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Existence and non-existence results for elliptic exterior problems with nonlinear boundary conditions

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Abstract

Existence and non-existence results are established for elliptic problems with nonlinear boundary conditions and lack of compactness. The proofs combine variational methods with the geometrical feature, due to the competition between the different growths of the non-linearities. Our paper completes previous results obtained by R. Filippucci, P. Pucci and V. Rădulescu in 2008.

1 Introduction and the main results

This paper is motivated by recent advances in elastic mechanics and electrorheological fluids (sometimes referred to as "smart fluids") where some processes are modeled by nonhomogeneous quasilinear operators (see Acerbi and Mingione [1], Diening [7], Halsey [9], Ruzicka [16], Zhikov [21, 22], and the references therein). We refer mainly to the $p(x)$-Laplace operator $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$, where p is a continuous non-constant function. This differential operator is a natural generalization of the p- Laplace operator $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearities that the p-Laplace operator,

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due to the fact that $\Delta_{p(x)}$ is not homogeneous. Recent qualitative properties of solutions to quasilinear problems in Sobolev spaces with variable exponent have been obtained by Alves and Souto [2], Chabrowski and Fu [5], Mihăilescu and Rădulescu [10] and Rădulescu [14].

Let Ω be a smooth exterior domain in $\mathbf{R}^N$, that is, Ω is the complement of a bounded domain with $C^{1,\delta}$ boundary $(0 < \delta < 1)$. Assume that p is a real number satisfying $1 < p < N$, $a \in L^\infty(\Omega) \bigcap C^{0,\delta}(\overline{\Omega})$ is a positive function, and $b \in L^\infty(\Omega) \bigcap C(\Omega)$ is non-negative. Let $p^* := Np/(N-p)$ denote the critical Sobolev exponent. In Yu [20] it is studied the following quasilinear problem

$$\begin{cases} -div(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = g(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \quad \partial\Omega, \\ lim_{|x|\to\infty} u(x) = 0, \end{cases} \qquad (P)$$

where $p < r < p^*$ and $g \in L^\infty(\Omega) \bigcap L^{p_0}(\Omega)$, with $p_0 := p^*/(p^* - r)$, is a non-trivial potential which is positive on some non-empty open subset of Ω. Under these assumptions, Yu proved in [20] that problem (P) has a weak positive solution u of class $C^{1,\alpha}(\overline{\Omega} \bigcap B_R(0))$ for any $R > 0$ and some $\alpha = \alpha(R) \in (0,1)$. Problems of this type are motivated by mathematical physics (see Reed and Simon [15] and Strauss [19]), where certain stationary waves in nonlinear Klein-Gordon or Schrödinger equations can be reduced to this form.

Actually, a weak solution of (P) satisfies for all $\varphi \in E$ the identity

$$\int_\Omega (a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + b(x)|u|^{p-2}u\varphi)dx = \int_\Omega g(x)|u|^{r-2}u\varphi dx, \qquad (1)$$

where E is the completion of C_0^∞ under the underlying norm

$$||u||_{a,b} = \left(\int_\Omega [a(x)|\nabla u|^p + b(x)|u|^p]dx\right)^{1/p}.$$

By Lemma 2 of [20] every weak solution u of (P) is in $L^q(\Omega)$ for every $q \in [p^*, \infty)$ and approaches 0 as $|x| \to \infty$. Of course $E \sim H_0^{1,p}(\Omega)$ whenever $0 < b_0 \le b(x) \in L^\infty(\Omega)$. Taking $\varphi = u$ in (1) we get $||u||_{a,b}^p = ||u||_{L^r(\Omega;g)}^r$, so that (P) does not admit nontrivial weak solutions whenever $g \le 0$ a.e. in Ω.

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}), h(x) > 1 \quad for \quad all \quad x \in \overline{\Omega}\}.$$

For $h \in C_+(\overline{\Omega})$, let

$$h^- = ess \inf_{x\in\Omega} h(x), \quad h^+ = ess \sup_{x\in\Omega} h(x)$$

With the same hypotheses on Ω, a, g, and r, we consider the problem

$$\begin{cases} -div(a(x)|\nabla u|^{p(x)-2}\nabla u) + |u|^{q-2}u = \lambda g(x)|u|^{r-2}u & in \quad \Omega \\ a(x)|\nabla u|^{p(x)-2}\partial_\nu u + b(x)|u|^{p(x)-2}u = 0 & on \quad \partial\Omega \end{cases} \tag{2}$$

where $p \in C_+(\overline{\Omega})$, λ is a real parameter and ν is the unit vector of the outward normal on $\partial\Omega$. More precisely, we first assume

$(H1)$ $g \in L^\infty(\Omega) \bigcap L^{p_0}(\Omega)$, with $p_0 := p^*/(p^* - r)$, $p^+ < r < q < p^*$, is a non-negative function which is positive on a non-empty open subset of Ω, where $p^* := Np^+/(N - p^+)$;

$(H2)$ b is a continuous positive function on $\Gamma = \partial\Omega$.

By a weak (non-trivial) solution of problem (2) we mean a non-trivial function $u \in X = E \bigcap L^q(\Omega)$ verifying for all $\varphi \in X$ the identity

$$\int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi dx + \int_\Gamma b(x)|u|^{p(x)-2}u\varphi d\sigma \tag{3}$$

$$+ \int_\Omega |u|^{q-2}u\varphi dx = \lambda \int_\Omega g(x)|u|^{r-2}u\varphi dx,$$

where now E is the completion of the restriction on Ω of functions of $C_0^\infty(\mathbf{R}^N)$ with respect to the norm

$$||u||_{a,b} = \left(\int_\Omega a(x)|\nabla u|^{p^+} dx + \int_\Gamma b(x)|u|^{p^+} d\sigma \right)^{1/p^+},$$

and X is the reflexive Banach space endowed with the norm

$$||u|| = \left\{ ||u||_{a,b}^{p^+} + ||u||_{L^q(\Omega)}^{p^+} \right\}^{1/p^+}.$$

Hence, by $(H1) - (H2)$, all the integrals in (3) are well defined and converge.

The loss of compactness of the Sobolev imbeddings on unbounded domains renders variational techniques more delicate. Some of the papers treating problems on unbounded domains use special function spaces where the compactness is preserved, such as spaces of radially symmetric functions. We point out that even if Ω is unbounded, standard compact imbeddings still remain true, e.g., if Ω is thin at infinity, in the sense that

$$\lim_{R\to\infty} \sup\{\mu(\Omega \cap B(x, 1)) : x \in \mathbf{R}^N, |x| = R\} = 0,$$

where μ denotes the Lebesgue measure and $B(x, 1)$ is the unit ball centered at x. Such arguments cannot be applied to our general unbounded domain Ω. In this case, since Ω is not "thin" and it looks like $\mathbf{R}^N$ at infinity (because

Ω is an exterior domain), the analysis of the compactness failure shows that a Palais-Smale sequence of the associated energy functional (see Bahri and Lions [4]) differs from its weak limit by "waves" that go to infinity. However, the definition of X, combined with the main assumption $p^+ < r < p^*$, ensures that

$(H3)$ the function space X is compactly embedded into the weighted Lebesgue space $L^r(\Omega; g)$.

Taking $\varphi = u$ in (3), we have that any weak solution u of (2) satisfies the equality

$$\int_\Omega a(x)|\nabla u|^{p(x)}dx + \int_\Gamma b(x)|u|^{p(x)}d\sigma + ||u||^q_{L^q(\Omega)} = \lambda ||u||^r_{L^r(\Omega;g)}, \tag{4}$$

so that problem (2) does not have any nontrivial solution whenever $\lambda \le 0$. We first prove that the result still remains true for sufficiently small values of $\lambda > 0$ when $p^+ < r < q < p^*$, that is, the term $|u|^{q-2}u$ "dominates" the right hand-side and makes impossible the existence of a solution to our problem (2). On the other hand, if $\lambda > 0$ is sufficiently large, then (2) admits weak solutions. The precise statement of this result is the following.

Theorem 1.1 *(The case $p^+ < r < q < p^*$). Under the assumptions $(H1)$ and $(H2)$ there exists $\lambda^* > 0$ such that*

(i) if $\lambda < \lambda^*$, then problem (2) does not have any weak solution;

(ii) if $\lambda \ge \lambda^*$, then problem (2) has at least one weak solution u, with the properties

(a) $u \in L^\infty_{loc}(\Omega)$;

(b) $u \in C^{1,\alpha}(\Omega \cap B_R)$, $\alpha = \alpha(R) \in (0, 1)$;

(c) $u > 0$ in Ω;

(d) $u \in L^m(\Omega)$ for all $p^* \le m < \infty$ and $\lim_{|x|\to\infty} u(x) = 0$.

In the second part of the paper we consider condition $(H1)'$, which is exactly assumption $(H1)$, with the only exception that condition $p^+ < r < q < p^*$ is replaced by

$$p^+ < q < r < p^*.$$

Theorem 1.2 *Under the assumptions $(H1)'$ and $(H2)$*

(i) problem (2) does not have any weak solution for any $\lambda \le 0$;

(ii) problem (2) has at least one weak solution u, with the properties $(a)-(d)$ of Theorem 1.1 for all $\lambda > 0$.

2 Proof of Theorem 1.1

We point out in what follows the main ideas of the proof:

(a) There is some $\lambda^* > 0$ such that problem (2) does not have any solution for any $\lambda < \lambda^*$. This means that if a solution exists then λ must be sufficiently large. One of the key arguments in this proof is based on the assumption $q > r$. In particular, this proof yields an energy lower bound of solutions in term of λ which will be useful to conclude that problem (2) has a non-trivial solution if $\lambda = \lambda^*$.

(b) There exists $\lambda^{**} > 0$ such that problem (2) has at least one solution for any $\lambda > \lambda^{**}$. Next, by the properties of λ^* and λ^{**} we deduce that $\lambda^{**} = \lambda^*$. The proof uses variational arguments and is based on the coercivity of the corresponding energy functional defined on X by

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} a(x)|\nabla u|^{p(x)} dx$$

$$+ \int_\Gamma \frac{1}{p(x)} b(x)|u|^{p(x)} d\sigma + \frac{1}{q}||u||^q_{L^q(\Omega)} - \frac{\lambda}{r}||u||^r_{L^r(\Omega;g)}.$$

We show that the minimum of J is achieved by a weak solution of (2). In order to obtain that this global minimizer is not trivial, we prove that the corresponding energy level is negative provided λ is sufficiently large.

Step 1. Non-existence for $\lambda > 0$ small. It is enough to swow that, if there is a weak solution of problem (2), then λ must be sufficiently large. Assume that u is a weak solution of (2), then by (3) we get (4). Since $r < q$ and $g^{q/(q-r)}$ is in $L^1(\Omega)$ by $(H1)$, applying the Young inequality we deduce that

$$\lambda||u||^r_{L^r(\Omega;g)} \le \frac{(q-r)\lambda^{q/(q-r)}}{q} \int_\Omega g(x)^{q/(q-r)} dx + \frac{r}{q}||u||^q_{L^q(\Omega)}. \tag{5}$$

Next, by (4), (5) and the fact that u is non-trivial,

$$\begin{aligned} 0 < ||u||^{p^+}_{a,b} &\le \frac{q-r}{q}\lambda^{q/(q-r)} \int_\Omega g(x)^{q/(q-r)} dx + \frac{r-q}{q}||u||^q_{L^q(\Omega)} \\ &\le \frac{q-r}{q}\lambda^{q/(q-r)} \int_\Omega g(x)^{q/(q-r)} dx := \lambda^{q/(q-r)} A < \infty. \end{aligned} \tag{6}$$

The continuity of the imbedding $X \hookrightarrow L^r(\Omega; g)$ implies that there exists $C = C(\Omega, g, p^+, q, r) > 0$ such that

$$C||v||^{p^+}_{L^r(\Omega;g)} \le ||v||^{p^+}_{a,b} \tag{7}$$

for any $v \in X$. Thus, by (4) and (7), we have $C||u||^{p^+}_{L^r(\Omega;g)} \le \lambda||u||^r_{L^r(\Omega;g)}$. Since $p^+ < r < q$, $\lambda > 0$ and $||u||_{L^r(\Omega;g)} > 0$ by (4), we deduce that

$$\lambda \ge C||u||^{p^+-r}_{L^r(\Omega;g)} \ge CC^{-1+r/p^+}||u||^{p^+-r}_{a,b} \ge C^{r/p^+}\lambda^{q(p^+-r)/p^+(q-r)} A^{(p^+-r)/p^+}.$$

It follows that $\lambda \geq (A^{p^+ - r}C^r)^{(q-r)/r(q-p^+)}$, which also implies that $\lambda^* \leq (A^{p^+ - r}C^r)^{(q-r)/r(q-p^+)}$. This concludes the proof of (i).

In particular, Step 1 shows that if for some $\lambda > 0$ problem (2) has a weak solution u, then

$$(C^r/\lambda^{p^+})^{1/(r-p^+)} \leq ||u||_{a,b}^{p^+} \leq \lambda^{q/(q-r)}A, \tag{8}$$

where $C = C(\Omega, g, p^+, q, r) > 0$ is the constant given in (7).

Step 2. Coercivity of J. It follows by $(H1)$. Indeed, for any $u \in X$ and all $\lambda > 0$

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} a(x)|\nabla u|^{p(x)} dx + \int_\Gamma \frac{1}{p(x)} b(x)|u|^{p(x)} d\sigma$$
$$+\frac{1}{2q}||u||_{L^q(\Omega)}^q + \frac{1}{2q}||u||_{L^q(\Omega)}^q - \frac{\lambda}{r}||u||_{L^r(\Omega;g)}^r.$$

By Hölder inequality and $(H1)$ we have

$$J_\lambda(u) \geq \frac{1}{p^+}||u||_{a,b}^{p^+} + \frac{1}{2q}||u||_{L^q(\Omega)}^q + \frac{1}{2q}||u||_{L^q(\Omega)}^q - \frac{\lambda}{r}||g||_{L^{q/(q-r)}(\Omega)}||u||_{L^q(\Omega)}^r. \tag{9}$$

Now, since for any positive numbers α, β, q and r, with $r < q$, the function $\Phi : \mathbf{R}_+ \to \mathbf{R}$ defined by $\Phi(t) = \alpha t^r - \beta t^q$, achieves its positive global maximum

$$\Phi(t_0) = \frac{q-r}{q}\left(\frac{r}{q}\right)^{r/(q-r)} \alpha^{q/(q-r)}\beta^{r/(r-q)} > 0$$

at $t_0 = (\alpha r/\beta q)^{1/(q-r)} > 0$, we have $\alpha t^r - \beta t^q \leq C(q,r)\alpha^{q/(q-r)}\beta^{r/(r-q)}$, where $C(q,r) = (q-r)(r^r/q^q)^{1/(q-r)}$. Returning to (9) and using the above inequality, with $\alpha = \lambda||g||_{L^{q/(q-r)}(\Omega)}/r$, $\beta = 1/2q$ and $t = ||u||_{L^q(\Omega)}$, we deduce that

$$J_\lambda(u) \geq \frac{1}{p^+}||u||_{a,b}^{p^+} + \frac{1}{2q}||u||_{L^q(\Omega)}^q - C(\lambda, q, r, g),$$

where $C(\lambda, q, r, g) = 2^{r/(q-r)}(q-r)\left(\lambda||g||_{L^{q/(q-r)}(\Omega)}\right)^{q/(q-r)}/qr$. This implies the claim.

Let $n \to u_n$ be a minimizing sequence of J_λ in X, which is bounded in X by Step 2. Without loss of generality, we may assume that $(u_n)_n$ is non-negative, converges weakly to some u in X and converges also pointwise.

Step 3. The non-negative weak limit $u \in X$ is a weak solution of (2). To prove this, we shall show that

$$J_\lambda(u) \leq \lim_{n\to\infty} \inf J_\lambda(u_n).$$

By the weak lower semicontinuity of the norm $||\cdot||$ we have

$$\frac{1}{p^+}||u||^{p^+}_{a,b} + \frac{1}{q}||u||^q_{L^q(\Omega)} \le \lim_{n\to\infty} \inf\left(\frac{1}{p^+}||u_n||^{p^+}_{a,b} + \frac{1}{q}||u_n||^q_{L^q(\Omega)}\right).$$

Next, the boundedness of $(u_n)_n$ in X implies with the same argument that

$$||u||_{L^r(\Omega;g)} = \lim_{n\to\infty} ||u_n||_{L^r(\Omega;g)}$$

by $(H3)$. Hence u is a global minimizer of J_λ in X.

Step 4. The weak limit u is a non-negative weak solution of (2) if $\lambda > 0$ is sufficiently large. Clearly $J_\lambda(0) = 0$. Thus, by Step 3 it is enough to show that there exists $\Lambda > 0$ such that

$$\inf_{u\in X} J_\lambda(u) < 0 \quad for \quad all \quad \lambda > \Lambda.$$

Consider the constrained minimization problem

$$\Lambda := \inf\left\{\frac{1}{p^+}||w||^{p^+}_{a,b} + \frac{1}{q}||w||^q_{L^q(\Omega)} : w \in X \quad and \quad ||w||^r_{L^r(\Omega;g)} = r\right\}. \tag{10}$$

Let $n \to v_n \in X$ be a minimizing sequence of (10), which is clearly bounded in X, so that we can assume, without loss of generality, that it converges weakly to some $v \in X$, with $||v||^r_{L^r(\Omega;g)} = r$ and

$$\Lambda = \frac{1}{p^+}||v||^{p^+}_{a,b} + \frac{1}{q}||v||^q_{L^q(\Omega)}$$

by the weak lower semicontinuity of $||\cdot||$. Thus, $J_\lambda(v) = \Lambda - \lambda < 0$ for any $\lambda > \Lambda$.

Now put

$\lambda^* := \sup\{\lambda > 0 :$ problem (2) does not admit any weak solution$\}$,

$\lambda^{**} := \inf\{\lambda > 0:$ problem (2) admits a weak solution$\}$.

Of course $\Lambda \ge \lambda^{**} \ge \lambda^* > 0$. To complete the proof of Theorem 1.1 it is enough to argue the following essential facts: (a) problem (2) has a weak solution for any $\lambda > \lambda^{**}$; (b) $\lambda^{**} = \lambda^*$ and problem (2) admits a weak solution when $\lambda = \lambda^*$.

Step 5. Problem (2) has a weak solution for any $\lambda > \lambda^{**}$ and $\lambda^{**} = \lambda^*$. Fix $\lambda > \lambda^{**}$. By the definition of λ^{**}, there exists $\mu \in (\lambda^{**}, \lambda)$ such that J_μ has a non-trivial critical point $u_\mu \in X$. Of course, u_μ is a sub-solution of (2). In order to find a super-solution of (2) which dominates u_μ, we consider the constrained minimization problem

$$\inf\left\{\frac{1}{p^+}||u||^{p^+}_{a,b} + \frac{1}{q}||w||^q_{L^q(\Omega)} - \frac{\lambda}{r}||w||^r_{L^r(\Omega;g)} : w \in X \quad and \quad w \ge u_\mu\right\}.$$

Arguments similar to those used in Step 4 show that the above minimization problem has a solution $u_\lambda \geq u_\mu$ which is also a weak solution of problem (2), provided $\lambda > \lambda^{**}$.

We already know that $\lambda^{**} \geq \lambda^*$. But, by the definition of λ^{**} and the above remark, problem (2) has no solutions for any $\lambda < \lambda^{**}$. Passing to the supremum, this forces $\lambda^{**} = \lambda^*$ and completes the proof.

Step 6. Problem (2) admits a non-negative weak solution when $\lambda = \lambda^*$. Let $n \to \lambda_n$ be a decreasing sequence converging to λ^* and let $n \to u_n$ be a corresponding sequence of non-negative weak solutions of (2). As noted in Step 2, the sequence $(u_n)_n$ is bounded in X, so that, without loss of generality, we may assume that it converges weakly in X, strongly in $L^r(\Omega; g)$, and pointwise to some $u^* \in X$, with $u^* \geq 0$. By (3), for all $\varphi \in X$,

$$\int_\Omega a(x)|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi dx + \int_\Gamma b(x)|u_n|^{p(x)-2}u_n\varphi d\sigma + \int_\Omega |u_n|^{q-2}u_n\varphi dx$$

$$= \lambda_n \int_\Omega g(x)|u_n|^{r-2}u_n\varphi dx,$$

and passing to the limit as $n \to \infty$ we deduce that u^* verifies (3) for $\lambda = \lambda^*$, as claimed.

It remains to argue that $u^* \neq 0$. A key ingredient in this argument is the lower bound energy given in (8). Hence, since u_n is a non-trivial weak solution of problem 2 corresponding to λ_n, we have $||u_n||_{a,b}^{p^+} \geq (C^r/\lambda^{p^+})^{1/(r-p^+)}$ by (8), where $C > 0$ is the constant given in (7) and not depending on λ_n. Next, since $\lambda_n \searrow \lambda^*$ as $n \to \infty$ and $\lambda^* > 0$, it is enough to show that

$$||u_n - u^*||_{a,b} \to 0 \quad as \quad n \to \infty. \tag{11}$$

Since u_n and u^* are weak solutions of (2) corresponding to λ_n and λ^*, we have by (3), with $\varphi = u_n - u^*$,

$$\begin{aligned}\int_\Omega a(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u^*|^{p(x)-2}\nabla u^*) \cdot \nabla(u_n - u^*)dx \\ + \int_\Gamma b(x)(|u_n|^{p(x)-2}u_n - |u^*|^{p(x)-2}u^*)(u_n - u^*)d\sigma \\ + \int_\Omega (|u_n|^{q-2}u_n - |u^*|^{q-2}u^*)(u_n - u^*)dx \\ = \int_\Omega g(x)(\lambda_n|u_n|^{r-2}u_n - \lambda^*|u^*|^{r-2}u^*)(u_n - u^*)dx.\end{aligned} \tag{12}$$

Elementary monotonicity properties imply that

$$\int_\Omega (|u_n|^{q-2}u_n - |u^*|^{q-2}u^*)(u_n - u^*)dx \geq 0 \quad and \quad \langle I'(u_n^*) - I'(u^*), u_n - u^*\rangle \geq 0,$$

where

$$I(u) := ||u||_{a,b}^{p^+}/p^+.$$

Since $\lambda_n \searrow \lambda^*$ as $n \to \infty$ and X is compactly embedded in $L^r(\Omega; g)$, for all $p^+ > 1$ relation (12) implies

$$0 \leq \langle I'(u_n^*) - I'(u^*), u_n - u^*\rangle \leq \int_\Omega g(x)\left[\lambda_n u_n^{r-1} - \lambda^*(u^*)^{r-1}\right](u_n - u^*)dx \to 0 \tag{13}$$

as $n \to \infty$.

Now, we distinguish the cases $p^+ \geq 2$ and $1 < p^+ < 2$ and we use the following elementary inequalities (see [18, formula (2.2)]): for all ξ, $\zeta \in \mathbf{R}^N$

$$|\xi - \zeta|^{p^+} \leq c(|\xi|^{p^+-2}\xi - |\zeta|^{p^+-2}\zeta)(\xi - \zeta) \quad for \quad p^+ \geq 2; \tag{14}$$

$$|\xi - \zeta|^{p^+} \leq c\langle |\xi|^{p^+-2}\xi - |\zeta|^{p^+-2}\zeta, \xi - \zeta\rangle^{p^+/2}(|\xi|^{p^+} + |\zeta|^{p^+})^{(2-p^+)/2}$$

for $1 < p^+ < 2$, where c is a positive constant.

Case 1: $p^+ \geq 2$. By (14) and (13), we immediately conclude that

$$||u_n - u^*||_{a,b}^{p^+} \leq c\langle I'(u_n^*) - I'(u^*), u_n - u^*\rangle = o(1) \quad as \quad n \to \infty.$$

Case 2: $1 < p^+ < 2$. Since by convexity for all $\gamma \geq 1$

$$(v + w)^\gamma \leq 2^{\gamma-1}(v^\gamma + w^\gamma) \quad for \quad all \quad v, w \in \mathbf{R}_+ \tag{15}$$

then, for $\gamma = 2/p^+$, we have

$$||u_n - u^*||_{a,b}^2 \leq 2^{(2-p^+)/p^+}\left(\int_\Omega a(x)|\nabla(u_n - u^*)|^{p^+}dx\right)^{2/p^+}$$

$$+2^{(2-p^+)/p^+}\left(\int_\Gamma b(x)|u_n - u^*|^{p^+}d\sigma\right)^{2/p^+}.$$

Thus, in order to conclude that (11) holds, it is enough to show that

$$\int_\Omega a(x)|\nabla(u_n - u^*)|^{p^+}dx \to 0 \quad and \quad \int_\Gamma b(x)|u_n - u^*|^{p^+}d\sigma \to 0$$

as $n \to \infty$. Indeed, combining (14) and (15), we have

$$\int_\Omega a(x)|\nabla(u_n - u^*)|^{p^+}dx$$
$$\le c\int_\Omega a(x)\{(|\nabla u_n|^{p^+-2}\nabla u_n - |\nabla u^*|^{p^+-2}\nabla u^*)\cdot\nabla(u_n-u^*)\}^{p^+/2}$$
$$(|\nabla u_n|^{p^+} + |\nabla u^*|^{p^+})^{(2-p^+)/2}dx$$
$$\le c\left(\int_\Omega a(x)(|\nabla u_n|^{p^+-2}\nabla u_n - |\nabla u^*|^{p^+-2}\nabla u^*)\cdot\nabla(u_n-u^*)dx\right)^{p^+/2}$$
$$(||u_n||_{a,b}^{p^+} + ||u^*||_{a,b}^{p^+})^{(2-p^+)/2}$$
$$\le c\left(\int_\Omega a(x)(|\nabla u_n|^{p^+-2}\nabla u_n - |\nabla u^*|^{p^+-2}\nabla u^*)\cdot\nabla(u_n-u^*)dx\right)^{p^+/2}$$
$$||u_n||_{a,b}^{(2-p^+)p^+/2} + ||u^*||_{a,b}^{(2-p^+)p^+/2})$$
$$\le C_1\left(\int_\Omega a(x)(|\nabla u_n|^{p^+-2}\nabla u_n - |\nabla u^*|^{p^+-2}\nabla u^*)\cdot\nabla(u_n-u^*)dx\right)^{p^+/2}$$

where $C_1 = 2c(\lambda^{q/(q-r)}A)^{(2-p^+)/2}$ by (8) and C_1 is independent of n by (6). Similar arguments yield

$$\int_\Gamma b(x)(u_n-u^*)^{p^+}d\sigma \le C_2\left(\int_\Gamma b(x)[u_n^{p^+-1} - (u^*)^{p^+-1}](u_n-u^*)dx\right)^{p^+/2},$$

with an appropiate positive constant C_2 independent of n. Combining the above two inequalities with (13) we conclude that $||u_n - u^*||_{a,b} = o(1)$ as $n \to \infty$, that is (11) holds and u^* is a non-trivial non-negative weak solution of problem (2) corresponding to $\lambda = \lambda^*$.

Theorem 2.2 in Pucci and Servadei [13], based on the Moser iteration, shows that u satisfies (a), since $u \in W^{1,p^+}_{loc}(\Omega)$, being $u \in X$, $A(x,u,\xi) = -a(x)|\xi|^{p^+-2}\xi$ and $B(x,u,\xi) = \lambda g(x)|u|^{r-2}u - |u|^{q-2}u$ clearly verifies inequality (2.18) of [13] by $(H1)$; for other applications see also [12]. Next, again by the main assumptions on the coefficient $a = a(x)$, an applications of [6, Corollary on p. 830] due to DiBenedetto shows that the weak solution u verifies also property (b). Finally, (c) follows immediately by the strong maximum principle since u is a C^1 non-negative weak solution of the differential inequality $div(a(x)|\nabla u|^{p^+-2}\nabla u) - |u|^{q-2}u \le 0$ in Ω, with $q > p^+$, see, for instance, Section 4.8 of Pucci and Serrin [11] and the comments thereby. Property (d) follows using similar arguments as in the proof of Lemma 2 of [20], which is based on Theorem 1 of Serrin [17]. ■

3 Proof of Theorem 1.2

Taking $\varphi = u$ in (3), we see that any weak solution u of (2) satisfies the equality (4), and the conclusion (i) of Theorem 1.2 follows at once.

We next show that C^1 energy functional $J_\lambda : X \to \mathbf{R}$ satisfies the assumptions of the Mountain Pass theorem of Ambrosetti and Rabinowitz [3]. Fix $w \in X \setminus \{0\}$. Since $p^+ < q < r$ then

$$J_\lambda(tw) = \int_\Omega \frac{t^{p(x)}}{p(x)} a(x)|\nabla w|^{p(x)}dx + \int_\Gamma \frac{t^{p(x)}}{p(x)} b(x)|w|^{p(x)}d\sigma$$
$$+\frac{t^q}{q}||w||^q_{L^q(\Omega)} - \frac{\lambda t^r}{r}||w||^r_{L^r(\Omega;g)} < 0$$

provided t is sufficiently large. Next, by ($H3$), (7) and the fact that $p^+ < q < r$ we observe that

$$J_\lambda(u) \geq \frac{1}{q}||u||^{p^+} - \frac{\lambda}{r}||u||^r_{L^r(\Omega;g)} \geq \frac{1}{q}||u||^{p^+} - \frac{\lambda}{rC^{r/p^+}}||u||^r \geq \alpha > 0,$$

whenever $||u|| = \eta$ and $\eta > 0$ is sufficiently small. Set

$$\Upsilon = \{\gamma \in C([0,1];X) : \gamma(0) = 0, \gamma(1) \neq 0 \quad and \quad J_\lambda(\gamma(1)) \leq 0\},$$

and put

$$c = \inf_{\gamma \in \Upsilon} \max_{t \in [0,1]} J_\lambda(\gamma(t)).$$

Applying the Mountain Pass theorem without the Palais-Smale condition we find a sequence $n \to u_n \in X$ such that

$$J_\lambda(u_n) \to c \quad and \quad J'_\lambda(u_n) \to 0 \tag{16}$$

as $n \to \infty$. Moreover, since $J_\lambda(|u|) \leq J_\lambda(u)$ for all $u \in X$, we can assume that $u_n \geq 0$ for any $n \geq 1$. In what follows we prove that $(u_n)_n$ is bounded in X. Indeed, since $J'_\lambda(u_n) \to 0$ in X', then

$$||u_n||^{p^+}_{a,b} + ||u_n||^q_{L^q(\Omega)} = \lambda ||u_n||^r_{L^r(\Omega;g)} + o(1)$$

as $n \to \infty$. Therefore,

$$c + o(1) = J_\lambda(u_n) \geq \frac{1}{q}||u_n||^{p^+} - \frac{\lambda}{r}||u_n||^r_{L^r(\Omega;g)} \geq \left(\frac{1}{q} - \frac{1}{r}\right)(||u_n||^{p^+} - 1) + o(1)$$

as $n \to \infty$. Thus, since $q < r$, we deduce that the Palais-Smale sequence $(u_n)_n$ is bounded in X. Hence, up to a subsequence, we can assume that

$(u_n)_n$ converges weakly in X and strongly in $L^r(\Omega; g)$ to some element, say $u^* \geq 0$. From now on, with the same arguments as in the proof of Theorem 1.1, we deduce that u^* is a weak solution of the problem (2) such that properties $(a) - (d)$ are fulfilled. Due to the mountain-pass geometry of our problem (2) generated by the assumption $p^+ < q < r < p^*$, we are able to give the following alternative proof in order to show that u^* is a weak solution of (2). Fix $\varphi \in C_0^\infty(\mathbf{R}^N)$. Since $J'_\lambda(u_n) \to 0$ in X', we have

$$\int_\Omega a(x)|\nabla u_n|^{p^+-2}\nabla u_n \cdot \nabla\varphi dx + \int_\Gamma b(x)u_n^{p^+-1}\varphi d\sigma$$

$$+ \int_\Omega u_n^{q-1}\varphi dx - \lambda \int_\Omega g(x)u_n^{r-1}\varphi dx = o(1)$$

as $n \to \infty$. Letting $n \to \infty$, we deduce that

$$\int_\Omega a(x)|\nabla u^*|^{p^+-2}\nabla u^* \cdot \nabla\varphi dx + \int_\Gamma b(x)(u^*)^{p^+-1}\varphi d\sigma$$

$$+ \int_\Omega (u^*)^{q-1}\varphi dx - \lambda \int_\Omega g(x)(u^*)^{r-1}\varphi dx = 0$$

and so by density u^* satisfies relation (3) for any $\varphi \in X$. It remains to show that $u^* \neq 0$. Indeed, by (16) and n is sufficiently large we obtain

$$\begin{aligned} 0 < \frac{c}{2} &\leq J_\lambda(u_n) - \frac{1}{p^+}\langle J'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{q} - \frac{1}{p^+}\right)||u||^q_{L^q(\Omega)} - \lambda\left(\frac{1}{r} - \frac{1}{p^+}\right)||u||^r_{L^r(\Omega;g)} \\ &\leq \frac{\lambda(r-p^+)}{r}||u||^r_{L^r(\Omega;g)}, \end{aligned}$$

since $p^+ < q < r$. This implies that $||u^*||^r_{L^r(\Omega;g)} > 0$ and in turn $u^* \neq 0$, as required.

Finally, u^* verifies properties $(a) - (d)$, as shown in the proof of Theorem 1.1. ■

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