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Remarks on the Spectrum of Bounded and Normal Operators on Hilbert Spaces

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Abstract

Let H be a complex Hilbert space H. Let T be a bounded opertor on H, and let λ be a scalar. We set $T_\lambda := T - \lambda I$. We introduce the concept of $T_\lambda-$spectral sequence in order to discuss the nature of λ when λ belongs to the spectrum of T. This concept is used to make new proofs of some classical and well-known results from general spectral theory. This concept is also used to give a new classification of the spectral points λ of any normal and bounded operator T in terms of properties of their associated spectral sequences. This classification should be compared with the classical one (see for example [4]) based on the properties of the ranges of the operators T_λ.

1 Introduction

1.1

In all what follows, H will be a complex Hilbert space, endowed with its inner product denoted by $\langle \cdot \mid \cdot \rangle$, and associated norm denoted by $\|.\|$. Let $T \in \mathcal{B}(H)$ (the Banach algebra of all bounded linear operators on H). The spectrum $\sigma(T)$ of T is the collection of complex numbers λ such that $T - \lambda I_H$ has no (continuous linear) inverse. We know that $\sigma(T)$ has three disjoint components:

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T),$$

where

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$\sigma_p(T)$ is the *discrete spectrum*, that is the collection of complex numbers λ such that $T-\lambda I_H$ fails to be injective (i.e. $\sigma_p(T)$ is the collection of eigenvalues of T);

$\sigma_c(T)$ is the *continuous spectrum*, that is the collection of complex numbers λ such that $T-\lambda I_H$ is injective, does have dense image, but fails to be surjective;

$\sigma_r(T)$ is the *residual spectrum*, that is the collection of complex numbers λ such that $T-\lambda I_H$ is injective and fails to have dense image.

The *approximate spectrum* of T will be denoted by $\sigma_{ap}(T)$. It is defined as being the collection of complex numbers λ for which there exists a sequence $(x_n)_n$ in H satisfying the following two properties:

(i) x_n is a unit vector for each n,

(ii) $\lim_{n\to\infty}\|Tx_n-\lambda x_n\|=0$.

One can easily prove the following inclusions :

$$\sigma_p(T)\cup\sigma_c(T)\subset\sigma_{ap}(T)\subset\sigma(T).$$

It is well-known that the spectrum of a normal operator has a simple structure. More precisely, if $T\in\mathcal{B}(H)$ is normal, then we have

$$\sigma_p(T)\cup\sigma_c(T)=\sigma(T)=\sigma_{ap}(T). \tag{1.1}$$

Remark. Next we give a new proof of the equalities (1.1).

For sake of completeness, we end this subsection by recalling the following important classification of the elements λ in the spectrum of a bounded and normal operator T ([4], p. 112) which is based on the use of the ranges $\mathcal{R}(T_\lambda)$ of the operators $T_\lambda:=T-\lambda I_H$.

Theorem 1 *Let $(H,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Let $T\in\mathcal{B}(H)$ be a normal operator and let $\lambda\in\mathbb{C}$. Then we have:*

1) $\rho(T)=\{\lambda\in\mathbb{C}:\mathcal{R}(T_\lambda)=H\}$.

2) $\sigma_p(T)=\{\lambda\in\mathbb{C}:\overline{\mathcal{R}(T_\lambda)}\neq H\}$, where $\overline{\mathcal{R}(T_\lambda)}$ means the closure of $\mathcal{R}(T_\lambda)$.

3) $\sigma_c(T)=\{\lambda\in\mathbb{C}:\overline{\mathcal{R}(T_\lambda)}=H$ and $\mathcal{R}(T_\lambda)\neq H\}$.

4) $\sigma_r(T)$ is empty.

1.2

To state and prove our results, we need to introduce the following definition.

Definition 1 *Let $S \in \mathcal{B}(H)$, and let $(x_n)_n$ be a sequence of elements of H. We say that $(x_n)_n$ is an $S-$spectral sequence, if it satisfies the following properties:*

(i) x_n is a unit vector for each n, and

(ii) $\lim_{n\to\infty} \|Sx_n\| = 0$.

Let $T \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$. We denote by $\mathcal{S}_T(\lambda)$ the set of $T_\lambda-$spectral sequences, where $T_\lambda := T - \lambda I_H$.

For any $T \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$, we have the following observations :

(a) $\mathcal{S}_T(\lambda) \neq \emptyset \Longleftrightarrow \lambda \in \sigma_{ap}(T)$.

(b) If $(x_n)_n$ belongs to $\mathcal{S}_T(\lambda)$, then any subsequence of $(x_n)_n$ belongs also to $\mathcal{S}_T(\lambda)$.

1.3

Let $T \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$. In Theorem 2.1 of Section 2, we prove that $\lambda \in \sigma_p(T)$ if and only if there exists a $T_\lambda-$spectral sequence which does not converge weakly to zero. We apply this result to recapture some well-known results concerning compact and normal operators. In Section 3, we make a remark concerning the elements of the residual spectrum of T. In Section 4, we make a remark concerning the elements of the continuous spectrum of T. In Sections 5 and 6, we suppose that T is normal. In Theorem 5.1, we provide some characterizations of the continuous spectrum of T. In particular, $\lambda \in \rho(T)$ (the resolvent set of T) if and only if $\mathcal{S}_T(\lambda)$ is empty. In Theorem 6.1, we give a classification of the spectral points $z \in \sigma(T)$ in terms of their associated T_z-spectral sequences.

2 Characterization of the eigenvalues of a bounded operator and applications

We start by our first result which provides a characterization of the point spectrum of a bounded operator on a Hilbert space.

Theorem 2 *Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Let $T \in \mathcal{B}(H)$ and let $\lambda \in \mathbb{C}$. Then the following statements are equivalent:*

(i) $\lambda \in \sigma_p(T)$.

(ii) There exists a T_λ-spectral sequence $(x_n)_n$ which is strongly converging in H.

(iii) There exists a T_λ-spectral sequence $(x_n)_n$ which is not weakly converging to zero.

Proof. The implications $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$ are evident.

$(iii) \Longrightarrow (i)$ Let $\lambda \in \mathbb{C}$ and let $(x_n)_n$ be a λ-sequence which is not weakly converging to zero. Then we can find z a nonzero vector in H and a subsequence $(y_k := x_{n_k})_k$ of $(x_n)_n$ which converges weakly to z. Thus the sequence $(y_k)_k$ satisfies the following conditions :

(a) y_k is a unit vector for each k, and $\lim_{k\to\infty} \|Ty_k - \lambda y_k\| = 0$, (i.e., $(y_k)_k$ is a T_λ-sequence) and

(b) $(y_k)_k$ converges weakly to z, as $k \to \infty$.

By using Banach-Saks Theorem (see [1] and [2], p. 154), we can find a subsequence $(z_m := y_{k_m})_m$ of $(y_k)_k$ for which the sequence $(\tilde{z}_m)_m$ is converging strongly to z, where $\tilde{z}_m$ are the arithmetic means given by

$$\tilde{z}_m := \frac{1}{m}\sum_{j=1}^{m} z_j = \frac{1}{m}\sum_{j=1}^{m} y_{k_j}, \quad \forall m \geq 1.$$

Since $(y_{k_j})_j$ is a T_λ-sequence, then by using Cesaro's means convergence theorem, we obtain

$$\|T(\tilde{z}_m) - \lambda\tilde{z}_m\| = \frac{1}{m}\left\|\sum_{j=1}^{m} T(y_{k_j}) - \lambda y_{k_j}\right\| \leq$$

$$\leq \frac{1}{m}\sum_{j=1}^{m} \left\|T(y_{k_j}) - \lambda y_{k_j}\right\| \longrightarrow 0, \text{ as } \quad m \to \infty.$$

Since T is continuous, we get

$$\|Tz - \lambda z\| = \lim_{m\to\infty} \|T(\tilde{z}_m) - \lambda\tilde{z}_m\| = 0.$$

We conclude that λ is an eigenvalue. Thus our result is proved.

As a first application of Theorem 2, we give a new proof of the following classical and well-known result.

Theorem 3 *Let $T \in \mathcal{B}(H)$ be a compact operator. Then we have*

$$\sigma_{ap}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}.$$

Proof. Let $\lambda \in \sigma_{ap}(T) \setminus \{0\}$ and suppose that $\lambda \notin \sigma_p(T)$. Let $(x_n)_n$ be a λ-spectral sequence Then by Theorem 2.1, necessarily, this sequence must converge weakly to zero. Since T is compact, then, by Riesz Theorem (see [2],

p. 150), the sequence $(T(x_n))_n$ will converge strongly to zero. Since $\lambda \neq 0$ and since $(x_n)_n$ is a λ-sequence, then it follows that $(x_n)_n$ converges strongly to zero. We note also that the following holds

$$\lim_{n\to\infty} \langle T(x_n) \mid x_n \rangle = \lambda.$$

Now, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|T(x_n) - \lambda x_n\|^2 = \\ &= \lim_{n\to\infty} \left(\|T(x_n)\|^2 - 2\Re(\overline{\lambda} \langle T(x_n) \mid x_n \rangle) + |\lambda|^2 \right) = \\ &= |\lambda|^2. \end{aligned}$$

Thus we get $\lambda = 0$, a contradiction. This completes the proof.

We know, that, if T is normal, then $\sigma(T) = \sigma_{ap}(T)$. Therefore, we have the following result.

Corollary 1 *Let $T \in \mathcal{B}(H)$ be a normal and compact operator. Then we have*

$$\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}.$$

We end this section by proving the following result which says that the point spectrum of any normal operator is not empty.

Theorem 4 *Let $T \in \mathcal{B}(H)$ be a normal operator. Then the following assertions hold true.*
(i) There exists $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$ (i.e., $\sigma(T) \cap \{z \in \mathbb{C} : |z| = \|T\|\}$ is not empty).
(ii) If, in addition, T is compact then there exists $\lambda \in \sigma_p(T)$ such that $|\lambda| = \|T\|$ (i.e., $\sigma_p(T) \cap \{z \in \mathbb{C} : |z| = \|T\|\}$ is not empty).

Proof. We can suppose that T is not zero. Since T is normal, then (see, for example, [3], p. 310) we have

$$\|T\| = \sup_{\|x\|=1} |\langle T(x) \mid x \rangle|.$$

It follows that there exists a sequence $(x_n)_n$ of unit vectors such that

$$\lim_{n\to\infty} |\langle T(x_n) \mid x_n \rangle| = \|T\|.$$

We can suppose that the sequence of numbers $(\langle T(x_n) \mid x_n \rangle)_n$ is convergent (otherwise, one can take a subsequence of $(x_n)_n$). Let λ be the limit of this sequence. Then $|\lambda| = \|T\|$. To prove that λ belongs to the spectrum of T, it is sufficient to show that $(x_n)_n$ is a T_λ-spectral sequence. To see this, we use the following inequalities :

$$\begin{aligned}\|T(x_n) - \lambda x_n\|^2 &= \|T(x_n)\|^2 - 2\Re(\overline{\lambda}\,\langle T(x_n) \mid x_n\rangle) + |\lambda|^2\|x_n\|^2 = \\ &= \|T(x_n)\|^2 - 2\Re(\overline{\lambda}\,\langle T(x_n) \mid x_n\rangle) + |\lambda|^2 \leq \\ &\leq 2|\lambda|^2 - 2\Re(\overline{\lambda}\,\langle T(x_n) \mid x_n\rangle) \longrightarrow \\ &\quad \longrightarrow 2|\lambda|^2 - 2|\lambda|^2 = 0, \text{ as } n \to \infty.\end{aligned}$$

Thus $\lambda \in \sigma_{ap}(T) \setminus \{0\} \subset \sigma(T)$. If in addition T is compact, then, by Theorem 2.2, we deduce that $\lambda \in \sigma_{ap}(T) \setminus \{0\} \subset \sigma_p(T) \setminus \{0\}$. This completes the proof of (i) and (ii).

3 A remark on the residual spectrum of a bounded operator

Let H be a complex Hilbert space as above. Let $T \in \mathcal{B}(H)$ and let $\lambda \in \mathbb{C}$. We recall that $\lambda \in \sigma_r(T)$ if and only if (a) $T_\lambda := T - \lambda I_H$ is injective, and (b) the closure $\overline{\mathcal{R}(T_\lambda)}$ of the range $\mathcal{R}(T_\lambda)$ is not equal to H.

We have the following proposition.

Proposition 1 *Let $T \in \mathcal{B}(H)$. Let $\lambda \in \mathbb{C}$. Suppose that the set $\mathcal{S}_T(\lambda)$ is empty and T_λ is not surjective. Then $\lambda \in \sigma_r(T)$.*

Proof. Since $\mathcal{S}_T(\lambda)$ is empty, then $\epsilon := \inf_{x \in S_H} \|Tx - \lambda x\| > 0$, where $S_H := \{x \in H : \|x\| = 1\}$. Therefore, we have

$$\|Tx - \lambda x\| \geq \epsilon\|x\|, \quad \forall x \in H. \tag{3.1}$$

(3.1) shows that T_λ is injective and that its range $\mathcal{R}(T_\lambda)$ is closed in H. Since T_λ is not surjective, we conclude that $\mathcal{R}(T_\lambda)$ is not dense in H. Thus, $\lambda \in \sigma_r(T)$.

4 A remark on the continuous spectrum of a bounded operator

Let H be a complex Hilbert space as above. Let $T \in \mathcal{B}(H)$ and let $\lambda \in \mathbb{C}$. We recall that $\lambda \in \sigma_c(T)$ if and only if (a) $T_\lambda := T - \lambda I_H$ is injective, (b) T_λ is not surjective, and (c) the range $\mathcal{R}(T_\lambda)$ is dense in H.

Proposition 2 *Let $T \in \mathcal{B}(H)$. Let $\lambda \in \mathbb{C}$. Suppose that $\lambda \in \sigma_c(T)$. Then:*

(i) The set $\mathcal{S}_T(\lambda)$ is not empty.

(ii) Each T_λ-spectral sequence converges weakly to zero.

(iii) Each T_λ-spectral sequence is not strongly convergent in H.

Proof. Since $\lambda \in \sigma_c(T)$, then $\lambda \in \sigma_{ap}(T)$, therefore $\mathcal{S}_T(\lambda)$ is not empty. Since λ is not an eigenvalue of T, then, by (iii) of Theorem 2.1, we deduce that every T_λ-spectral sequence converges weakly to zero. Also, by (ii) of Theorem 2.1, we deduce that every T_λ-spectral sequence does not converge strongly in H.

5 Characterizations of the continuous spectrum of a normal operator

Let H be a complex Hilbert space as above. In the next result, we present some characterizations of the continuous spectrum of any bounded and normal operator on H.

Theorem 5 *Let $T \in \mathcal{B}(H)$ be a normal operator. Let $\lambda \in \mathbb{C}$. Then the following statements are equivalent:*

(i) $\lambda \in \sigma_c(T)$.

(ii) $\lambda \in \sigma(T) \setminus \sigma_p(T)$.

(iii) $T - \lambda I_H$ is injective and the image $(T - \lambda I_H)(H)$ is not closed.

(iv) The set $\mathcal{S}_T(\lambda)$ is not empty and every T_λ-sequence converges weakly to zero.

(v) The set $\mathcal{S}_T(\lambda)$ is not empty and every T_λ-sequence is not strongly convergent in H.

Proof. $(ii) \Longrightarrow (i)$ Since $\lambda \in \sigma(T)\setminus\sigma_p(T)$, then $T-\lambda I_H$ is injective but fails to be surjective. Suppose that the image $(T - \lambda I_H)(H)$ is not dense in H. Then there exists at least a nonzero vector z in the orthogonal of $(T - \lambda I_H)(H)$. Hence, by using well-known identities, we have

$$z \in (T - \lambda I_H)(H)^\perp = \ker(T^* - \overline{\lambda} I_H) = \ker(T - \lambda I_H),$$

a contradiction. We conclude that $\lambda \in \sigma_c(T)$.

$(i) \Longrightarrow (iii)$ is evident from the definition of the continuous spectrum.

$(iii) \Longrightarrow (ii)$ Since $T - \lambda I_H$ is injective, then $\lambda \notin \sigma_p(T)$. Suppose that $\lambda \notin \sigma(T)$. Then there exists a linear (invertible) map $S \in \mathcal{B}(H)$ such that $S(T - \lambda I_H)(x) = x$ for every $x \in H$. In particular, we have

$$\frac{1}{\|S\|} \|x\| \leq \|(T - \lambda I_H)(x)\|, \quad \forall x \in H. \tag{5.1}$$

It follows from (5.1) that $(T - \lambda I_H)(H)$ is complete and thereby closed in H, which is a contradiction. We conclude that $\lambda \in \sigma(T) \setminus \sigma_p(T)$.

The equivalences $(ii) \iff (iv) \iff (v)$ are ensured by Theorem 2.1. Hence, our result is completely proved.

As consequence of Theorem 5.1, we recapture the following well-known result (which was recalled in Section 1).

Corollary 2 *Let $T \in \mathcal{B}(H)$ be a normal operator. Then the residual spectrum $\sigma_r(T)$ is empty and $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) = \sigma_{ap}(T)$.*

6 Classification of the spectral points of a bounded normal operator

As a conclusion of our study, we have the following classification of the spectral points of bounded normal operators on Hilbert spaces in terms of their associated spectral sequences.

Theorem 6 *Let $(H, \langle \cdot, \cdot \rangle)$ be as above and let $T \in \mathcal{B}(H)$ be a normal operator. Let $\lambda \in \mathbb{C}$. Then we have.*

1) $\lambda \in \rho(T)$ if and only if the set $\mathcal{S}_T(\lambda)$ is not empty.

2) The following statements are equivalent:
(i) $\lambda \in \sigma_p(T)$.
(ii) There exists a T_λ-sequence $(x_n)_n$ which is strongly converging in H.
(iii) There exists a T_λ-sequence $(x_n)_n$ which is not weakly converging to zero.

3) The following statements are equivalent:
(i) $\lambda \in \sigma_c(T)$.
(ii) The set $\mathcal{S}_T(\lambda)$ is not empty and every T_λ-sequence converges weakly to zero.
(iii) The set $\mathcal{S}_T(\lambda)$ is not empty and every T_λ-sequence is not strongly convergent in H.

4) $\sigma_r(T)$ is empty.

References

[1] S. Banach and S. Saks, *Sur la convergence forte dans les champs L^p*, Studia Math.,**2** (1930), 51-57.

[2] K. Maurin, *Methods of Hilbert Spaces*, PWN - Polish Scientific Publishers, Warszawa, Poland, 1972.

[3] W. Rudin, *Functional Analysis*, Tata McGraw-Hill Publishing Company Ltd, New Delhi, 1974.

[4] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, D. Reidel Publishing Company, Dordrecht, Holland, 1987.

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