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LARGE EQUIVALENCE OF d^h-MEASURES

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Abstract

We extend the definition of d^h-measures introduced by Lee and Baek to the more general setting of compact metric spaces and prove that two d^h-measures are equivalent if and only if their respective measure functions are equivalent.

1 Introduction

Let us begin with the definition of the $d^{\rho,h}$-measure introduced by Lee and Baek[4, 5]. Let E be a bounded set in $\mathbb{R}^n$ and h be a measure function, i.e. $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and continuous function with $h(0+) = 0$. The pre d^h-measure of E is:

$$D^h(E) = \liminf_{r\to 0} N_r(E)h(r),$$

where $N_r(E)$ is the minimum number of closed balls with diameter r, needed to cover E. Then we employ Method I by Munroe to obtain an outer measure d^h of $E \subset X$:

$$d^h(E) = inf\{\sum_{i=1}^{\infty} D^h(E_i)|E \subset \cup E_i, E_i \subset \mathbb{R}^n\}.$$

If $h(t) = t^s$, then the d^h-measure induces the modified lower box dimension [4, 5].

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In this paper, we extend the definition of d^h-measures to the more general setting of compact metric spaces and prove that two d^h-measures are equivalent if and only if their respective measure functions are equivalent. Let (X, ρ) be a compact metric space. We define the pre $d^{\rho,h}$-measure of E with respect to the metric ρ, by

$$D^{\rho,h}(E) = \liminf_{r \to 0} N_r(E) h(|B(x,r)|_\rho),$$

where $N_r(E)$ is the minimum number of closed balls$\{B(x,r)\}$ with radius r, needed to cover E and $|B(x,r)|_\rho$ denotes the diameter of $B(x,r)$ with respect to the metric ρ. Then we employ Method I by Munroe to obtain an outer measure $d^{\rho,h}$ of $E \subset X$:

$$d^{\rho,h}(E) = inf\{\sum_{i=1}^{\infty} D^{\rho,h}(E_i) | E \subset \cup E_i, E_i \subset X\}.$$

Remark 1 The definition of $d^{\rho,h}$ remains unchanged if we put $E = \cup E_i$ in the place of $E \subset \cup E_i$.

Remark 2 By the definitions, we can see that $d^{\rho,h} \leq D^{\rho,h}$.

Recall that two measure functions g and h are said to be equivalent if there are constants $c \geq 1$ and $\delta > 0$ such that

$$c^{-1} h(t) \leq g(t) \leq c h(t)$$

for any $0 < t \leq \delta$. Two Borel measures μ and ν on (X, ρ) are said to be equivalent if there is a constant $c \geq 1$ such that

$$c^{-1} \mu(A) \leq \nu(A) \leq c \mu(A)$$

for all Borel sets A.

2 Main results and proofs

Proposition 1 *$d^{\rho,h}$ is a metric outer measure.*

Proof. It is sufficient to proof that $d^{\rho,h}(E \cup F) = d^{\rho,h}(E) + d^{\rho,h}(F)$ whenever $E, F \subset X$ with $dist(E, F) > 0$. Suppose that $dist(E, F) > 0$ for $E, F \subset X$. Then $dist(E, F) > 2\varepsilon > 0$ for some positive constant ε. Noting that $N_\varepsilon(E \cup F) = N_\varepsilon(E) + N_\varepsilon(F)$, we have

$$D^{\rho,h}(E \cup F) \geq D^{\rho,h}(E) + D^{\rho,h}(F). \tag{1}$$

Hence, for E and F with $dist(E,F) > 0$,

$$\begin{aligned} d^{\rho,h}(E \cup F) &= inf\{\sum_{i=1}^{\infty} D^{\rho,h}(E_i) | E \cup F = \cup E_i, E_i \subset X\} \\ &= inf\{\sum_{i=1}^{\infty} D^{\rho,h}((E_i \cap E) \cup (E_i \cap F)) | E \cup F = \cup E_i, E_i \subset X\} \\ &\geq inf\{\sum_{i=1}^{\infty} D^{\rho,h}(E_i \cap E) + D^{\rho,h}(E_i \cap F) | E \cup F = \cup E_i, E_i \subset X\} \\ &\geq inf\{\sum_{i=1}^{\infty} D^{\rho,h}(E_i \cap E) | E \cup F = \cup E_i, E_i \subset X\} \\ &+ inf\{\sum_{i=1}^{\infty} D^{\rho,h}(E_i \cap E) | E \cup F = \cup E_i, E_i \subset X\} \\ &\geq d^{\rho,h}(E) + d^{\rho,h}(F). \end{aligned}$$

The second inequality is obtained by (1).

On the other hand, we have $d^{\rho,h}(E \cup F) \leq d^{\rho,h}(E) + d^{\rho,h}(F)$ by subadditivity of $d^{\rho,h}$. This completes the proof. □

The measure $d^{\rho,h}$ is close related to Hausdorff measure. More precisely, we have the following proposition which can be deduced by the definitions(see also [5]).

Proposition 2 *For a subset E of (X, ρ), $\mathcal{H}^{\rho,h}(E) \leq d^{\rho,h}(E)$, where $\mathcal{H}^{\rho,h}(E)$denotes the Hausdorff h-measure of E.*

For details about Hausdorff h-measure, see [1, 2, 3, 8].

By the definitions, $d^{\rho,g}$ and $d^{\rho,h}$ are equivalent, if g and h are equivalent measure functions. Conversely, can we get from the equivalence of $d^{\rho,g}$ and $d^{\rho,h}$ that g and h are equivalent?

The theorem below answers this question.

Theorem A *Let g, h be any two measure functions. If $d^{\rho,g}$ and $d^{\rho,h}$ are equivalent for any compact metric space (X, ρ), then g and h are equivalent.*

Proof. Suppose g and h are not equivalent. We are going to construct a compact metric space (X, ρ) such that $0 < d^{\rho,h}(X) < \infty$ and $d^{\rho,g}(X) = 0$, which shows that $d^{\rho,g}$ and $d^{\rho,g}$ are not equivalent. The proof consists of four steps.

Step 1. Constructing (X, ρ). Let $\frac{1}{2} < \lambda < 1$ and $a_n = \lambda^{2^{-n}} (n \in \mathbb{N})$, then $a_1 a_2 \cdots\ a_n > \lambda$ for any $n \geq 1$. Assume that g and h are not equivalent, then by the definition, there exists a sequence $\{\delta_n\}_{n\geq 0} \searrow 0$ such that

$$\text{either } \lim_{n\to\infty} \frac{g(\delta_n)}{h(\delta_n)} = 0 \text{ or } \lim_{n\to\infty} \frac{g(\delta_n)}{h(\delta_n)} = \infty.$$

We only discuss the case $\lim_{n\to\infty} \frac{g(\delta_n)}{h(\delta_n)} = 0$. The case $\lim_{n\to\infty} \frac{g(\delta_n)}{h(\delta_n)} = \infty$ can be treated in the same way.

Since $\lim_{n\to\infty} h(\delta_n) = 0$, we may suppose further the sequence $\{\delta_n\}$ is chosen to satisfy

$$h(\delta_n) \leq (1 - a_n) h(\delta_{n-1}), \quad n \in \mathbb{N}.$$

Take

$$k_n = [\frac{h(\delta_{n-1})}{h(\delta_n)}], \quad n \in \mathbb{N},$$

where $[x]$ denotes the integer part of x, then we have

$$k_n \geq [\frac{1}{1 - a_n}] \geq 2, \quad k_1 \cdots k_n \leq \frac{h(\delta_0)}{h(\delta_n)} \tag{2}$$

and

$$k_1 k_2 \cdots k_n \geq (\frac{h(\delta_0)}{h(\delta_1)} - 1)(\frac{h(\delta_1)}{h(\delta_2)} - 1) \cdots (\frac{h(\delta_{n-1})}{h(\delta_n)} - 1) \geq \frac{\lambda h(\delta_0)}{h(\delta_n)}. \tag{3}$$

Let $F_0 = [0, 1]$. We construct a compact subset X of the interval $[0, 1]$ in the following way. Take k_1 disjoint closed subintervals of the unit interval $[0, 1]$ of positive length, and denote by F_1 the union of these k_1 intervals. For every element I of F_1, take k_2 disjoint closed subintervals of I of positive length to obtain $k_1 k_2$ disjoint closed intervals of $[0, 1]$, and denote by F_2 the union of these $k_1 k_2$ intervals. Continuing the above procedure, we obtain a sequence $F_0 \supset F_1 \supset \cdots \supset F_n \cdots$. Set

$$X = \cap_{n=1}^{\infty} F_n.$$

By the above construction, X is a nonempty compact subset of [0,1]. Every element of F_n is called a basic interval of level-n. Denote by d_n the largest length of the basic intervals of level-n, we may require

$$\lim_{n\to\infty} d_n = 0.$$

Let $x, y \in X$ with $x \neq y$. Denote by $n(x, y)$ the highest level of the basic interval containing x and y, thus, there exists an interval I of level $n(x, y)$ which contains both x and y, but any basic interval does not contain simultaneously x and y, if its level is higher than $n(x, y)$. We define another metric ρ on X by letting

$$\rho(x,y) = \begin{cases} 0, & if\, x = y, \\ \delta_{n(x,y)}, & if\, x \neq y. \end{cases}$$

Step 2. (X, ρ) is a compact metric space.

Now X has two topologies, the relative topology as a subset of the real line and the metric topology defined by the metric ρ. Let $(X, |\cdot|)$ be the subspace of real line and it is a compact metric space. Consider the identical mapping, $I(x)$, from $(X, |\cdot|)$ to (X, ρ). We will prove $I(x)$ is continuous and obtain (X, ρ) is compact by the fact that the continuous image of compact metric space is compact. Let $x \in X$ and $\varepsilon > 0$. We can choose n so large that $d_n < \varepsilon$. Then all point y of X with $|x - y| \leq d_n$ lie in the same basic interval of level-n as x, and so satisfy $\rho(I(x), I(y)) \leq d_n < \varepsilon$, which implies $I(x)$ is continuous.

Step 3. Estimating $d^{\rho,h}(X)$.

Let $n \geq 1$ and let I be a basic interval of level-n. Let $|I \cap X|_\rho$ denote the diameter of $I \cap X$ under the metric ρ, then we have $|I \cap X|_\rho = \delta_n$. In fact, for any $x, y \in I$, since $n(x, y)$ is the highest level of the basic interval containing x and y, we have $n(x, y) \geq n$ and in which the equality holds for some pair $x, y \in I$, so $|I \cap X|_\rho = \delta_n$ by the definition of the metric ρ.

First, we conclude that $d^{\rho,h}(X) < \infty$. It is sufficient to prove $D^{\rho,h}(X) < \infty$. Indeed,

$$D^{\rho,h}(X) \leq \lim_{n\to\infty} N_{\delta_n}(E)h(\delta_n) \leq \lim_{n\to\infty} k_1 \cdots k_n h(\delta_n) \leq \frac{h(\delta_0)}{h(\delta_n)} \cdot h(\delta_n) = h(\delta_0) < \infty.$$

So

$$d^{\rho,h}(X) \leq h(\delta_0) < \infty. \tag{4}$$

Let μ be the natural measure on X, that is, μ is the unique probability measure satisfying

$$\mu(I_n) = \frac{1}{k_1 \cdots k_n}$$

for all basic intervals I_n of level-n and for all n. Let U be a subset of X with $0 < |U| < \delta_0$ and n the positive integer with $\delta_n \leq |U| < \delta_{n-1}$. By the definition of the metric ρ,we have $|U| = \delta_n$, so there is a basic interval of level-n I_n such that $U \subset I_n$. Thus we have from(3)

$$\mu(U) \leq \mu(I_n) = \frac{1}{k_1 \cdots k_n} \leq \frac{h(|U|)}{\lambda h(\delta_0)},$$

which yields from mass distribution principle

$$\lambda h(\delta_0) \leq \mathcal{H}^{\rho,h}(X).$$

Then by proposition2 and (4), we have

$$0 < \lambda h(\delta_0) \leq d^{\rho,h}(X) \leq h(\delta_0) < \infty.$$

Step 4. Estimating $d^{\rho,g}(X)$. $D^{\rho,g}(X) \leq \lim_{n\to\infty} N_{\delta_n}(E)g(\delta_n) \leq \lim_{n\to\infty} k_1 \cdots k_n g(\delta_n) \leq \lim_{n\to\infty} \frac{h(\delta_0)}{h(\delta_n)} \cdot g(\delta_n) = \lim_{n\to\infty} \frac{g(\delta_n)}{h(\delta_n)} \cdot h(\delta_0) = 0$. So $d^{\rho,g}(X) = 0$.

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