APPROXIMATION OF GENERALIZED HOMOMORPHISMS IN QUASI-BANACH ALGEBRAS

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Abstract

Let A be a quasi–Banach algebra with quasi-norm $\|.\|_A$ and B be a p-Banach algebra with p-norm $\|.\|_B$. A linear mapping $f : A \to B$ is called a generalized homomorphism if there exists a homomorphism $h' : A \to B$ such that $f(ab) = f(a)h'(b)$ for all $a, b \in A$.
In this paper, we investigate generalized homomorphisms on quasi–Banach algebras, associated with the following functional equation

$$rf(\frac{a+b}{r}) = f(a) + f(b).$$

Moreover, we prove the generalized Hyers–Ulam–Rassias stability and superstability of generalized homomorphisms in quasi–Banach algebras.

1 Introduction

We recall some basic facts concerning quasi–Banach spaces and some preliminary results.

Definition 1.1. ([4, 20]). Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:
(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda.x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

Key Words: Hyers–Ulam–Rassias stability; quasi–Banach algebra; p-Banach algebra; Homomorphism.
Mathematics Subject Classification: 46B03, 47Jxx, 47B48, 39B52.
Received: January, 2009
Accepted: September, 2009

The pair $(X, \|.\|)$ is called a quasi–normed space if $\|.\|$ is a quasi-norm on X . A quasi–Banach space is a complete quasi–normed space. A quasi-norm $\|.\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi–Banach space is called a p-Banach space.
Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [20] (see also [4]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.2. ([2]). Let $(A, \|.\|)$ be a quasi–normed space. The quasi–normed space $(A, \|.\|)$ is called a quasi–normed algebra if A is an algebra and there is a constant $K > 0$ such that

$$\|xy\| \leq K\|x\|.\|y\|$$

for all $x, y \in A$.

A quasi–Banach algebra is a complete quasi–normed algebra. If the quasi-norm $\|.\|$ is a p-norm then the quasi–Banach algebra is called a p-Banach algebra.
The stability problem of functional equations originated from a question of Ulam [21] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *, d)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality

$$d(h(x.y), h(x) * h(y)) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [13] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. Th. M. Rassias [19] succeeded in extending the result of Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(\|x\|^p + \|y\|^p)$, $p \in [0, 1)$ to be unbounded. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers–Ulam stability problem forms. A number of mathematicians were attracted to the pertinent stability results of Th. M. Rassias [19], and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Th. M. Rassias is called Hyers–Ulam–Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [6]-[11] and [16, 17, 18]).

Definition 1.3. A $\mathbb{C}$-linear mapping $h : A \to B$ is called a homomorphism in quasi–Banach algebras if $h(xy) = h(x)h(y)$ for all $x, y \in A$.

Definition 1.4. A $\mathbb{C}$-linear mapping $h : A \to B$ is called a generalized homomorphism in quasi–Banach algebras if there exists a homomorphism $h' : A \to B$ such that $h(xy) = h(x)h'(y)$ for all $x, y \in A$

For example, every homomorphism is a generalized homomorphism, but the converse is false, in general. For instance, let A be an algebra over $\mathbb{C}$ and let $h : A \to A$ be a non-zero homomorphism on A. Then, we have $ih(ab) = ih(a)h(b)$. This means that ih is a generalized homomorphism. It is easy to see that ih is not a homomorphism.
D. G. Bourgin [5] is the first mathematician dealing with stability of (ring) homomorphism $f(xy) = f(x)f(y)$. The topic of approximate homomorphisms was studied by a number of mathematicians, see [3],[7] ,[12] and [15], and references therein.

This paper is organized as follows: we prove the generalized Hyers–Ulam–Rassias stability and superstability of generalized homomorphisms in quasi–Banach algebras.

2 Main result

Throughout this paper, assume that A is a quasi–Banach algebra with quasi-norm $\|.\|_A$ and that B is a p-Banach algebra with p-norm $\|.\|_B$. In addition,

we assume r to be constant positive integer.
We will use the following Lemma in this section.

Lemma 2.1. *([14]). Let X and Y be linear spaces and let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} ; |\mu| = 1\}$. Then the mapping f is $\mathbb{C}$-linear.*

Now we prove the generalized Hyers–Ulam–Rassias stability of generalized homomorphisms in quasi–Banach algebras.

Theorem 2.2. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A \times A \times A \times A \to \mathbb{R}^+$ such that*

$$\|rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)\|_B \le \varphi(a, b, c, d), \tag{2.1}$$

$$\|g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)\|_B \le \varphi(a, b, c, d) \tag{2.2}$$

and

$$\tilde{\varphi}(a, b, c, d) := \sum_{i=0}^{\infty} \frac{\varphi(2^i a, 2^i b, 2^i c, 2^i d)}{2^i} < \infty \tag{2.3}$$

for all $a, b, c, d \in A$ and all $\mu \in T^1$. Then there exists a unique generalized homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \le \frac{1}{2}\tilde{\varphi}(a, a, 0, 0) \tag{2.4}$$

for all $a \in A$.

Proof. Putting $c = d = 0$ and $r = \mu = 1$ in (2.1), we get

$$\|f(a + b) - f(a) - f(b)\|_B \le \varphi(a, b, 0, 0) \tag{2.5}$$

for all $a, b \in A$. If we replace b in (2.5) by a and multiply both sides of (2.5) by $\frac{1}{2}$, we get

$$\|\frac{f(2a)}{2} - f(a)\|_B \le \frac{\varphi(a, a, 0, 0)}{2} \tag{2.6}$$

for all $a \in A$. Now we use the Rassias' method on inequality (2.6) ([8]). One can use induction on n to show that

$$\|\frac{f(2^n a)}{2^n} - f(a)\|_B \le \frac{1}{2}\sum_{i=0}^{n-1} \frac{\varphi(2^i a, 2^i a, 0, 0)}{2^i} \tag{2.7}$$

for all $a \in A$ and all non-negative integers n. Hence

$$\|\frac{f(2^{n+m}a)}{2^{n+m}}-\frac{f(2^m a)}{2^m}\|_B \leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{\varphi(2^{i+m}a,2^{i+m}a,0,0)}{2^{i+m}} = \frac{1}{2}\sum_{i=m}^{n+m-1}\frac{\varphi(2^i a,2^i a,0,0)}{2^i} \tag{2.8}$$

for all non-negative integers n and m with $n \geq m$ and all $a \in A$. It follows from the convergence (2.3) that the sequence $\{\frac{f(2^n a)}{2^n}\}$ is Cauchy. Due to the completeness of B, this sequence is convergent. Set

$$h(a) := \lim_{n\to\infty}\frac{f(2^n a)}{2^n}. \tag{2.9}$$

Putting $c = d = 0$, $r = 1$ and replacing a, b by $2^n a, 2^n b$, respectively, in (2.1) and multiply both sides of (2.1) by $\frac{1}{2^n}$, we get

$$\|\frac{f(2^n(\mu a+\mu b))}{2^n} - \mu\frac{f(2^n a)}{2^n} - \mu\frac{f(2^n b)}{2^n}\|_B \leq \frac{\varphi(2^n a, 2^n b, 0, 0)}{2^n} \tag{2.10}$$

for all $a, b \in A$, $\mu \in \mathbb{T}^1$ and all non-negative integers n. Taking the limit as $n \to \infty$ in (2.10), we obtain

$$h(\mu a + \mu b) = \mu h(a) + \mu h(b)$$

for all $a, b \in A$ and all $\mu \in \mathbb{T}^1$. By Lemma 2.3 the mapping h is $\mathbb{C}$- linear. Moreover, it follows from (2.7) and (2.9) that $\|h(a) - f(a)\|_B \leq \frac{1}{2}\tilde{\varphi}(a, a, 0, 0)$ for all $a \in A$. It is known that the additive mapping h satisfying (2.4) is unique [?]. Putting $r = \mu = 1$ and $a = b = 0$ in (2.1), we get

$$\|f(cd) - f(c)g(d)\|_B \leq \varphi(0, 0, c, d) \tag{2.11}$$

for all $c, d \in A$. If we replacing c and d in (2.11) by $2^n c$ and $2^n d$ respectively, and multiply both sides of (2.11) by $\frac{1}{2^{2n}}$, we get

$$\|\frac{f(2^{2n}cd)}{2^{2n}} - \frac{f(2^n c)}{2^n}\frac{g(2^n d)}{2^n}\|_B \leq \frac{\varphi(0,0,2^n c, 2^n d)}{2^{2n}}, \tag{2.12}$$

for all $c, d \in A$ and all non-negative integers n. By (2.9), it follows that $h(a) = \lim_{n\to\infty}\frac{f(2^n a)}{2^n}$ and by the convergence of series (2.3), $\lim_{n\to\infty}\frac{\varphi(0,0,2^n c,2^n d)}{2^{2n}} = 0$. Hence the sequence $\{\frac{g(2^n c)}{2^n}\}$ is convergent. Set $h'(d) := \lim_{n\to\infty}\frac{g(2^n d)}{2^n}$ for all $d \in A$. Let n tend to ∞ in (2.12). Then

$$h(cd) = h(c)h'(d) \tag{2.13}$$

for all $c, d \in A$. Next we claim that h' is a homomorphism. Putting $b = d = 1$ and replacing a, c in (2.2) $2^n a, 2^n c$ respectively, and multiply both sides of (2.2) by $\frac{1}{2^n}$, we get

$$\left\|\frac{g(2^n(\mu a + \mu c))}{2^n} - \mu\frac{g(2^n a)}{2^n} - \mu\frac{g(2^n c)}{2^n}\right\|_B \leq \frac{\varphi(2^n a, 1, 2^n c, 1)}{2^n} \tag{2.14}$$

for all $a, c \in A$ and all $\mu \in \mathbb{T}^1$. Let n tend to ∞ in (2.14). Then

$$h'(\mu a + \mu c) = \mu h'(a) + \mu h'(c)$$

for all $a, c \in A$ and all $\mu \in \mathbb{T}^1$. Hence by Lemma 2.3, h' is $\mathbb{C}$-linear. Now, letting $c = d = 0$, $\mu = 1$ in (2.2), we get

$$\|g(ab) - g(a)g(b)\|_B \leq \varphi(a, b, 0, 0) \tag{2.15}$$

for all $a, b \in A$. If we replacing a and b in (2.15) by $2^n a$ and $2^n b$ respectively, and multiply both sides of (2.15) by $\frac{1}{2^{2n}}$, we get

$$\left\|\frac{g(2^{2n}ab)}{2^{2n}} - \frac{g(2^n a)}{2^n}\frac{g(2^n b)}{2^n}\right\|_B \leq \frac{\varphi(2^n a, 2^n b, 0, 0)}{2^{2n}} \tag{2.16}$$

for all $a, b \in A$ and all non-negative integers n. Hence by letting $n \to \infty$ in (2.16), we conclude that $h'(ab) = h'(a)h'(b)$ for all $a, b \in A$. It then follows from (2.13) that h is a generalized homomorphism. $\square$

One can get easily the stability of Hyers–Ulam–Rassias by the following Corollary.

Corollary 2.3. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist constant $\epsilon > 0$, $p \neq 1$ and a map $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ such that*

$$\left\|rf\left(\frac{\mu a + \mu b + cd}{r}\right) - \mu f(a) - \mu f(b) - f(c)g(d)\right\|_B \leq \epsilon(\|a\|_A^p + \|b\|_A^p + \|c\|_A^p + \|d\|_A^p), \tag{2.17}$$

$$\|g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)\|_B \leq \epsilon(\|a\|_A^p + \|b\|_A^p + \|c\|_A^p + \|d\|_A^p) \tag{2.18}$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique generalized homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \leq \frac{2\,\epsilon}{|2^p - 2|}\|a\|_A^p \tag{2.19}$$

for all $a \in A$.

Proof. It follows from Theorem 2.2 by putting $\varphi(a,b,c,d) = \epsilon(\|a\|_A^p + \|b\|_A^p + \|c\|_A^p + \|d\|_A^p)$. $\square$

By Corollary 2.3, we solve the following Hyers–Ulam stability problem for generalized homomorphisms.

Corollary 2.4. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist constant $\delta > 0$ and a map $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ such that*

$$\|rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)\|_B \leq \delta, \tag{2.22}$$

$$\|g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)\|_B \leq \delta \tag{2.23}$$

for all $a,b,c,d \in A$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique generalized homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \leq \frac{\delta}{2} \tag{2.24}$$

for all $a \in A$.

Proof. Letting $p = 0$ and $\epsilon := \frac{\delta}{4}$ in Corollary 2.3, we obtain the result. $\square$

Theorem 2.5. *Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist a map $g : A \to B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A \times A \times A \times A \to \mathbb{R}^+$ such that satisfying the inequality* (2.1), (2.2) *and*

$$\tilde{\varphi}(a,b,c,d) := \sum_{i=1}^{\infty} 2^i \varphi(\frac{a}{2^i}, \frac{b}{2^i}, \frac{c}{2^i}, \frac{d}{2^i}) < \infty \tag{2.20}$$

for all $a,b,c,d \in A$ and all $\mu \in T^1$. Then there exists a unique generalized homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \leq \frac{1}{2}\tilde{\varphi}(a,a,0,0) \tag{2.21}$$

for all $a \in A$.

Proof. The proof is similar to the proof of Theorem 2.2. $\square$

Now we prove the superstability of the generalized homomorphisms as follows.

Theorem 2.6. *Let $p \neq 1$ and ϵ be constant positive integer. Suppose $f : A \to B$ is a mapping with $f(0) = 0$ for which there exists a map $g : A \to B$ with $g(0) = 0$ and $g(1) = 1$ such that*

$$\|rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)\|_B \leq \epsilon \|f(c)\|_B, \tag{2.25}$$

$$\|g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)\|_B \leq \epsilon(\|a\|_A^p + \|b\|_A^p + \|c\|_A^p + \|d\|_A^p) \tag{2.26}$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then $f : A \to B$ is a generalized homomorphism.

Proof. Assume $p < 1$. By putting $c = d = 0$ and $r = 1$ in (2.25), we get

$$\|f(\mu a + \mu b) - \mu f(a) - \mu f(b)\|_B \leq \epsilon \|f(0)\|_B = 0 \tag{2.27}$$

for all $a, b \in A$ and all $\mu \in \mathbb{T}^1$. Thus we have

$$f(\mu a + \mu b) = \mu f(a) + \mu f(b)$$

for all $a, b \in A$ and all $\mu \in \mathbb{T}^1$. By Lemma 2.3, the mapping f is $\mathbb{C}$- linear. Putting $a = b = 0$ and $r = \mu = 1$ in (2.25), we get

$$\|f(cd) - f(c)g(d)\|_B \leq \epsilon \|f(c)\|_B \tag{2.28}$$

for all $c, d \in A$. If we replacing c and d in (2.28) by $2^n c$ and $2^n d$, respectively, and multiply both sides of (2.28) by $\frac{1}{2^{2n}}$, we get

$$\|\frac{f(2^{2n}cd)}{2^{2n}} - \frac{f(2^n c)}{2^n}\frac{g(2^n d)}{2^n}\|_B \leq \frac{\epsilon}{2^{2n}}\|f(2^n c)\|_B,$$

for all $c, d \in A$ and all non-negative integers n. Hence

$$\|f(cd) - f(c)\frac{g(2^n d)}{2^n}\|_B \leq \frac{\epsilon}{2^n}\|f(c)\|_B \tag{2.29}$$

for all $c, d \in A$ and all non-negative integers n.
Let n tend to ∞ in (2.29). Then

$$f(cd) = f(c) \lim_{n\to\infty} \frac{g(2^n d)}{2^n}$$

for all $c, d \in A$. By Hyers' Theorem, the sequence $\{\frac{g(2^n d)}{2^n}\}$ is convergent. Set $h'(d) := \lim_{n\to\infty} \frac{g(2^n d)}{2^n}$ for all $d \in A$. Hence

$$f(cd) = f(c)h'(d) \tag{2.30}$$

for all $c, d \in A$.
Next we claim that h' is a homomorphism. Putting $b = d = 1$ and replacing a, c in (2.26) by $2^n a$, $2^n c$, respectively, and multiply both sides of (2.26) by $\frac{1}{2^n}$, we get

$$\|\frac{g(2^n(\mu a + \mu c))}{2^n} - \mu\frac{g(2^n a)}{2^n} - \mu\frac{g(2^n c)}{2^n}\|_B \leq \frac{\epsilon}{2^n}(\|2^n a\|_A^p + \|2^n\|_A^p + \|2^n c\|_A^p + \|2^n\|_A^p) \tag{2.31}$$

for all $a, c \in A$ and all $\mu \in \mathbb{T}^1$. Let n tend to ∞ in (2.31). Then

$$h'(\mu a + \mu c) = \mu h'(a) + \mu h'(c) \tag{2.32}$$

for all $a, c \in A$ and all $\mu \in \mathbb{T}^1$. Hence by Lemma 2.3, h' is $\mathbb{C}$-linear. Now, letting $a = b = 0$ and $\mu = 1$ in (2.26), we get

$$\|g(cd) - g(c)g(d)\|_A \leq \epsilon(\|c\|_A^p + \|d\|_A^p) \tag{2.33}$$

for all $c, d \in A$. If we replacing c and d in (2.33) by $2^n c$ and $2^n d$ respectively, and multiply both sides of (2.33) by $\frac{1}{2^{2n}}$, we get

$$\|\frac{g(2^{2n}cd)}{2^{2n}} - \frac{g(2^n c)}{2^n}\frac{g(2^n d)}{2^n}\|_B \leq \frac{\epsilon}{2^{2n}}(\|2^n c\|_A^p + \|2^n d\|_A^p) \tag{2.34}$$

for all $c, d \in A$ and all non-negative integer n. Hence by letting $n \to \infty$ in (2.34), we conclude that $h'(cd) = h'(c)h'(d)$ for all $c, d \in A$. It then follows from (2.30) that f is a generalized homomorphism. Similarly, one can show the result for the case $p > 1$. □

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