An. Şt. Univ. Ovidius Constanţa Vol. 17(1), 2009, 137–144

An Extension of the Szász-Mirakjan Operators

C. MORTICI

Abstract

The paper is devoted to defining a new class of linear and positive operators depending on a certain function φ These operators generalize the Szász-Mirakjan operators (case in which φ is the exponential function). Furthermore, conditions when these operators have properties of monotony and convexity are given.

1 Introduction

One of the main purpose of the approximation theory is to find how functions can be approximated by simpler functions. A direction is to use the linear, positive operators and consequently, a large number of authors have established new properties of them. We discuss here the Szász-Mirakjan operators

$$S_n : C^2([0,\infty)) \to C^\infty([0,\infty)) \quad , \quad n \in \mathbb{N},$$

given by the law

$$(S_n f)(x) = \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \quad , \quad f \in C^2([0,\infty)).$$

Key Words: Szasz-Mirakjan operators; Popoviciu-Bohman-Korovkin theorem; Positivity; Linearity; Convexity.
Mathematics Subject Classification: 41A36.
Received: October, 2008
Accepted: April, 2009

Obviously, these operators are linear and if $f \geq 0$, then $S_n f \geq 0$, so they are also positive. The Szász-Mirakjan operators was defined for the first time by Otto Szász in the paper [9], where the original notation was

$$P(u,x) = \mathrm{e}^{-xu} \sum_{v=1}^{\infty} \frac{(ux)^v}{v!} f\left(\frac{v}{u}\right) \quad , \quad u > 0.$$

These operators was also discussed in the paper [3], from a different point of view, while in the paper [10] the convergence of $P(x,u)$ to $f(x)$ as $u \to \infty$ was established. This fact was considered a generalization for the interval $0 \leq x \leq \infty$ of the well-known properties of S. Bernstein's approximation polynomials in a finite interval, established in 1912.

The Szász-Mirakjan operators play a central role in the theory of approximation, so they are intensively studied. For various extensions and further properties and proofs, see for example [1], [5], [6], [7], [11]. Recently, one direction for study more general versions of the Szász-Mirakjan operators was given in [8], where a sequence of positive real numbers $(\alpha_n)_{n \geq 0}$ was considered to define the operators

$$(S_n^{\alpha} f)(x) = \mathrm{e}^{-\frac{nx}{\alpha_n}} \sum_{k=0}^{\infty} \left(\frac{nx}{\alpha_n}\right)^k \frac{1}{k!} f\left(\frac{k}{n}\right) \quad , \quad f \in C([0,\infty)).$$

In case $\alpha_n = 1$, the classical Szász-Mirakjan operators are obtained. Our idea is to consider an analytic function $\varphi : \mathbb{R} \to [0,\infty)$ and to define the operators

$$\varphi S_n : C^2([0,\infty)) \to C^{\infty}([0,\infty)) \quad , \quad n \in \mathbb{N},$$

given by the formula

$$(\varphi S_n f)(x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k f\left(\frac{k}{n}\right) \quad , \quad f \in C^2([0,\infty)). \qquad (1)$$

We will call (φS_n) the φ-Szász-Mirakjan operators. In case $\varphi(y) = \mathrm{e}^y$, the classical Szász-Mirakjan operators are obtained.

2 The Main Result

We remind first the following basic theorem, also called Popoviciu-Bohman-Korovkin theorem. This result was first published by the Romanian mathematician Tiberiu Popoviciu in [9] - unfortunately a local journal which was not so known in the mathematics world of that time. After this, the result was found independently by the Danish mathematician H. Bohman in [2], while the result was clear published by the Russian mathematician P.P. Korovkin in his book [4]. Denote by

$$e_0(x) = 1 \quad , \quad e_1(x) = x \quad , \quad e_2(x) = x^2$$

the test functions. The result we are talking about is the following:
Theorem 2.1. *Let $L_n : C([a,b]) \to C([a,b])$, $n \in \mathbb{N}$ be a sequence of linear, positive operators such that*

$$\lim_{n\to\infty} (L_n e_j)(x) = e_j(x) \quad , \quad j = 0, 1, 2, \tag{2}$$

uniformly on $[a,b]$. Then for every $f \in C([a,b])$,

$$\lim_{n\to\infty} (L_n f)(x) = f(x),$$

uniformly on $[a,b]$.

Now, in order to establish the approximations properties of the new defined operators $(\varphi S_n)_{n\in\mathbb{N}}$, we give the following
Lemma 2.2. *The φ-Szász-Mirakjan operators satisfy the following relations:*
a) $(\varphi S_n e_0)(x) = e_0(x)$
b) $(\varphi S_n e_1)(x) = \dfrac{\varphi'(nx)}{\varphi(nx)} \cdot x$
c) $(\varphi S_n e_2)(x) = \dfrac{\varphi''(nx)}{\varphi(nx)} \cdot x^2 + \dfrac{1}{n} \cdot \dfrac{\varphi'(nx)}{\varphi(nx)} \cdot x.$
Proof. From the fact that the function φ is analytic, it results that

$$\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \cdot y^k = \varphi(y),$$

then by derivation,

$$\varphi'(y) = \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} \cdot y^{k-1} \qquad \text{and} \qquad \varphi''(y) = \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} \cdot y^{k-2}.$$

a) We have

$$(\varphi S_n e_0)(x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k = \frac{1}{\varphi(nx)} \cdot \varphi(nx) = 1.$$

b) We have

$$(\varphi S_n e_1)(x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \cdot \frac{k}{n} =$$

$$= \frac{x}{\varphi(nx)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!}(nx)^{k-1} = \frac{\varphi'(nx)}{\varphi(nx)} \cdot x.$$

c) We have

$$(\varphi S_n e_2)(x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(nx)^k \cdot \left(\frac{k}{n}\right)^2 =$$

$$= \frac{1}{n^2\varphi(nx)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(nx)^k \cdot [k(k-1)+k] =$$

$$= \frac{1}{n^2\varphi(nx)} \left[(nx)^2 \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!}(nx)^{k-2} + nx \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!}(nx)^{k-1}\right] =$$

$$= \frac{1}{n^2\varphi(nx)} \left[(nx)^2\varphi''(nx) + nx\varphi'(x)\right] = \frac{\varphi''(x)}{\varphi(x)} \cdot x^2 + \frac{1}{n} \cdot \frac{\varphi'(x)}{\varphi(x)} \cdot x. \square$$

From this lemma, it results the following

Theorem 2.3. *Let $\varphi : \mathbb{R} \to (0, \infty)$ be such that*

$$\lim_{y\to\infty} \frac{\varphi'(y)}{\varphi(y)} = 1 \quad \textit{and} \quad \lim_{y\to\infty} \frac{\varphi''(y)}{\varphi(y)} = 1. \tag{3}$$

Then the φ-Szász-Mirakjan operators satisfy

$$\lim_{n\to\infty} (\varphi S_n)e_j = e_j \quad , \quad j = 0, 1, 2,$$

uniformly on compact intervals and according with the Theorem 2.1, for every $f \in C([a, b])$, it holds: $\lim_{n\to\infty} (\varphi S_n)f = f$, uniformly on $[a, b]$.

Proof. From the Lemma 2.2 and from the hypotesis (3), it results

$$\lim_{n\to\infty} (\varphi S_n)e_1(x) = \lim_{n\to\infty} \left(\frac{\varphi'(nx)}{\varphi(nx)} \cdot x\right) = x$$

and

$$\lim_{n\to\infty} (\varphi S_n)e_2(x) = \lim_{n\to\infty} \left(\frac{\varphi''(nx)}{\varphi(nx)} \cdot x^2 + \frac{1}{n} \cdot \frac{\varphi'(nx)}{\varphi(nx)} \cdot x\right) = x^2. \square$$

Remark that the exponential function $\varphi(y) = \mathrm{e}^y$ satisfies the hypotesis (3), so we have obtained the results from [8] in a general background. Furthermore, note that our extension is consistent, if we take into account that there is a large class of functions with the property (3), for example

$$\varphi(y) = y^i \, \mathrm{e}^y \quad , \quad i \in \mathbb{N},$$

or more general, the functions of the form $\varphi(y) = P(y)\,e^y$, where P is any polynomial function with non-negative coefficients. Other interesing particular φ-Szász-Mirakjan operators can be obtained. In general, by the product differentiation formula, we have

$$\varphi^{(k)}(y) = e^y \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(y),$$

so

$$\varphi^{(k)}(0) = \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(0).$$

Now, by replacing in (1), we obtain the following class of operators:

$$(PS_n f)(x) = \frac{1}{P(nx)\,e^{nx}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=0}^{k} \binom{k}{i} P^{(i)}(0) \right) (nx)^k f\left(\frac{k}{n}\right),$$

where P is any polynomial function with non-negative coefficients.

Other ideas for extensions is to consider certain function P in the previous relation, not necessary a polynomial function.

3 Hereditary Properties

According with the usual procedures, we will study in this section the hereditary properties (monotony and convexity) of the φ-Szász-Mirakjan operators.

Theorem 3.1. *Assume that the analytic function $\varphi : [a,b] \to (0,\infty)$ with $\varphi^{(k)}(0) \geq 0$, for all integers $k \geq 0$, satisfies*

$$\sup_{y\in[a,b]} \frac{y\varphi'(y)}{\varphi(y)} \leq 1. \tag{4}$$

Then if f is positive, then $(\varphi S_n f)$ is also positive and increasing on $\left[\frac{a}{n}, \frac{b}{n}\right]$.

Proof. For $\frac{a}{n} \leq x_2 \leq x_1 \leq \frac{b}{n}$, we have

$$(\varphi S_n) f(x_1) - (\varphi S_n) f(x_2) =$$

$$= \frac{1}{\varphi(nx_1)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx_1)^k f\left(\frac{k}{n}\right) - \frac{1}{\varphi(nx_1)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx_1)^k f\left(\frac{k}{n}\right) =$$

$$= \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} f\left(\frac{k}{n}\right) \left[\frac{(nx_1)^k}{\varphi(nx_1)} - \frac{(nx_2)^k}{\varphi(nx_2)} \right] \geq 0,$$

because the function $y \mapsto \dfrac{y}{\varphi(y)}$ is increasing on $[a,b]$ and moreover,

$$y \mapsto \frac{y^k}{\varphi(y)}$$

with $k \geq 2$ is also increasing.□

Theorem 3.2. *Assume that the analytical function $\varphi : \mathbb{R} \to (0,\infty)$ with $\varphi^{(k)}(0) \geq 0$, for all integers $k \geq 0$ is such that $y \mapsto y/\varphi(y)$ is convex and increasing. Then if f is positive, then $(\varphi S_n f)$ is convex.*

Proof. The functions $y \mapsto y^{k-1}$ and $y \mapsto y/\varphi(y)$ are convex and increasing, so their product $y \mapsto y^k/\varphi(y)$ is also convex. For x, $y > 0$, we have:

$$(\varphi S_n f)\left(\frac{x+y}{2}\right) = \frac{1}{\varphi\left(\frac{x+y}{2}\right)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \left(\frac{x+y}{2}\right)^k f\left(\frac{k}{n}\right) =$$

$$= \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} f\left(\frac{k}{n}\right) \frac{\left(\frac{x+y}{2}\right)^k}{\varphi\left(\frac{x+y}{2}\right)} \leq$$

$$\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} f\left(\frac{k}{n}\right) \left[\frac{x^k}{\varphi(x)} + \frac{y^k}{\varphi(y)}\right] =$$

$$= \frac{1}{2} \cdot \frac{1}{\varphi(x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} x^k f\left(\frac{k}{n}\right) + \frac{1}{2} \cdot \frac{1}{\varphi(y)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} y^k f\left(\frac{k}{n}\right) =$$

$$= \frac{(\varphi S_n f)(x) + (\varphi S_n f)(y)}{2}.$$

We obtained that

$$(\varphi S_n f)\left(\frac{x+y}{2}\right) \leq \frac{(\varphi S_n f)(x) + (\varphi S_n f)(y)}{2}$$

and by continuity arguments, $(\varphi S_n f)$ is convex.□

References

[1] M. Becker, *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J., **27**(1978), no. 1, 127–142.

[2] H. Bohman, *On approximation of continuous and of analytic functions*, Ask. Mat. (2), **3**(1951), 43-51.

[3] M. Kac, *Une remarque sur les polynomes de M. S. Bernstein,* Studia Math., **7**(1938), 49-51.

[4] P. P. Korovkin, *Linear Operators and Approximation Theory*, Delhi, 1960.

[5] H. G. Lehnhoff, *On a Modified Szász-Mirakjan Operator*, J. Approx. Theory, **42**(1984), 278–282.

[6] M. Leśniewicz, L. Rempulska, *Approximation by some operators of the Szász-Mirakjan type in exponential weight spaces*, Glas. Math. Ser. III, **32**(1997), 57–69.

[7] C. Mortici, I. Oancea, *A nonsmooth extension for the Bernstein-Stancu operators and an application,* Studia Math., **2**(2003), 69-82.

[8] C. M. Muraru, *On the sequence of Kantorovich type operators,* Int. J. Pure Appl. Math., **45**(2008), no. 3, 439-444.

[9] T. Popoviciu, *Asupra demonstraţiei teoremei lui Weierstrass cu ajutorul polinoamelor de interpolare*, Lucrările Sesiunii Generale Ştiinţifice Acad. R.P.R. (1950), 1664 - 1667.

[10] O. Szász, *Generalization of S. Bernstein's polynomials to the infinite interval,* J. of Research of the National Bureau of Standards, **45**(1950), no. 3, 239-244.

[11] Z. Walczak, *On the rate of convergence for some linear operators,* Hiroshima Math. J., **35**(2005), 115-124.

Department of Mathematics
Valahia University of Târgovişte,
Bd. Unirii 18, 130082 Târgovişte
ROMANIA
e-mail: cmortici@valahia.ro