,ds\ h.$$

By virtue of the equalities $w_i(t,-1) = w_i(t,0)$, $w_i(t,N+1) = w_i(t,N)$, we have

$$\sum_{\ell=0}^{N}\big\{D_i(t,\ell+1)w_i(t,\ell+1) - 2D_i(t,\ell+1/2)w_i(t,\ell) + D_i(t,\ell)w_i(t,\ell-1)\big\}w_i(t,\ell) =$$

$$= -\sum_{\ell=0}^{N} D_i(t,\ell+1)\big(w_i(t,\ell+1) - w_i(t,\ell)\big)^2 \le 0.$$

Next we have

$$\sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))\tilde{\beta}_i(w_i(t,\ell))w_i(t,\ell)\,h \ge \underline{\alpha}_i\beta_i\sum_{\ell=0}^{N}\big(w_i(t,\ell)\big)^2 h = \underline{\alpha}_i\beta_i\|w_i(t,\cdot)\|^2;$$

$$\begin{aligned}
&\sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))w_i(t,\ell)\sum_{j=1}^{m}a_{ij}\tilde{f}_j(w_j(t,\ell))\,h \le\\
&\le \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|\sum_{\ell=0}^{N}|w_i(t,\ell)|\,F_j\,|w_j(t,\ell)|\,h \le\\
\le\ &\overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|F_j\left(\sum_{\ell=0}^{N}\big(w_i(t,\ell)\big)^2 h\right)^{1/2}\times\left(\sum_{\ell=0}^{N}\big(w_j(t,\ell)\big)^2 h\right)^{1/2} =\\
&= \overline{\alpha}_i\sum_{j=1}^{m}|a_{ij}|F_j\|w_i(t,\cdot)\|\,\|w_j(t,\cdot)\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))w_i(t,\ell)\sum_{j=1}^{m}b_{ij}\tilde{g}_j(w_j(t-\tau_{ij}(t),\ell))\,h \le\\
&\le \overline{\alpha}_i\sum_{j=1}^{m}|b_{ij}|G_j\|w_i(t,\cdot)\|\,\|w_j(t-\tau_{ij}(t),\cdot)\|
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\ell=0}^{N}\tilde{\alpha}_i(w_i(t,\ell))w_i(t,\ell)\sum_{j=1}^{m}c_{ij}\int_0^{+\infty}K_{ij}(s)\tilde{h}_j(w_j(t-s,\ell))\,ds\,h \le\\
&\le \overline{\alpha}_i\sum_{j=1}^{m}|c_{ij}|H_j\|w_i(t,\cdot)\|\int_0^{+\infty}K_{ij}(s)\|w_j(t-s,\cdot)\|\,ds.
\end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_i(t,\cdot)\|^2 \le -\underline{\alpha}_i\beta_i\|w_i(t,\cdot)\|^2 + \overline{\alpha}_i\sum_{j=1}^{m}\Bigg\{&|a_{ij}|F_j\|w_i(t,\cdot)\|\,\|w_j(t,\cdot)\| +\\
+ |b_{ij}|G_j\|w_i(t,\cdot)\|\,\|w_j(t-\tau_{ij}(t),\cdot)\| + |c_{ij}|H_j\|w_i(t,\cdot)\|&\int_0^{+\infty}K_{ij}(s)\|w_j(t-s,\cdot)\|\,ds\Bigg\}
\end{aligned}$$

or

$$D^+\|w_i(t,\cdot)\| \le -\underline{\alpha}_i\beta_i\|w_i(t,\cdot)\| + \overline{\alpha}_i \sum_{j=1}^{m} \Big\{ |a_{ij}|F_j\|w_j(t,\cdot)\| + \tag{7}$$

$$+ \quad |b_{ij}|G_j\|w_j(t-\tau_{ij}(t),\cdot)\| + |c_{ij}|H_j \int_0^{+\infty} K_{ij}(s)\|w_j(t-s,\cdot)\|\,ds \Big\},$$

where D^+ denotes the upper left Dini derivative.

If we introduce the notation

$$v_i(t) = \|w_i(t,\cdot)\|,\ p_{ij} = -\underline{\alpha}_i\beta_i\delta_{ij} + \overline{\alpha}_i|a_{ij}|F_j,\ q_{ij} = \overline{\alpha}_i|b_{ij}|G_j,\ r_{ij} = \overline{\alpha}_i|c_{ij}|H_j,$$

then (7) takes the form

$$D^+v_i(t) \le \sum_{j=1}^{m} \Big\{ p_{ij}v_j(t) + q_{ij}v_j(t-\tau_{ij}(t)) + r_{ij}\int_0^{+\infty} K_{ij}(s)v_j(t-s)\,ds \Big\}. \tag{8}$$

Lemma 3 *Let $a < b < +\infty$. Suppose that $v(t) = \big(v_1(t),\dots,v_m(t)\big)^T \in C\big((a,b],\mathbb{R}^m\big)$ satisfies* (8) *and $v_i(s)$ are piecewise continuous on $(-\infty,b]$ with possible discontinuities at a finite number of points at which they are continuous from the left.*

If $v_i(t) \le \kappa\xi_i e^{-\lambda(t-a)}$, $\kappa \ge 0$, $t \in (-\infty,a]$, and $v_i(a^+) \le \kappa\xi_i$, $i = \overline{1,m}$, where $\lambda > 0$ and $\xi = (\xi_1,\dots,\xi_m)^T \ge 0$ satisfy

$$\sum_{j=1}^{m} \left[\lambda\delta_{ij} + p_{ij} + q_{ij}e^{\lambda\tau_{ij}} + r_{ij}k_{ij}(\lambda)\right]\xi_j < 0, \tag{9}$$

then $v_i(t) \le \kappa\xi_i e^{-\lambda(t-a)}$ for $t \in (a,b]$, $i = \overline{1,m}$.

Proof of Lemma 3. Let ε be an arbitrary positive number. Denote $V_i(t) = (\kappa+\varepsilon)\xi_i e^{-\lambda(t-a)}$. We shall prove that $v_i(t) \le V_i(t)$, $t \in (a,b]$, $i = \overline{1,m}$.

Denote $t^* = \sup\{B \mid B \in (a,b),\ v_i(t) \le V_i(t),\ t \in [a,B],\ i = \overline{1,m}\}$. If $t^* = b$, the assertion is proved. Otherwise, $t^* \in (a,b)$, $v_i(t) \le V_i(t)$, $t \in [a,t^*]$, $i = \overline{1,m}$ and there exists $i_0 \in \{1,\dots,m\}$ such that $v_{i_0}(t^*) = V_{i_0}(t^*)$ and

$D^+v_{i_0}(t^*) \geq \dot{V}_{i_0}(t^*)$. Further on, we have

$$D^+v_{i_0}(t^*) \leq$$

$$\leq \sum_{j=1}^{m} \left[p_{i_0j}v_j(t^*) + q_{i_0j}v_j(t^* - \tau_{i_0j}(t^*)) + r_{i_0j}\int_0^{+\infty} K_{i_0j}(s)v_j(t^* - s)\,ds \right] \leq$$

$$\leq \sum_{j=1}^{m} \Big[p_{i_0j}(\kappa+\varepsilon)\xi_j e^{-\lambda(t^*-a)} + q_{i_0j}(\kappa+\varepsilon)\xi_j e^{-\lambda\left(t^*-\tau_{i_0j}(t^*)-a\right)} +$$

$$+ r_{i_0j}\int_0^{+\infty} K_{i_0j}(s)(\kappa+\varepsilon)\xi_j e^{-\lambda(t^*-s-a)}\,ds \Big] \leq$$

$$\leq (\kappa+\varepsilon)e^{-\lambda(t^*-a)}\sum_{j=1}^{m} \left[p_{i_0j} + q_{i_0j}e^{\lambda\tau_{i_0j}} + r_{i_0j}k_{i_0j}(\lambda) \right]\xi_j.$$

From (9), it follows that

$$\sum_{j=1}^{m} \left[p_{i_0j} + q_{i_0j}e^{\lambda\tau_{i_0j}} + r_{i_0j}k_{i_0j}(\lambda) \right]\xi_j < -\lambda\xi_{i_0} < 0,$$

thus

$$D^+v_{i_0}(t^*) < -(\kappa+\varepsilon)e^{-\lambda(t^*-a)}\lambda\xi_{i_0} = \dot{V}_{i_0}(t^*),$$

which is a contradiction.

Thus we proved that $v_i(t) \leq (\kappa+\varepsilon)\xi_i e^{-\lambda(t-a)}$, $t \in (a,b]$, $i = \overline{1,m}$. Now the assertion of the lemma follows for $\varepsilon \to 0$. □

Let ξ and λ be as in (5), then (9) is satisfied. If we denote

$$\kappa = \sup_{s\leq 0}\sum_{i=1}^{m} \|\phi_i(s,\cdot) - u_i^*\| / \min_{i=\overline{1,m}}\{\xi_i\},$$

it is easy to see that $v_i(t) \leq \kappa\xi_i e^{-\lambda t}$ for $t \in (-\infty, t_0]$, $t_0 = 0$.

From Lemma 3 it follows that $v_i(t) \leq \kappa\xi_i e^{-\lambda t}$ for $t \in (t_0, t_1]$.

Now suppose that

$$v_i(t) \leq \kappa\eta_0\eta_1\cdots\eta_{k-1}\xi_i e^{-\lambda t} \quad \text{for} \quad t \in (t_{k-1}, t_k], \tag{10}$$

where $\eta_0 = 1$. Then

$$v_i(t_k^+) \leq \gamma_{ik}v_i(t_k) \leq \eta_k v_i(t_k) \leq \kappa\eta_0\eta_1\cdots\eta_{k-1}\xi_i e^{-\lambda t_k}.$$

Since $\eta_k \geq 1$, from (10), it follows that

$$v_i(t) \leq \kappa\eta_0\eta_1\cdots\eta_{k-1}\eta_k\xi_i e^{-\lambda t_k}e^{-\lambda(t-t_k)} \quad \text{for} \quad t \leq t_k.$$

From Lemma 3, it follows that

$$\begin{aligned} v_i(t) &\leq \kappa\eta_0\eta_1\cdots\eta_{k-1}\eta_k\xi_i e^{-\lambda t_k}e^{-\lambda(t-t_k)} = \\ &= \kappa\eta_0\eta_1\cdots\eta_{k-1}\eta_k\xi_i e^{-\lambda t} \quad \text{for} \quad t\in(t_k,t_{k+1}]. \end{aligned}$$

Thus we have proved (10) by induction.

By condition (b) of Theorem 1, we have $\eta_k \leq e^{\eta(t_k-t_{k-1})}$. Then

$$\eta_0\eta_1\cdots\eta_{k-1} \leq e^{\eta t_1}e^{\eta(t_2-t_1)}\cdots e^{\eta(t_{k-1}-t_{k-2})} = e^{\eta t_{k-1}} \leq e^{\eta t} \ \text{ for } \ t\in(t_{k-1},t_k].$$

Now (10) implies $\|u_i(t,\cdot)-u_i^*\| \leq \kappa\xi_i e^{-(\lambda-\eta)t}$ for $t\geq 0$, $i=\overline{1,m}$. Hence

$$\sum_{i=1}^m \|u_i(t,\cdot)-u^*\| \leq M\sup_{s\leq 0}\sum_{i=1}^m \|\phi_i(s,\cdot)-u^*\|e^{-(\lambda-\eta)t}, \quad \text{for all } \ t\geq 0$$

with $M = \left(\sum\limits_{i=1}^m \xi_i\right) / \min\limits_{i=\overline{1,m}}\{\xi_i\} > 1.$ □

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