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On common fixed point theorems of Meir and Keeler type

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Abstract

Two common fixed point results of Meir and Keeler type for four weakly compatible mappings are obtained which complement, improve and extend various previous ones existing in the literature especially the result of [2].

1 Introduction and Preliminaries

Let $\mathcal{S}$ and $\mathcal{T}$ be two self mappings of a metric space $(\mathcal{X}, d)$. The pair $\{\mathcal{S}, \mathcal{T}\}$ is called *compatible* [3] if

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$. Note that compatibility is a generalization of commutativity and weak commutativity.

The same pair is said to be *compatible of type* (A) [5] if

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) = 0.$$

The notions of compatible and compatible mappings of type (A) are independent (see [5]).

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The above $\mathcal{S}$ and $\mathcal{T}$ are called *compatible of type* (B) [8] if

$$\begin{aligned}\lim_{n\to\infty} d(\mathcal{ST}x_n, \mathcal{T}^2x_n) &\le \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{ST}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right] \\ &\text{and} \\ \lim_{n\to\infty} d(\mathcal{TS}x_n, \mathcal{S}^2x_n) &\le \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{TS}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right].\end{aligned}$$

If the pair $\{\mathcal{S}, \mathcal{T}\}$ is compatible of type (A), then it is compatible of type (B). However, the converse is not true in general.

The same pair is said to be *compatible of type* (P) [6] if it satisfies the equality

$$\lim_{n\to\infty} d(\mathcal{S}^2x_n, \mathcal{T}^2x_n) = 0.$$

These two mappings are compatible of type (C) [7] if

$$\begin{aligned}\lim_{n\to\infty} d(\mathcal{ST}x_n, \mathcal{T}^2x_n) &\le \frac{1}{3}\Big[\lim_{n\to\infty} d(\mathcal{ST}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{T}^2x_n) \\ &\quad + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\Big], \\ &\text{and} \\ \lim_{n\to\infty} d(\mathcal{TS}x_n, \mathcal{S}^2x_n) &\le \frac{1}{3}\Big[\lim_{n\to\infty} d(\mathcal{TS}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{S}^2x_n) \\ &\quad + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\Big].\end{aligned}$$

It is easy to see that compatible mappings of type (A) are compatible of type (C) but the converse is false in general. Note that the notions above are equivalent under the condition of continuity.

Recently, G. Jungck [4] generalized compatibility by giving the notion of weak compatibility. $\mathcal{S}$ and $\mathcal{T}$ are weakly compatible if $\mathcal{S}t = \mathcal{T}t$ for some $t \in \mathcal{X}$ implies $\mathcal{ST}t = \mathcal{TS}t$. The following example shows that the converse is not true in general.

Example 1.1. Let $\mathcal{X} = [0, \infty)$ with the usual metric. Define mappings $\mathcal{S}$, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{S}x = \begin{cases} x \text{ if } 0 \le x < 2 \\ 2 \text{ if } x = 2 \\ 4 \text{ if } 2 < x < \infty \end{cases}, \quad \mathcal{T}x = \begin{cases} 4 - x \text{ if } 0 \le x < 2 \\ 2 \text{ if } x = 2 \\ 7 \text{ if } 2 < x < \infty. \end{cases}$$

Note that $\mathcal{S}$ and $\mathcal{T}$ are weakly compatible. Moreover, if $\{x_n\}$ is a sequence in

$\mathcal{X}$ such that $x_n = 2 - \dfrac{1}{n}$ for $n = 1, 2, \dots$. Then

$$\begin{aligned} \mathcal{S}x_n &= x_n \to 2,\ \mathcal{T}x_n = 4 - x_n \to 2, \\ \mathcal{S}\mathcal{T}x_n &= 4,\ \mathcal{T}\mathcal{S}x_n = 4 - x_n, \\ \mathcal{S}^2x_n &= x_n,\ \mathcal{T}^2x_n = 7, \end{aligned}$$

$$\begin{aligned} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) &\to 2,\ d(\mathcal{S}^2x_n, \mathcal{T}^2x_n) \to 5, \\ d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) &\to 3,\ d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) \to 0, \end{aligned}$$

$$\begin{aligned} 3 &= \lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \\ &\nleq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right] = 1, \\ 3 &= \lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \\ &\nleq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{T}^2x_n)\right] = \frac{7}{3}. \end{aligned}$$

Therefore the mappings $\mathcal{S}$ and $\mathcal{T}$ are neither compatible, nor compatible of type (A) (resp. type (B), (C) and (P)).

The following lemma will be needed.

Lemma 1.1. [1] *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ be self mappings of a metric space $(\mathcal{X}, d)$ such that $\mathcal{A}\mathcal{X} \subset \mathcal{T}\mathcal{X}$, $\mathcal{B}\mathcal{X} \subset \mathcal{S}\mathcal{X}$. Assume further that given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in $\mathcal{X}$*

$$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(\mathcal{A}x, \mathcal{B}y) \le \epsilon,$$

and

$$d(\mathcal{A}x, \mathcal{B}y) < M(x, y), \text{ whenever } M(x, y) > 0,$$

where

$$M(x, y) = \max\{d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y), [d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{A}x, \mathcal{T}y)]/2\}.$$

Then for each x_0 in $\mathcal{X}$, the sequence $\{y_n\}$ in $\mathcal{X}$ defined by the rule

$$y_{2n} = \mathcal{A}x_{2n} = \mathcal{T}x_{2n+1},\ y_{2n+1} = \mathcal{B}x_{2n+1} = \mathcal{S}x_{2n+2}$$

is a Cauchy sequence.

Now, we state the following theorem due to [2].

Theorem 1.1. *Let $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ be compatible pairs of self mappings of a complete metric space $(\mathcal{X}, d)$ such that*

(i) $\mathcal{AX} \subset \mathcal{TX}$, $\mathcal{BX} \subset \mathcal{SX}$,
(ii) given $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* x, y *in* $\mathcal{X}$,

$$\epsilon \leq M(x,y) < \epsilon + \delta \Rightarrow d(\mathcal{A}x, \mathcal{B}y) < \epsilon, \text{ and}$$

(iii)

$$d(\mathcal{A}x, \mathcal{B}y) < k\left[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{A}x, \mathcal{S}x) + d(\mathcal{B}y, \mathcal{T}y) + d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{A}x, \mathcal{T}y)\right],$$

for $0 \leq k \leq \dfrac{1}{3}$.

If one of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ *and* $\mathcal{T}$ *is continuous, then* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point.*

Remark. Note that
(1) If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ *and* $\mathcal{T}$ have a common fixed point z in $\mathcal{X}$, then (iii) becomes an impossible inequality $(0 < 0)$. So, inequality $<$ should be replaced by $\leq$.
(2) The four mappings cannot have a unique common fixed point if $k = \dfrac{1}{3}$, therefore, in this case k must be strictly lower than $\dfrac{1}{3}$.

2 Main Results

Now, we give our first main result which corriges, improves and extends the above result because we deleted the continuity and one weakened the compatibility to the weak one.

Theorem 2.1. *Let* $(\mathcal{A}, \mathcal{S})$ *and* $(\mathcal{B}, \mathcal{T})$ *be weakly compatible pairs of self mappings of a complete metric space* $(\mathcal{X}, d)$ *such that the following conditions hold:*
(a) $\mathcal{AX} \subseteq \mathcal{TX}$ *and* $\mathcal{BX} \subseteq \mathcal{SX}$,
(b) one of $\mathcal{AX}$, $\mathcal{BX}$, $\mathcal{SX}$ *or* $\mathcal{TX}$ *is closed,*
(c) for each $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\epsilon < M(x,y) < \epsilon + \delta \Rightarrow d(\mathcal{A}x, \mathcal{B}y) \leq \epsilon,$$

(c')

$$x,\ y \in \mathcal{X},\ M(x,y) > 0 \Rightarrow d(\mathcal{A}x, \mathcal{B}y) < M(x,y),$$

where

$$M(x,y) = \max\left\{d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y), \frac{1}{2}(d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{A}x, \mathcal{T}y))\right\},$$

(d)

$$d(\mathcal{A}x, \mathcal{B}y) \leq k\left[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{A}x, \mathcal{S}x) + d(\mathcal{B}y, \mathcal{T}y) + d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{A}x, \mathcal{T}y)\right]$$

for $0 \leq k < \dfrac{1}{3}$.

Then, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. Let x_0 be an arbitrary element in $\mathcal{X}$, then, by virtue of (a), we can define inductively a sequence

$$(*)\ \ \{\mathcal{A}x_0, \mathcal{B}x_1, \mathcal{A}x_2, \mathcal{B}x_3, ..., \mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}, ...\}$$

such that

$y_{2n} = \mathcal{A}x_{2n} = \mathcal{T}x_{2n+1}$ and $y_{2n+1} = \mathcal{B}x_{2n+1} = \mathcal{S}x_{2n+2}$ for $n \in \mathbb{N} = \{0, 1, 2, ...\}$.

By Lemma 1.1 of [1] it follows that $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, $\{y_n\}$ and its subsequences $\{\mathcal{A}x_{2n}\} = \{\mathcal{T}x_{2n+1}\}$, $\{\mathcal{S}x_{2n}\} = \{\mathcal{B}x_{2n-1}\}$ and $\{\mathcal{S}x_{2n+2}\} = \{\mathcal{B}x_{2n+1}\}$ converge to some point $z \in \mathcal{X}$.
Suppose that $\mathcal{A}\mathcal{X}$ is closed. Then since $\mathcal{A}\mathcal{X} \subseteq \mathcal{T}\mathcal{X}$, there exists a point $u \in \mathcal{X}$ such that $z = \mathcal{T}u$. Using inequality (d), we have

$$\begin{aligned} d(\mathcal{A}x_{2n}, \mathcal{B}u) \quad \leq \quad & k\left[d(\mathcal{S}x_{2n}, \mathcal{T}u) + d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}) + d(\mathcal{B}u, \mathcal{T}u)\right. \\ & \left. + d(\mathcal{S}x_{2n}, \mathcal{B}u) + d(\mathcal{A}x_{2n}, \mathcal{T}u)\right]. \end{aligned}$$

At infinity, we get

$$d(z, \mathcal{B}u) \leq 2kd(z, \mathcal{B}u) < d(z, \mathcal{B}u),$$

which is a contradiction. Thus, $z = \mathcal{T}u = \mathcal{B}u$ and, by the weak compatibility of $(\mathcal{B}, \mathcal{T})$, it follows that $\mathcal{B}\mathcal{T}u = \mathcal{T}\mathcal{B}u$ and so $\mathcal{B}z = \mathcal{B}\mathcal{T}u = \mathcal{T}\mathcal{B}u = \mathcal{T}z$.
We claim that z is a common fixed point of $\mathcal{B}$ and $\mathcal{T}$. Assume not, then, by assumption (d), we obtain

$$\begin{aligned} d(\mathcal{A}x_{2n}, \mathcal{B}z) \quad \leq \quad & k\left[d(\mathcal{S}x_{2n}, \mathcal{T}z) + d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}) + d(\mathcal{B}z, \mathcal{T}z)\right. \\ & \left. + d(\mathcal{S}x_{2n}, \mathcal{B}z) + d(\mathcal{A}x_{2n}, \mathcal{T}z)\right]. \end{aligned}$$

When n tends to infinity, it gives

$$d(z, \mathcal{B}z) \leq 3kd(z, \mathcal{B}z) < d(z, \mathcal{B}z),$$

which implies that $z = \mathcal{B}z = \mathcal{T}z$.
Now, since $\mathcal{B}\mathcal{X} \subseteq \mathcal{S}\mathcal{X}$, there exists a point $v \in \mathcal{X}$ such that $z = \mathcal{S}v$. Then, from (d), we have

$$d(\mathcal{A}v, \mathcal{B}z) \leq k\left[d(\mathcal{S}v, \mathcal{T}z) + d(\mathcal{A}v, \mathcal{S}v) + d(\mathcal{B}z, \mathcal{T}z) + d(\mathcal{S}v, \mathcal{B}z) + d(\mathcal{A}v, \mathcal{T}z)\right],$$

it follows that

$$d(\mathcal{A}v, z) \leq 2kd(\mathcal{A}v, z) < d(\mathcal{A}v, z),$$

a contradiction, which demands that $\mathcal{A}v = z$. Also, since $\mathcal{A}v = \mathcal{S}v = z$, by the weak compatibility of $\mathcal{A}$ and $\mathcal{S}$, it follows that $\mathcal{SA}v = \mathcal{AS}v$ and so $\mathcal{S}z = \mathcal{SA}v = \mathcal{AS}v = \mathcal{A}z$.
Again the use of inequality (d) gives

$$d(\mathcal{A}z, \mathcal{B}z) \leq k\left[d(\mathcal{S}z, \mathcal{T}z) + d(\mathcal{A}z, \mathcal{S}z) + d(\mathcal{B}z, \mathcal{T}z) + d(\mathcal{S}z, \mathcal{B}z) + d(\mathcal{A}z, \mathcal{T}z)\right];$$

i.e.,

$$d(\mathcal{A}z, z) \leq 3kd(\mathcal{A}z, z) < d(\mathcal{A}z, z),$$

a contradiction. Consequently, we have $\mathcal{A}z = z = \mathcal{S}z$. Hence, z is a common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$.
Finally, we prove the uniqueness of z. Indeed, suppose that w is a second distinct common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$. Then, again using inequality (d), we get

$$d(\mathcal{A}z, \mathcal{B}w) \leq k\left[d(\mathcal{S}z, \mathcal{T}w) + d(\mathcal{A}z, \mathcal{S}z) + d(\mathcal{B}w, \mathcal{T}w) + d(\mathcal{S}z, \mathcal{B}w) + d(\mathcal{A}z, \mathcal{T}w)\right];$$

that is,

$$d(z, w) \leq 3kd(z, w) < d(z, w),$$

this contradiction implies that $w = z$.
Similarly, we can obtain this conclusion by supposing $\mathcal{BX}$ (resp. $\mathcal{SX}$, $\mathcal{TX}$) is closed.

For our second main result, we need the following lemma.

Lemma 2.1. *[3] (resp. [5], [6], [8]) Let $\mathcal{S}$ and $\mathcal{T}$ be compatible, compatible of type (A) (resp. (B), (P), (C)) self mappings of a metric space $(\mathcal{X}, d)$. If $\mathcal{S}t = \mathcal{T}t$ for some $t \in \mathcal{X}$, then $\mathcal{ST}t = \mathcal{TS}t$.*

From the previous theorem, our result is immediate.

Corollary 2.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$* and *$\mathcal{T}$ be mappings from a complete metric space $(\mathcal{X}, d)$ into itself satisfying conditions (a), (b), (c), (c') and (d) of Theorem* 2.1. *Further, if the pairs $(\mathcal{A}, \mathcal{S})$* and *$(\mathcal{B}, \mathcal{T})$ are compatible, compatible of type (A) (resp. (B), (P) and (C)), then the four mappings have a unique common fixed point $z \in \mathcal{X}$.*

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