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ON LEVEL HYPERSURFACES OF THE COMPLETE LIFT OF A SUBMERSION

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Abstract

Suppose that (M, G) is a Riemannian manifold and $f : M \to \mathbb{R}$ is a submersion. Then the complete lift of f, $f^c : TM \to \mathbb{R}$ defined by $f^c = \frac{\partial f}{\partial x^i} y^i$ is also a submersion. This interesting case leads us to the investigation of the level hypersurfaces of f^c as a submanifold of tangent bundle TM. In addition, we prolonge the level hypersurfaces of f to $\bar{N} = (f^c)^{-1}(0)$. Also, under the condition $\hat{\nabla} f$ is a constant, we show that $\bar{N}$ has a lightlike structure with induced metric $\bar{G}$ from G^c.

1 Introduction

We denote by $\Im_0^0(M)$ the algebra of smooth functions on M. For $f \in \Im_0^0(M)$, the complete lift of f to tangent bundle TM is defined by $f^c = y^i \frac{\partial f}{\partial x^i}$. From the local expression of f^c we realize that f^c is induced by f. In that case some geometrical relations must be between found the level hypersurfaces of f and f^c. In addition, the level hypersurfaces of f^c can be investigated depending on f.

To do these investigations we need some tools. These are vertical and complete lifts of differentiable elements defined on M. The notion of vertical and complete lift was introduced by K. Yano and S. Kobayashi in [12]. By using these lifts, in [10], M. Tani introduced the notion of prolongations of the hypersurfaces to tangent bundle.

In [10], Tani showed that there exist some geometrical relations between the geometry of S in M and TS in TM for a given hypersurface S.

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Many authors have studied lightlike hypersurfaces of semi-Riemannian manifolds [1], [6], [7], [9] and others.

In this paper, we discuss the relationships between the geometry of level surfaces of a real-valued function and its complete lift. The importance of this paper is that, differently from [10], we find a class of submanifolds in tangent bundle TM such that these are derived from hypersurfaces in M. Because, in [10], the obtained submanifold is tangent to original submanifold in M, but it isn't so in this work.

In addition, as we know from literature, an application of vertical and complete lifts to lightlike geometry was not studied yet. We can do these applications in the present paper. In last section, we establish lightlike structure on a level hypersurface of complete lift of f and see that fundamental notions of degenerate submanifold geometry were obtained by a natural way. That is, we needn't to any strong condition. This case shows that the problem studied here is completely suitable and interesting.

In Section 2, we shall give an introductory information. In Section 3, we shall show that the complete lift of a submersion is also a submersion and its any level set is a hypersurface (denoted by $\bar{N}$) in the tangent bundle. In Section 4, we obtain Gauss and Weingarten formulas for $\bar{N}$. In addition, it is obtained that $\bar{N}$ is a semi-Riemannian submanifold with index n with respect to G^c (G is a Riemannian metric on M). By using vertical and complete lifts, in Section 5, the level hypersurfaces of f have been prolonged to $\bar{N}$. In Section 6 we give a lightlike (null) structure on $\bar{N}$. In addition, as in Section 5, considering the lightlike structure on $\bar{N}$ we obtain some geometrical relations between the level hypersurfaces of f and $\bar{N}$ as well.

2 Notations and Preliminaries

For any differentiable manifold M, we denote by TM its tangent bundle with the projection $\pi_M : TM \longrightarrow M$ and by $T_p(M)$ its tangent space at a point p of M. $\Im_s^r(M)$ is the space of tensor fields of class C^∞ and of type (r,s). An element of $\Im_0^0(M)$ is a C^∞ function defined on M. We denote by $\Im(M)$ the tensor algebra on M.

Let M be an n-dimensional differentiable manifold and V be a coordinate neighborhood in M and (x^i) , $1 \leq i \leq n$, are certain local coordinates defined in V. We introduce a system of coordinates (x^i, y^i) in $\pi_M^{-1}(V)$ such that (y^i) are cartesian coordinates in each tangent space $T_p(M)$, p being an arbitrary point of V, with respect to the natural frame $(\frac{\partial}{\partial x^i})$ of local coordinates (x^i). We call (x^i, y^i) the coordinates induced in $\pi_M^{-1}(V)$ from (x^i). We suppose that all the used maps belong to the class C^∞ and we shall adopt the Einstein summation convention through this paper.

Now we must recall the definition of vertical and complete lifts of differentiable elements defined on M. Let f, X, w, G and $\hat{\nabla}$ be a function, a vector field, a 1-form, a tensor field of type $(0,2)$ and a linear connection, respectively. We denote by f^v, X^v, w^v and G^v the vertical lifts and by f^c, X^c, w^c, G^c and $\hat{\nabla}^c$ the complete lifts, respectively. For a function f on M, we have

$$f^v = f \circ \pi_M$$

$$f^c = y^i \frac{\partial f}{\partial x^i}$$

with respect to induced coordinates. Moreover these lifts have the properties:

$$\begin{aligned}
(fX)^v &= f^v X^v, \qquad\qquad (fX)^c = f^v X^c \ + f^c X^v \\
X^v f^c &= X^c f^v = (Xf)^v \\
w^v(X^c) &= w^c X^v = (w(X))^v \\
w^v(X^v) &= 0, \ \ w^c(X^c) = (w(X))^c \\
[X,Y]^c &= [X^c, Y^c], \ \ [X,Y]^v = [X^v, Y^c] = [X^c, Y^v], \\
[X^v, Y^v] &= 0, \\
G^c(X^c, Y^c) &= (G(X,Y))^c, \\
G^c(X^v, Y^c) &= G^c(X^c, Y^v) = (G(X,Y))^v, \\
G^c(X^v, Y^v) &= 0 \\
\hat{\nabla}^c_{X^c} Y^c &= (\hat{\nabla}_X Y)^c \quad \hat{\nabla}^c_{X^v} Y^c = \hat{\nabla}^c_{X^c} Y^v = (\hat{\nabla}_X Y)^v \\
\hat{\nabla}^c_{X^v} Y^v &= 0
\end{aligned} \tag{1}$$

(cf. [11]). Hence it is easily seen that if G is a Riemannian metric on M then G^c is a semi-Riemannian metric on TM and the index of G is equal to the dimension of M. Thus if (M, G) is a Riemannian manifold then (TM, G^c) is a semi-Riemannian manifold. Let $\hat{\nabla}$ be a metrical connection on M with respect to G. In this case, considering equalities in (1) we can say that $\hat{\nabla}^c$ is a metrical connection on TM with respect to G^c. Through this paper, as a semi-Riemannian structure on TM, we shall consider $(TM, G^c, \hat{\nabla}^c)$.

Let $f : M \to \mathbb{R}$ be a submersion. In this case for each $t \in range f$, $f^{-1}(t) = S_t$ is a level hypersurfaces in M , i.e. S_t is $(n-1)-$ dimensional submanifold of M [3]. Let F be a foliation by level sets of f so that

$$\Im^1_0(F) = \{X \in \Im^1_0(M) : X(f) = 0\}.$$

If (M, G) is a Riemannian manifold, then we write $\Im^1_0(F)^\perp = Span\{\hat{\nabla} f\}$, where $\hat{\nabla} f$ is gradient vector field of f. We also state that $X \in T(F)$ if and only if $G(X, \hat{\nabla} f) = 0$.

Consequently, if D is a distribution on M, defined by $D_p = \ker df_p$, from Frobenious Theorem D is integrable and determines the foliation F. It follows that the integral manifolds of D are the level sets of f or connected components of these level sets.

We know that a vector field on M belongs to D if $X_p \in D_p$ for each $p \in M$. When this happens we write $X \in \Gamma(D)$. Thus, if $X \in \Gamma(D)$, we call X is tangent to F, i.e. for any $p \in S_t$, $X_p \in T_pS_t$. According to these, we can write the following equality,

$$\Gamma(D) = \Im^1_0(F),$$

and we get a local basis of $\Gamma(D)$ or $\Im^1_0(F)$ in that form,

$$\{X_i : 1 \leq i \leq n-1,\ X_i \in \Gamma(D)\}.$$

For the purpose of this paper we must write the tangent bundle of F. Naturally, the tangent bundle of F is the disjoint union of tangent bundles of the level hypersurfaces (or integral manifolds of D) that is ,

$$TF = \bigcup_{t \in range(f)} TS_t \quad .$$

Then $dim\, TF = 2(n-1)$ and $codim\, TF = 2$. We denote by $\Im^1_0(TF)$ the vector fields on M being tangent to the foliation F, from [10] and [11],

$$\Im^1_0(TF) = Span\{X^c_1, ..., X^c_{n-1}, X^v_1, ..., X^v_{n-1}\}, \tag{2}$$

and

$$\Im^1_0(TF)^{\perp} = Span\{\xi^c, \xi^v\} \tag{3}$$

where $\xi = \frac{\hat{\nabla} f}{|\hat{\nabla} f|}$ is a unit normal of the foliation F. Thus we have ,

$$\Im^1_0(TM) \mid_{TF} = \Im^1_0(TF) \oplus \Im^1_0(TF)^{\perp}.$$

From (1), (2) and (3) as a local basis for $\Im^1_0(TM)$ along TF, we get

$$\{X^c_1, ..., X^c_{n-1}, X^v_1, ..., X^v_{n-1}, \xi^c, \xi^v\}.$$

3 Level Hypersurfaces of f^c

In this section, we will consider a special level hypersurface of f^c. If f is an element of $\Im^0_0(M)$ and $domain(f) = U$ is an open set of M, then the complete lift of f is defined on TU.

If $f : M \to \mathbb{R}$ is a submersion, then f^c is also. Indeed, let $f : M \to \mathbb{R}$ is a submersion, then f has rank one for each p in U . This means that, for at least i, $(1 \le i \le n)$, $\frac{\partial f}{\partial x^i} \mid_p \neq 0$, $p \in U$. Furthermore, we can write the Jacobian matrix of f^c as follows,

$$J(f^c) \mid_{v_p} = \left[\begin{array}{cc} \frac{\partial^2 f}{\partial x^i \partial x^j} \mid_p v^i & \frac{\partial f}{\partial x^i} \mid_p \end{array} \right]$$

for a point $v_p \in T_pM$. It follows that f^c has rank one.

If we put $\bar{N} = \bigcup\limits_{p \in U} (\ker df \mid_p)$, from definition of f^c we say that the restriction of f^c to $\bar{N}$ is identically zero. Thus, we can write

$$\bar{N} = (f^c)^{-1}(0)$$

that is, $\bar{N}$ is a level hypersurface of f^c.

Let (V, φ) be a coordinate neighbourhood in M. Then $(\hat{V} = \pi^{-1}(V), d\varphi)$ is a coordinate neighbourhood in TM. Let us construct the differentiable structure of $\bar{N}$:

$$\begin{aligned} \bar{N} \cap \hat{V} &= \bar{V} \\ &= \left\{ (p, v) \in \hat{V} : p \in V, v_p \in \ker df_p \right\} \\ &= \left\{ (p, v) \in \hat{V} : p \in V, f^c(v_p) = 0 \right\}. \end{aligned}$$

Thus, a local coordinate system on $\bar{V}$ is written to be $\bar{\varphi} = (x^i, \bar{v}^a)$, $(1 \le a \le n-1)$ and we take $\{\bar{V}_\alpha, \bar{\varphi}_\alpha\}_{\alpha \in I}$ as a differentiable structure on $\bar{N}$. In addition we can also say that $(\bar{N}, \bar{\pi}, M, \mathbb{R}^{n-1})$ has a vector bundle structure and by this structure it is a vector subbundle of TM, where $\bar{\pi}$ is restriction of π_M to $\bar{N}$.

Let $\bar{\imath} : \bar{N} \to TM$ be a natural injection in terms of local coordinates (x^i, y^i), $\bar{\imath}$ has the following local expressions

$$x^i = x^i, \qquad y^i = M^i_a \bar{v}^a \ , \qquad rank(M^i_a) = n-1.$$

Lemma 1 *The tangent bundle of each leaf of F is included in $\bar{N}$ as a submanifold of dimension 2(n-1).*

Proof. We know that the each leaf of F is a hypersurface of M. Let S be a leaf of the foliation F. From [10], TS is a submanifold of TM with dimension of $2(n-1)$. For every $v_p \in TS$,

$$\begin{aligned} f^c(v_p) &= \frac{\partial f}{\partial x^i} v^i \\ &= df_p(v_p) \\ &= 0. \end{aligned}$$

Hence $TS \subset \bar{N}$. Let $\tilde{\imath} : TS \to \bar{N}$ be a natural injection, then, in terms of local coordinates $(x^i, \bar{v}^a)$, $\tilde{\imath}$ has local expressions

$$x^i = x^i(u^a), \qquad \bar{v}^a = v^a,$$

where (u^a, v^a) are local coordinates on TS.

This case occurs for all leaves of F. For shortness we say that $\bar{N}$ includes the tangent bundle of F. We also say that $\Gamma(TF) \subset \Im_0^1(\bar{N})$. Hence we write the following decomposition

$$\Im_0^1(\bar{N}) = \Im_0^1(TF) \oplus Span\{Z\}$$

where Z is a tangent vector field to $\bar{N}$.

Lemma 2 *The gradient vector field of f^c with respect to semi Riemannian metric G^c is the complete lift of $\hat{\nabla} f$. We shall denote $\left(\hat{\nabla} f\right)^c$ as $\hat{\nabla}^c f^c$.*

Proof. If G has its matrix expression $[g_{ij}]$, then the matrix expression of G^c is as follows:

$$\begin{bmatrix} (g_{ij})^c & (g_{ij})^v \\ (g_{ij})^v & 0 \end{bmatrix} \ (see\ [11]).$$

We can find the inverse of this matrix, say

$$\begin{bmatrix} 0 & (g^{ij})^v \\ (g^{ij})^v & (g^{ij})^c \end{bmatrix}.$$

From the definition of the gradient vector field, we can have

$$\hat{\nabla}^c f^c = \left(\hat{\nabla} f\right)^c.$$

The proof is complete.

Since the vector field $\left(\hat{\nabla} f\right)^c$ is orthogonal to the submanifold $\bar{N}$ and thus the vector field $\frac{(\hat{\nabla} f)^c}{\left|(\hat{\nabla} f)^c\right|}$ is a unit normal vector field of $\bar{N}$, we have the lemma.

Lemma 3 *For each $f \in \Im_0^0(M)$, we have the the following properties :*

$$a) \ \left|\hat{\nabla} f\right|^v = \xi^v[f^c]; \qquad b) \ \left|\hat{\nabla} f\right|^c = \xi^c[f^c]. \tag{4}$$

Theorem 1 *An orthonormal basis for $\Gamma(TF)^{\perp}$ in (TM, G^c) is*

$$\left\{ Z = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\sigma^c}{\sigma^v}} \xi^v - \sqrt{\frac{\sigma^v}{\sigma^c}} \xi^c \right), \bar{Z} = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\sigma^c}{\sigma^v}} \xi^v + \sqrt{\frac{\sigma^v}{\sigma^c}} \xi^c \right) \right\},$$

such that $Z \in \Im_0^1(\bar{N})$ and $\bar{Z} \in \Im_0^1(\bar{N})^{\perp}$,where $\sigma = \left|\hat{\nabla} f\right|$.

Proof. From (3), a vector field normal to TF must be a linear combination of ξ^v and ξ^c. If N is an element of $\Im_0^1(\bar{N})$ and ortogonal to TF, then we can write

$$N = \alpha\xi^v + \beta\xi^c, \qquad \text{for } \alpha, \beta \in \Im_0^0\left(\bar{N}\right).$$

In this case

$$G^c(X^c, N) = 0,$$

for each $X^c \in \Im_0^1(TF)$. On the other hand, since N is tangent to $\bar{N}$,

$$df^c(N\mid_A) = 0 \qquad \text{for } A \in \bar{N}.$$

By using (4), we obtain,

$$\begin{aligned} df^c\mid_A(N|_A) &= df^c\left(\alpha(A)\xi^v\mid_A + \beta(A)\xi^c|_A\right) \\ &= \alpha(A)\xi^v\mid_A[f^c] + \beta(A)\xi^c|_A\,[f^c] \\ &= \alpha(A)(\xi\,[f])^v(A) + \beta(A)(\xi\,[f])^c(A) \\ &= \alpha(A)\left|\hat{\nabla} f\right|^v(A) + \beta(A)\left|\hat{\nabla} f\right|^c(A). \end{aligned} \tag{5}$$

This means that $df^c(N\mid_A)$ is zero if and only if following equalities are satisfied

$$\alpha(A) = \left|\hat{\nabla} f\right|^c(A) \text{ and } \beta(A) = -\left|\hat{\nabla} f\right|^v(A). \tag{6}$$

Now, by virtue of (6), we get

$$\begin{aligned} N &= \left|\hat{\nabla} f\right|^c\xi^v - \left|\hat{\nabla} f\right|^v\xi^c \\ &= \sigma^c\xi^v - \sigma^v\xi^c. \end{aligned} \tag{7}$$

If we put $Z = \frac{N}{|N|}$, then Z is a timelike unit vector field tangent to $\bar{N}$ and orthogonal to TF. On the other hand, we recall that $(\hat{\nabla} f)^c = \hat{\nabla}^c f^c$ and thus $(\hat{\nabla} f)^c$ is normal to $\bar{N}$. We can write the following equalities:

$$\begin{aligned} \hat{\nabla} f &= \left|\hat{\nabla} f\right|\xi, \\ (\hat{\nabla} f)^c &= \left|\hat{\nabla} f\right|^c\xi^v + \left|\hat{\nabla} f\right|^v\xi^c, \end{aligned}$$

and we have the unit normal vector field to $\bar{N}$ in the form

$$\bar{Z} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\sigma^c}{\sigma^v}}\xi^v + \sqrt{\frac{\sigma^v}{\sigma^c}}\xi^c\right).$$

In addition, $\bar{Z}$ is a spacelike unit vector field and $G^c(Z, \bar{Z}) = 0$. Thus the proof is complete.

Corollary 1 *In the case σ is a constant real number we get $-Z = \bar{Z} = \xi^c$.*

Corollary 2 *As a basis for $\Im_0^1(\bar{N})$ we will take the set $\{X_1^c, ..., X_{n-1}^c, X_1^v, ..., X_{n-1}^v, Z\}$ such that the set $\{X_1, ..., X_{n-1}\}$ is a basis for the foliation F.*

4 The Submanifold Geometry of $\bar{N}$ in TM

In this section we assume that σ is not a constant real number. Also, we shall identify $d\bar{\iota}(\bar{X})$ with $\bar{X}$, for $\bar{X} \in \Im_0^1(\bar{N})$.

Let us consider a Riemannian structure $(M, G, \hat{\nabla})$. If we denote by $\bar{G}$ the induced metric on $\bar{N}$ from G^c, then by definition we have

$$\bar{G}(\bar{X}, \bar{Y}) = G^c(\bar{X}, \bar{Y}) \qquad \text{for } \bar{X}, \bar{Y} \in \Im_0^1(\bar{N})(\textit{see } [8]).$$

In addition, if we denote by $\bar{\nabla}$ the induced covariant differentiation on $\bar{N}$ from $\hat{\nabla}^c$, then by definition we have,

$$\hat{\nabla}^c_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + \bar{B}(\bar{X}, \bar{Y})\bar{Z}, \qquad \text{for } \bar{X}, \bar{Y} \in \Im_0^1(\bar{N}),$$

$\bar{Z}$ being the unit normal of $\bar{N}$ given in Theorem 1 and $\bar{B}$ being a certain tensor field of type $(0, 2)$ on $\bar{N}$. We call $\bar{B}$ the second fundamental form of $\bar{N}$ and we define the tensor field $\bar{H}$ of type $(1, 1)$ by

$$\bar{B}(\bar{X}, \bar{Y}) = \bar{G}(\bar{H}\bar{X}, \bar{Y}).$$

If $\bar{B}$ is identically zero, we say that $\bar{N}$ is a totally geodesic hypersurface of TM. The equation of Weingarten for $\bar{N}$ in TM is written as

$$\hat{\nabla}^c_{\bar{X}} \bar{Z} = -\bar{H}\bar{X},$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary elements of $\Im_0^1(\bar{N})$ and $\bar{H}$ is the shape operator of $\bar{N}$ (see [4]).

Theorem 2 *$\bar{N}$ is a non degenerate semi-Riemannian submanifold with index n of TM with respect to G^c.*

Proof. Let $\{X_1, ..., X_{n-1}\}$ be an orthonormal basis for the foliation F. Then it is easily seen that the basis $\{X_1^c, ..., X_{n-1}^c, X_1^v, ..., X_{n-1}^v\}$ consists of null vectors completely. Now, by using this basis we construct an orthonormal basis for TF as in [2] . If we put

$$E_i = \frac{1}{\sqrt{2}}(X_i^c - X_i^v), \qquad E_i^* = \frac{1}{\sqrt{2}}(X_i^c + X_i^v), \qquad 1 \leq i \leq n-1,$$

we get

$$G^c(E_i, E_j) = -(G(X_i, X_j))^v = -\delta_{ij}, \tag{8}$$

and

$$G^c(E_i^*, E_j^*) = (G(X_i, X_j))^v = \delta_{ij}. \tag{9}$$

By Theorem 1 in Section 3, (8) and (9), the set $\bar{\Phi} = \{E_1, ..., E_{n-1}, E_1^*, ..., E_{n-1}^*, Z\}$ is an orthonormal basis for $\bar{N}$ and $ind\bar{N} = n$.

Consequently, the restriction of G^c to $\bar{N}$ has this matrix form with respect to $\bar{\Phi}$:

$$G^c \mid_{\bar{N}} = \begin{bmatrix} -I_{n-1} & 0_{n-1} & 0 \\ 0_{n-1} & I_{n-1} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where 0_{n-1} and 0 are $(n-1) \times (n-1)$ and 1×1 zero matrices, respectively. Thus, $rankG^c \mid_{\bar{N}} = 2n-1$ and it follows that $\bar{N}$ is a non-degenerate semi-Riemannian submanifold of TM and $\bar{N}$ has index n. From Theorem 2 we understand that $\bar{N}$ has co-index 0.

Now, let us compute the second fundamental form $\bar{B}$ with respect to basis $\{X_1^c, ..., X_{n-1}^c, X_1^v, ..., X_{n-1}^v, Z\}$ such that $\{X_1, ..., X_{n-1}\}$ is an arbitrary basis for the foliation F. We know from [4] that

$$\begin{aligned} \bar{B}(X_i^c, X_j^c) &= \bar{B}_{i\,j} = G^c(\hat{\nabla}^c_{X_i^c} X_j^c, \bar{Z}) \\ \bar{B}(X_i^c, X_j^v) &= \bar{B}_{i\,\bar{j}} = G^c(\hat{\nabla}^c_{X_i^c} X_j^v, \bar{Z}) \\ \bar{B}(X_i^v, X_j^v) &= \bar{B}_{\bar{\imath}\,\bar{j}} = G^c(\hat{\nabla}^c_{X_i^v} X_j^v, \bar{Z}) \\ \bar{B}(X_i^c, Z) &= \bar{B}_{i\,0} = G^c(\hat{\nabla}^c_{X_i^c} Z, \bar{Z}) \\ \bar{B}(X_i^v, Z) &= \bar{B}_{\bar{\imath}\,0} = G^c(\hat{\nabla}^c_{X_i^v} Z, \bar{Z}) \\ \bar{B}(Z, Z) &= \bar{B}_{00} = G^c(\hat{\nabla}^c_Z Z, \bar{Z}). \end{aligned} \tag{10}$$

Thus we get

$$\begin{array}{rcl}
\bar{B}_{i\,j} & = & \frac{1}{\sqrt{2}}\left[\sqrt{\frac{\sigma^v}{\sigma^c}}g^c(H^cX_i^c,X_j^c)+\sqrt{\frac{\sigma^c}{\sigma^v}}g^c(H^vX_i^c,X_j^c)\right], \\
\bar{B}_{i\,\bar{j}} & = & \frac{1}{\sqrt{2}}\sqrt{\frac{\sigma^v}{\sigma^c}}(g^c(H^cX_i^c,X_j^v), \\
\bar{B}_{\bar{\imath}\,\bar{j}} & = & 0, \\
\bar{B}_{i\,0} & = & X_i^c[\ln\sqrt{\frac{\sigma^c}{\sigma^v}}] \\
\bar{B}_{\bar{\imath}\,0} & = & X_i^v[\ln\sqrt{\frac{\sigma^c}{\sigma^v}}] \\
\bar{B}_{00} & = & G^c(\hat{\nabla}^c_Z Z,\bar{Z}) \\
 & = & -G^c(Z,\hat{\nabla}^c_Z\bar{Z}) \\
 & = & -Z[\ln\sqrt{\frac{\sigma^c}{\sigma^v}}],
\end{array} \tag{11}$$

where H is the shape operator of F.

By virtue of (10) and (11) we have the following theorem.

Theorem 3 *If the following conditions are satisfied, then $\bar{N}$ is a totally geodesic hypersurface of TM:*
i) The foliation F is totally geodesic,
ii) σ^c/σ^v is a constant real number.

5 Prolongation of Level Hypersurfaces to $\bar{N}$

Let S be a level hypersurfaces or a leaf of f. In this section we investigate the geometry of TS in $\bar{N}$ and we find some relationships between the submanifold geometry of S and TS. This investigation is quite natural, because f^c is induced by f.

If we denote by g the induced metric on S from G, then we have

$$g(X,Y)=G(X,Y), \qquad X,Y\in\Im^1_0(S).$$

Let us consider the Riemannian covariant differentiation $\hat{\nabla}$ determined by G in M. Then we have along S

$$\hat{\nabla}_{BX}BY=B(\nabla_XY)+g(HX,Y)\xi$$

where ξ and H are a unit normal and shape operator of S, respectively. We know that from submanifold theory ∇ is determined by induced metric g.

The complete lift g^c of g to TS is a semi Riemannian metric on TS. Moreover, g^c is induced by G^c [10]. On the other hand, let us consider the induced metric tensor from the metric structure $(\bar{N},\bar{G})$ which is defined by,

$$\tilde{G}(\tilde{X},\tilde{Y})=\bar{G}(\tilde{B}\tilde{X},\tilde{B}\tilde{Y}),$$

where $\tilde{B}$ is the differential mapping of $\tilde{\imath}$.

Now, we can ask if "does there exist a relationship between g^c and $\tilde{G}$?" First of all, since T_pS is isomorphic to $\ker df\mid_p$, for an element A of T_pS , $B(A)$ is included in $\bar{N}$. This means that, $range(B)$ is a subset of $\bar{N}$. Then we have following the equality:

$$\begin{aligned}\tilde{G}(\tilde{X},\tilde{Y}) &= \bar{G}(\tilde{X},\tilde{Y}) \\ &= G^c(\tilde{X},\tilde{Y}) \\ &= g^c(\tilde{X},\tilde{Y}),\end{aligned} \tag{12}$$

for any $\tilde{X},\tilde{Y}\in\Im_0^1(TS)$. By virtue of (12) we have

$$\tilde{G}=g^c.$$

Since ∇ is the covariant differentiation determined by g, similarly ∇^c is the covariant differentiation determined by g^c (see [10]).

Thus, if $\tilde{\nabla}$ is the induced covariant differentiation from $\bar{\nabla}$, then $\tilde{\nabla}=\nabla^c$. In this case, for arbitrary elements X,Y of $\Im_0^1(S)$, we have

$$\begin{aligned}\hat{\nabla}^c_{X^c}Y^c &= \bar{\nabla}_{X^c}Y^c+\bar{G}(\bar{H}X^c,Y^c)\bar{Z}^c \\ &= \nabla^c_{X^c}Y^c+g^c(\tilde{H}X^c,Y^c)Z+\bar{G}(\bar{H}X^c,Y^c)\bar{Z} \\ &= \nabla^c_{X^c}Y^c+g^c(\tilde{H}X^c,Y^c)Z+\bar{G}(\bar{H}X^c,Y^c)\bar{Z},\end{aligned} \tag{13}$$

where $\tilde{H}$ is the shape operator of TS in

On the other hand, we have from [10],

$$\hat{\nabla}^c_{X^c}Y^c=\nabla^c_{X^c}Y^c+g^c(H^cX^c,Y^c)\xi^v+g^c(H^vX^c,Y^c)\xi^c. \tag{14}$$

The equalities (13) and (14) allow us to determine a relation among $\tilde{H},H^c$ and H^v. If $\bar{N}$ is a totally geodesic submanifold of TM, then we get the following equality :

$$\begin{aligned}\tilde{g}(\tilde{H}X^c,Y^c) &= \bar{G}(\tilde{H}X^c,Y^c) \\ &= \bar{G}(-\bar{\nabla}_{X^c}Z,Y^c) \\ &= G^c(-\bar{\nabla}_{X^c}Z,Y^c) \\ &= G^c(-\hat{\nabla}^c_{X^c}Z,Y^c) \\ &= -\frac{1}{\sqrt{2}}G^c(\sqrt{\frac{\sigma^c}{\sigma^v}}H^vX^c-\sqrt{\frac{\sigma^v}{\sigma^c}}H^cX^c,Y^c) \\ &= -\frac{1}{\sqrt{2}}G^c(\sqrt{\frac{\sigma^c}{\sigma^v}}H^vX^c-\sqrt{\frac{\sigma^v}{\sigma^c}}H^cX^c,Y^c) \\ &= \frac{1}{\sqrt{2}}g^c(\sqrt{\frac{\sigma^c}{\sigma^v}}H^cX^c-\sqrt{\frac{\sigma^v}{\sigma^c}}H^vX^c,Y^c).\end{aligned} \tag{15}$$

Theorem 4 *Let $\bar{N}$ be a totally geodesic submanifold of TM, $\tilde{H}$ be the shape operator of TS in $\bar{N}$ and H be the shape operator of S in M. In this case there exists the following relation among $\tilde{H}, H^c$ and H^v:*

$$\tilde{H} = \frac{1}{\sqrt{2}}(\sqrt{\frac{\sigma^v}{\sigma^c}}H^c - \sqrt{\frac{\sigma^c}{\sigma^v}}H^v). \tag{16}$$

Proof. Since g is nondegenerate, from (15) the proof is clear.

Theorem 5 *Let $\bar{N}$ be a totally geodesic submanifold of TM. If S is a totally geodesic submanifold, then TS is totally geodesic in $\bar{N}$.*

Proof. If we assume that S is totally geodesic, then the second fundamental tensor of S vanishes always identically. If we denote by $\tilde{B}$ the second fundamental tensor of TS , by virtue of (16) we have

$$\begin{aligned}
\tilde{B}(X^c, Y^c) &= g^c(\tilde{H}X^c, Y^c) \\
&= g^c(\frac{1}{\sqrt{2}}(\sqrt{\frac{\sigma^v}{\sigma^c}}H^c X^c - \sqrt{\frac{\sigma^c}{\sigma^v}}H^v X^c), Y^c) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\sigma^v}{\sigma^c}}g^c(H^c X^c, Y^c) - \sqrt{\frac{\sigma^c}{\sigma^v}}g^c(H^v X^c, Y^c)\right) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\sigma^v}{\sigma^c}}(g(HX, Y))^c - \sqrt{\frac{\sigma^c}{\sigma^v}}(g(HX, Y))^v\right) \\
&= 0,
\end{aligned}$$

for $X, Y \in \Im_0^1(S)$.

Corollary 3 *Let $\bar{N}$ be a totally geodesic submanifold of TM. If TS is totally geodesic, then S is totally geodesic if and only if the following differential equation is satisfied*

$$(g(HX, Y))^c \sigma^v = (g(HX, Y))^v \sigma^c.$$

If we denote by $\tilde{m}$ the mean curvature of TS in $\bar{N}$, $\tilde{m}$ is defined as follows,

$$\tilde{m} = \frac{1}{2(n-1)} trace(\tilde{H}) \ (see\ [4]).$$

Theorem 6 *If $\bar{N}$ is a totally geodesic submanifold of TM, then the mean curvature of TS in $\bar{N}$ is written under the form*

$$\tilde{m} = \frac{\sqrt{2}}{2(n-1)}\sqrt{\frac{\sigma^v}{\sigma^c}}m^v$$

where m is mean curvature of S.

Proof. By virtue of (15) we have

$$\tilde{m} = \frac{1}{2(n-1)} trace\left(\frac{1}{\sqrt{2}}(\sqrt{\frac{\sigma^v}{\sigma^c}}H^c - \sqrt{\frac{\sigma^c}{\sigma^v}}H^v)\right).$$

Since $traceH^c = 2(traceH)^v$ and $traceH^v = 0$, the proof is complete.

S is said to be totally umbilical in M, if there exists a scalar field λ such that

$$B(X,Y) = \lambda g(X,Y)$$

for arbitrary elements X, Y of $\Im_0^1(S)$ (see [4]), that is $H = \lambda I$, where I is identity on $\Im_0^1(S)$. Thus we have

$$H^c = \lambda^c I^v + \lambda^v I^c \qquad \text{and} \qquad H^v = \lambda^v I^v. \tag{17}$$

We recall that I^c is the identity on $\Im_0^1(TS)$ (see [11]).

Corollary 4 *Let $\bar{N}$ be a totally geodesic submanifold of TM. In this case S is minimal if and only if TS is minimal in $\bar{N}$.*

Theorem 7 *Let $\bar{N}$ be a totally geodesic submanifold of TM and S be totally umbilical and $H = \lambda I$ be the shape operator on S. In this case TS is totally umbilical in $\bar{N}$ if and only if following equality is satisfied,*

$$\lambda^v \sigma^c = \lambda^c \sigma^v \tag{18}$$

Proof. If we assume that the equality (17) is satisfied, by virtue of (15), we get

$$\begin{aligned} \tilde{H} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\sigma^v}{\sigma^c}}(\lambda^c I^v + \lambda^v I^c) - \sqrt{\frac{\sigma^c}{\sigma^v}}\lambda^v I^v\right) \\ &= \frac{1}{\sqrt{2}}\left((\sqrt{\frac{\sigma^v}{\sigma^c}}\lambda^c - \sqrt{\frac{\sigma^c}{\sigma^v}}\lambda^v)I^v + \sqrt{\frac{\sigma^v}{\sigma^c}}\lambda^v I^c\right) \\ &= \frac{1}{\sqrt{2}}\sqrt{\frac{\sigma^v}{\sigma^c}}\lambda^v I^c. \end{aligned}$$

Thus, $\tilde{H}$ is proportional to the identity transformation of $\Im_0^1(TS)$.

Conversely, we assume that TS is totally umbilical in $\bar{N}$. In this case, the shape operator of TS is written in the following form

$$\tilde{H} = \tilde{\lambda} I^c.$$

By means of (16) and (17) we have

$$\begin{aligned} \tilde{H} &= \frac{1}{\sqrt{2}}((\sqrt{\frac{\sigma^v}{\sigma^c}}\lambda^c - \sqrt{\frac{\sigma^c}{\sigma^v}}\lambda^v)I^v + \sqrt{\frac{\sigma^v}{\sigma^c}}\lambda^v I^c). \\ &= \tilde{\lambda} I^c, \end{aligned} \tag{19}$$

Because I^v and I^c are linearly independent, we deduce, from (19),

$$\sqrt{\frac{\sigma^v}{\sigma^c}}\lambda^c - \sqrt{\frac{\sigma^c}{\sigma^v}}\lambda^v = 0$$

and

$$\frac{1}{\sqrt{2}}\sqrt{\frac{\sigma^v}{\sigma^c}}\lambda^v = \tilde{\lambda}.$$

Thus, the equality (18) is satisfied.

6 Lightlike Structure on $\bar{\mathbf{N}}$

In this section, we investigate the lightlike submanifold structure of $\bar{N}$ in a semi-Riemannian manifold (TM, G^c). For this purpose we need some informations about the lightlike submanifold geometry.

Firstly, we note that the notation and fundamental formulas used in this study are the same as in [5]. Let $\bar{M}$ be an $(m+2)$-dimensional semi-Riemannian manifold with index $q \in \{1, ..., m+1\}$. Let M be a hypersurface of $\bar{M}$. Denote by g the induced tensor field by $\bar{g}$ on M. M is called a lightlike hypersurface if g is of constant rank m. Consider the vector bundles $TM^{\perp}$ and $Rad(TM)$ whose fibres are defined by

$$T_xM^{\perp} = \{Y_x \in T_XM |\ g_x(Y_x, X_x) = 0, \forall X_x \in T_xM\}, Rad(T_xM) = T_xM \cap T_xM^{\perp},$$

for any $x \in M$, respectively. Thus, a hypersurface M of $\bar{M}$ is lightlike if and only if $Rad(T_xM) \neq \{0\}$ for all $x \in M$.

If M is a lightlike hypersurface, then we consider the complementary distribution $S(TM)$ of $TM^{\perp}$in TM which is called a screen distribution. From [2], we know that it is nondegenerate. Thus we have a direct orthogonal sum

$$TM = S(TM) \perp TM^{\perp} \tag{20}$$

Since $S(TM)$ is non-degenerate with respect to $\bar{g}$ we have

$$T\bar{M} = S(TM) \perp S(TM)^{\perp}$$

where $S(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\bar{M}|_M$.

Now, we will give an important theorem about lightlike hypersurfaces which enables us to set fundamental equations of M.

Theorem 8 *(see [5]) Let $(M, g, S(TM))$ be a lightlike hypersurface of $\bar{M}$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that, for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $U \subset M$, there exists a unique section N of $tr(TM)$ on U satisfying*

$$\bar{g}(N, \xi) = 1$$

and

$$\bar{g}(N, N) = \bar{g}(N, W) = 0, \forall W \in \Gamma(S(TM)|_U).$$

From Theorem 8, we have

$$T\bar{M}|_M = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{21}$$

$tr(TM)$ is called the null transversal vector bundle of M with respect to $S(TM)$. Let ∇ be Levi-Civita connection on M. We have

$$\bar{\nabla}_X Y = \overset{*}{\nabla}_X Y + h(X, Y), \quad X, Y \in \Gamma(TM) \tag{22}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, X \in \Gamma(TM), V \in \Gamma(tr(TM)) \tag{23}$$

where $\overset{*}{\nabla}_X Y$, $A_V X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^t_X V \in \Gamma(tr(TM))$. ∇ is a symmetric linear connection on M which is called an induced linear connection, ∇^t is a linear connection on the vector bundle $tr(TM)$, h is a $\Gamma(tr(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V .

Locally, suppose $\{\xi, N\}$ is a pair of sections on $U \subset M$ in Theorem 8. Then define a symmetric $F(U)-$bilinear form B and a 1-form τ on U by

$$B(X, Y) = \bar{g}(h(X, Y), \xi), \forall X, Y \in (TM|_U)$$

and

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi)$$

Thus (22) and (23) locally become

$$\bar{\nabla}_X Y = \overset{*}{\nabla}_X Y + B(X, Y)N \tag{24}$$

and

$$\bar{\nabla}_X N = -A_N X + \tau(X)N \tag{25}$$

respectively.

Let denote P as the projection of TM on $S(TM)$. We consider the decomposition

$$\overset{*}{\nabla}_X PY = \nabla_X PY + C(X, PY)\xi \tag{26}$$

and

$$\overset{*}{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi \tag{27}$$

where $\nabla_X PY$, $A^*_\xi X$ belong to $S(TM)$ and C is a 1-form on U. Note that ∇ is not a metric connection (see [2]). We have the following equations

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0, \tag{28}$$

$$g(A^*_\xi X, PY) = B(X, PY), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{29}$$

for any $X, Y \in \Gamma(TM)$.

Now we will apply the above theory to the hypersurface $\bar{N}$.

Theorem 9 *If* $\left|\hat{\nabla} f\right|$ *is a constant real number on domain of f, then* $\bar{N}$ *is a lightlike hypersurface of TM.*

Proof. From (7) $\sigma^c = 0$, i.e. $\left|\hat{\nabla} f\right|$ is a constant, then we see that $Z = \xi^c$. Since $G^c(\xi^c, \xi^c) = 0$, the vector field Z is a lightlike vector field on $\bar{N}$. We recall that Z is a tangent vector field to $\bar{N}$.

On the other hand, for any $\bar{X} \in \Im^1_0(\bar{N})$,

$$\begin{aligned} \bar{G}(\bar{X}, Z) &= G^c(\bar{X}, Z) \\ &= G^c(\bar{X}, \xi^c) \\ &= G^c(\bar{X}, \frac{1}{\left|\hat{\nabla} f\right|}\hat{\nabla}^c f^c) \\ &= \frac{1}{\left|\hat{\nabla} f\right|} G^c(\bar{X}, \hat{\nabla}^c f^c) \\ &= 0 \end{aligned}$$

that is ξ^c is orthogonal to $\bar{N}$. Thus ξ^c is an element of $Rad(T\bar{N})$.

To describe a screen subspace of $T\bar{N}$, we must write the following decomposition from (20),

$$T_u\bar{N} = S(T_u\bar{N}) \perp Rad(T_u\bar{N}).$$

We also know that the restriction of G^c to $\Im_0^1(TF)$ has rank $2(n-1)$. It follows that, as a screen subspace of $T_u\bar{N}$ can be written following equality,

$$S(T_u\bar{N}) = T_u TS_t \qquad u \in TS_t \;\; t \in range(f).$$

From now on, as a screen subspace of $T\bar{N}$ we shall consider TF. In this case we get

$$\begin{aligned} \Im_0^1(\bar{N}) &= \Im_0^1(TF) \perp \Gamma(Rad(T\bar{N}) \\ &= Span\{X_1^c, ..., X_{n-1}^c, X_1^v, ..., X_{n-1}^v\} \perp Span\{\xi^c\}, \end{aligned}$$

where the set $\{X_1, ..., X_{n-1}\}$ is an orthonormal basis for $\Im_0^1(F)$.

On the other hand, from (21), we have the following decomposition for $\Im_0^1(TM)$,

$$\begin{aligned} \Im_0^1(TM)_{|_{\bar{N}}} &= \Gamma(S(T\bar{N})) \perp \Gamma(Rad(T\bar{N})) \oplus tr(T\bar{N})) \\ &= Span\{X_1^c, ..., X_{n-1}^c, X_1^v, ..., X_{n-1}^v, \xi^c\} \oplus tr(T\bar{N}). \end{aligned} \tag{30}$$

Using (1), we have the equalities:

$$G^c(\xi^v, \xi^v) = 0, \quad G^c(\xi^v, \xi^c) = 1$$

and

$$G^c(\xi^v, \bar{X}) = 0, \qquad \forall \bar{X} \in \Gamma(ST\bar{N} \mid_{\bar{U}}),$$

on a coordinate neighbourhood $\bar{U}$. Thus, from Theorem 8, the lightlike transversal bundle of $\bar{N}$ is as follows,

$$tr(T\bar{N} \mid_{\bar{U}}) = \bigcup_{u \in \bar{U}} Span\{\xi^v \mid_u\} \;,$$

with respect to $S(T\bar{N})$. By means of (20) and (21) for $\hat{X} \in \Im_0^1(TM)$ we can write the following decomposition

$$\hat{X} \mid_{\bar{U}} = \tilde{X} + \lambda\xi^c + \mu\xi^v, \qquad \tilde{X} \in \Im_0^1(TF), \quad \lambda, \mu \in \Im_0^0(\bar{N}),$$

on a neighbourhood $\bar{U}$.

7 The Induced Geometrical Objects

In this section we investigate the lightlike submanifold geometry of $\bar{N}$. By using (22) and (23), we get

$$\hat{\nabla}^c_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + \bar{h}(\bar{X}, \bar{Y}), \tag{31}$$

and

$$\hat{\nabla}^c_{\bar{X}} V = -A_V \bar{X} + \nabla^t_{\bar{X}} V \tag{32}$$

for any $\bar{X}, \bar{Y} \in \Im^1_0(\bar{N})$ and $V \in \Gamma(trT\bar{N})$. Here, $\bar{\nabla}$ and ∇^t are induced connections on $\bar{N}$ and $trT\bar{N}$ respectively, $\bar{h}$ and A_V are second fundamental form and shape operator of $\bar{N}$, respectively. The equalities (31) and (32) are the Gauss and Weingarten formulae, respectively [5].

Define a symmetric bilinear form $\bar{B}$ and a 1-form τ on $\bar{U} \subset \bar{N}$ by

$$\begin{array}{rcll} \bar{B}(\bar{X}, \bar{Y}) & = & G^c(\bar{h}(\bar{X}, \bar{Y}), \xi^c) & \forall \bar{X}, \bar{Y} \in \Im^1_0(\bar{N}). \\ \tau(\bar{X}) & = & G^c(\nabla^t_{\bar{X}} \xi^v, \xi^c) & \forall \bar{X} \in \Im^1_0(\bar{N}). \end{array}$$

It follows that

$$\bar{h}(\bar{X}, \bar{Y}) = \bar{B}(\bar{X}, \bar{Y}) \xi^v$$

and

$$\nabla^t_{\bar{X}} \xi^v = \tau(\bar{X}) \xi^v.$$

Hence, on $\bar{U}$, (24) and (25) become

$$\hat{\nabla}^c_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + \bar{B}(\bar{X}, \bar{Y}) \xi^v$$

and

$$\hat{\nabla}^c_{\bar{X}} \xi^v = -A_{\xi^v} \bar{X} + \tau(\bar{X}) \xi^v,$$

respectively.

On the other hand, if P denotes the projection of $\Im^1_0(\bar{N})$ to $\Im^1_0(TF)$ with respect to the decomposition $T_u\bar{N} = S(T_u\bar{N}) \perp Rad(T_u\bar{N})$, we obtain the local Gauss and Weingarten formulae on $S(T\bar{N})$

$$\begin{array}{rcll} \bar{\nabla}_{\bar{X}} P\bar{Y} & = & \tilde{\nabla}_{\bar{X}} P\bar{Y} + \tilde{C}(\bar{X}, P\bar{Y}) \xi^v & \\ \bar{\nabla}_{\bar{X}} \xi^c & = & -\tilde{A}_{\xi^c} \bar{X} - \tilde{\tau}(\bar{X}) \xi^c & \bar{X} \in \Im^1_0(\bar{N}), \quad \tilde{Y} \in \Im^1_0(TF) \end{array}$$

where $\tilde{C}$, $\tilde{A}_{\xi^c}$ and $\tilde{\nabla}$ are the local second form, the local shape operator and the linear connection on $S(T\bar{N})$.

Theorem 10 *The 1-form τ is identically zero.*

Proof. By virtue of (25), we have

$$\begin{aligned} \tau(\bar{X}) &= G^c(\hat{\nabla}^c_{\bar{X}}\xi^v + A_{\xi^v}\bar{X}, \xi^c) \\ &= G^c(\hat{\nabla}^c_{\bar{X}}\xi^v, \xi^c). \end{aligned} \tag{33}$$

On the other hand, for any $u \in \bar{N}$ and $\bar{X} \in \Im^1_0(\bar{N})$, from the decomposition (20), we can write,

$$\bar{X}_u = \tilde{X}_u + \lambda\xi^c_u, \qquad \lambda \in \mathbb{R},\ \tilde{X} \in \Im^1_0(TF),$$

and hence (33) becomes

$$\begin{aligned} \tau(\bar{X}_u) &= G^c(\hat{\nabla}^c_{\tilde{X}_u+\lambda\xi^c_u}\xi^v, \xi^c_u) \\ &= G^c(\hat{\nabla}^c_{\tilde{X}_u}\xi^v, \xi^c_u) + \lambda G^c(\hat{\nabla}^c_{\xi^c_u}\xi^v, \xi^c_u) \\ &= G^c(-H^v\tilde{X}_u, \xi^c_u) + \lambda(G(\hat{\nabla}_\xi\xi, \xi))^v_u \\ &= 0, \end{aligned}$$

where H is the shape operator of the foliation F. Recall from (14) that H^v is one of the shape operators of the tangent foliation of F in (TM, G^c) [10].

Corollary 5 *For any $\bar{X} \in \Im^1_0(\bar{N})$, the vector field $\hat{\nabla}^c_{\bar{X}}\xi^v$ is tangent to $\bar{N}$.*

Now, let us discuss the fundamental form of $\bar{N}$.

$$\begin{aligned} \bar{B}(\bar{X}, \bar{Y}) &= G^c(\bar{h}(\bar{X}, \bar{Y}), \xi^c) \\ &= G^c(\hat{\nabla}^c_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{X}}\bar{Y}, \xi^c) \\ &= G^c(\hat{\nabla}^c_{\bar{X}}\bar{Y}, \xi^c) \\ &= G^c(\hat{\nabla}^c_{\tilde{X}+\lambda\xi^c}\tilde{Y} + \mu\xi^c, \xi^c) \\ &= G^c(\hat{\nabla}^c_{\tilde{X}}\tilde{Y} + \tilde{X}[\mu]\xi^c + \mu\hat{\nabla}^c_{\tilde{X}}\xi^c + \lambda\hat{\nabla}^c_{\xi^c}\tilde{Y} \\ &\quad +\lambda\xi^c[\mu]\xi^c + \lambda\mu\hat{\nabla}^c_{\xi^c}\xi^c, \xi^c) \\ &= G^c(\hat{\nabla}^c_{\tilde{X}}\tilde{Y}, \xi^c) \\ &= G^c(\nabla^c_{\tilde{X}}\tilde{Y} + B^c(\tilde{X}, \tilde{Y})\xi^v + B^v(\tilde{X}, \tilde{Y})\xi^c, \xi^c) \\ &= B^c(\tilde{X}, \tilde{Y}) \\ &= B^c(P\bar{X}, P\bar{Y}), \end{aligned} \tag{34}$$

where B denotes the second fundamental form of F in M. From (34) we have following theorem.

Theorem 11 *If F is a totally geodesic foliation in M, then $\bar{N}$ is also a totally geodesic submanifold in TM.*

In addition, we get

$$\begin{aligned}
\hat{\nabla}^c_{\bar{X}} P\bar{Y} &= \bar{\nabla}_{\bar{X}} P\bar{Y} + \bar{h}(\bar{X}, \bar{Y}) \\
&= \tilde{\nabla}_{\bar{X}} P\bar{Y} + \tilde{h}(\bar{X}, P\bar{Y}) + \bar{h}(\bar{X}, \bar{Y}) \\
&= \tilde{\nabla}_{\bar{X}} P\bar{Y} + \tilde{C}(\bar{X}, P\bar{Y})\xi^c + \bar{B}(\bar{X}, \bar{Y})\xi^v \\
&= \tilde{\nabla}_{\tilde{X}+\lambda\xi^c} \tilde{Y} + \tilde{C}(\tilde{X} + \lambda\xi^c, \tilde{Y})\xi^c + \\
&\quad \bar{B}(\tilde{X} + \lambda\xi^c, \tilde{Y})\xi^v \\
&= \tilde{\nabla}_{\tilde{X}} \tilde{Y} + \lambda\tilde{\nabla}_{\xi^c} \tilde{Y} + [\tilde{C}(\tilde{X}, \tilde{Y}) + \lambda\tilde{C}(\xi^c, \tilde{Y})]\xi^c \\
&\quad +[\bar{B}(\tilde{X}, \tilde{Y}) + \lambda\bar{B}(\xi^c, \tilde{Y})]\xi^v \\
&= \tilde{\nabla}_{\tilde{X}} \tilde{Y} + \lambda\tilde{\nabla}_{\xi^c} \tilde{Y} + \tilde{C}(\tilde{X}, \tilde{Y})\xi^c + B^c(\tilde{X}, \tilde{Y})\xi^v.
\end{aligned} \tag{35a}$$

On the other hand, we can also write the following from

$$\begin{aligned}
\hat{\nabla}^c_{\bar{X}} P\bar{Y} &= \hat{\nabla}^c_{\bar{X}} \tilde{Y} \\
&= \hat{\nabla}^c_{\tilde{X}+\lambda\xi^c} \tilde{Y} \\
&= \hat{\nabla}^c_{\tilde{X}} \tilde{Y} + \lambda\hat{\nabla}^c_{\xi^c} \tilde{Y} \\
&= \nabla^c_{\tilde{X}} \tilde{Y} + B^v(\tilde{X}, \tilde{Y})\xi^c + B^c(\tilde{X}, \tilde{Y})\xi^v + \lambda\hat{\nabla}^c_{\xi^c} \tilde{Y}.
\end{aligned} \tag{36}$$

Thus by using (34), (35a) and (36) we can state the following corollary.

Corollary 6 *Assume that ∇ is the induced Riemannian connection on the foliation F,*
B is the second fundamental form of F,
P denotes the projection of $\Im^1_0(\bar{N})$ to $\Im^1_0(TF)$,
$\tilde{\nabla}$ is the linear connection on the screen bundle $S(T\bar{N}) = TF$ induced from $\hat{\nabla}$,
$\bar{B}$ is the second fundamental form of $\bar{N}$,
$\tilde{C}$ is the second fundamental form on $S(T\bar{N})$.
Then we have

i) $\nabla^c\tilde{X} \circ P = \tilde{\nabla}\bar{X}$,

ii) $B^v \circ (P \times P) = \tilde{C}$,

iii) $B^c \circ (P \times P) = \bar{B}$.

Example Let us consider an n-dimensional Euclidean space $\mathbb{E}^n$ with standard inner product G as a Riemannian metric and a function $f : \mathbb{E}^n \to \mathbb{R}$, $f(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x^i + b$, $\exists a_i \neq 0$, a_i, $b \in \mathbb{R}$, $1 \leq i \leq n$. It is easily seen that f is a submersion and the level hypersurfaces of f are $(n-1)$-planes of $\mathbb{E}^n$. In addition, the gradient vector field of f is given in the following form

$$\nabla f = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}.$$

Thus $|\nabla f|$ is a constant real number. A unit vector in direction of ∇f can be written as $\xi = \frac{\nabla f}{|\nabla f|}$. The complete lift of f is given by $f^c(x^i, y^i) = \sum_{i=1}^{n} a_i y^i$, where y^i's are cartesian coordinates on tangent bundle $T\mathbb{E}^n$. The level hypersurface of f^c at zero is given by

$$\bar{N} = \{(x^i, y^i) \mid \sum_{i=1}^{n} a_i y^i = 0\} = \mathbb{E}^n \times D^{n-1},$$

where D^{n-1} is an $(n-1)$-plane in $\mathbb{E}^n$. From Corollary 1, $\xi^c \in \Gamma(Rad(T\bar{N})$ and thus $\bar{N}$ is a lightlike hypersurface of $T\mathbb{E}^n$.

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