An. Şt. Univ. Ovidius Constanţa Vol. 16(2), 2008, 31–38

Infimum and supremum completeness properties of ordered sets without axioms

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Abstract

In this paper, by using the ideas of the second author, we establish several intimate connections among the most simple infimum and supremum completeness properties of a generalized ordered set. That is, an arbitrary set equipped with an arbitrary inequality relation.

In particular, we obtain straightforward extensions of some basic theorems on partially ordered sets. Due to the equalities $\inf(A) = \sup(\operatorname{lb}(A))$ and $\sup(A) = \inf(\operatorname{ub}(A))$ established first by the second author, the proofs given here are much shorter and more natural than the usual ones.

1 Prerequisites

Throughout this paper, X will denote an arbitrary set equipped with an arbitrary binary relation $\leq$. Thus, X may be considered as a generalized ordered set, or an ordered set without axioms.

For any $A \subset X$, the members of the families

$$\operatorname{lb}(A) = \left\{ x \in X : \quad \forall\ a \in A : \quad x \leq a \right\}$$

and

$$\operatorname{ub}(A) = \left\{ x \in X : \quad \forall\ a \in A : \quad a \leq x \right\}$$

are called the lower and upper bounds of A in X, respectively. And the members of the families

Key Words: Generalized ordered sets, infimum and supremum completenesses
Mathematics Subject Classification: 06A06, 06A23
Received: November, 2007
Accepted: August, 2008
*The research of the first author has been supported by the grant OTKA NK–68040.

$$\min(A) = A \cap \mathrm{lb}(A), \qquad \max(A) = A \cap \mathrm{ub}(A),$$
$$\inf(A) = \max\big(\mathrm{lb}(A)\big), \qquad \sup(A) = \min\big(\mathrm{ub}(A)\big)$$

are called the minima, maxima, infima and suprema of A in X, respectively.

Concerning the above basic tools, we shall only need here the following simple statements of [5]. Hints for the proofs are included for the reader's convenience.

Theorem 1.1 *We have*

(1) $\mathrm{lb}(\emptyset) = X$ *and* $\mathrm{ub}(\emptyset) = X$*;*

(2) $\mathrm{lb}(B) \subset \mathrm{lb}(A)$ *and* $\mathrm{ub}(B) \subset \mathrm{ub}(A)$ *for all* $A \subset B \subset X$.

Proof. It is convenient to note first that $\mathrm{lb}(A) = \bigcap_{a \in A} \mathrm{lb}(a)$, where $\mathrm{lb}(a) = \mathrm{lb}\big(\{a\}\big)$. Hence, the first statements of (1) and (2) are quite obvious.

Theorem 1.2 *If* $A \subset X$*, then*

(1) $A \subset \mathrm{ub}\big(\mathrm{lb}(A)\big)$ *and* $A \subset \mathrm{lb}\big(\mathrm{ub}(A)\big)$*;*

(2) $\mathrm{lb}(A) = \mathrm{lb}\Big(\mathrm{ub}\big(\mathrm{lb}(A)\big)\Big)$ *and* $\mathrm{ub}(A) = \mathrm{ub}\Big(\mathrm{lb}\big(\mathrm{ub}(A)\big)\Big)$.

Proof. It is convenient to note first that, for any $A, B \subset X$, we have $A \subset \mathrm{lb}(B)$ if and only if $B \subset \mathrm{ub}(A)$. Hence, by the inclusions $\mathrm{lb}(A) \subset \mathrm{lb}(A)$ and $\mathrm{ub}(A) \subset \mathrm{ub}(A)$, it is clear that (1) is true.

Now, from the first inclusion of (1), by Theorem 1.1, it is clear that $\mathrm{lb}\Big(\mathrm{ub}\big(\mathrm{lb}(A)\big)\Big) \subset \mathrm{lb}(A)$. Moreover, from the second inclusion of (1), by writing $\mathrm{lb}(A)$ in place of A, we can see that $\mathrm{lb}(A) \subset \mathrm{lb}\Big(\mathrm{ub}\big(\mathrm{lb}(A)\big)\Big)$. Therefore, the first statement of (2) is also true.

Theorem 1.3 *If* $A \subset X$*, then*

(1) $\inf(A) = \sup\big(\mathrm{lb}(A)\big)$*;* *(2)* $\sup(A) = \inf\big(\mathrm{ub}(A)\big)$.

Proof. By the corresponding definitions and Theorem 1.2, it is clear that $\inf(A) = \max\big(\mathrm{lb}(A)\big) = \mathrm{ub}\big(\mathrm{lb}(A)\big) \cap \mathrm{lb}(A) = \mathrm{ub}\big(\mathrm{lb}(A)\big) \cap \mathrm{lb}\Big(\mathrm{ub}\big(\mathrm{lb}(A)\big)\Big) = \min\Big(\mathrm{ub}\big(\mathrm{lb}(A)\big)\Big) = \sup\big(\mathrm{lb}(A)\big)$. Therefore, (1) is true.

Theorem 1.4 *We have*

(1) $\inf(\emptyset) = \mathrm{ub}(X) = \max(X) = \sup(X)$*;*

(2) $\sup(\emptyset) = \mathrm{lb}(X) = \min(X) = \inf(X)$.

Proof. By the corresponding definitions and Theorem 1.1, it is clear that $\inf(\emptyset) = \max\big(\operatorname{lb}(\emptyset)\big) = \max(X) = X \cap \operatorname{ub}(X) = \operatorname{ub}(X)$. Hence, by Theorem 1.3, it is clear that $\operatorname{ub}(X) = \inf(\emptyset) = \sup\big(\operatorname{lb}(\emptyset)\big) = \sup(X)$. Therefore, (1) is true.

2 Infimum and supremum completenesses

Definition 2.1 *We say that*

(1) X is inf-complete if $\inf(A) \neq \emptyset$ for all $A \subset X$;

(2) X is quasi-inf-complete if $\inf(A) \neq \emptyset$ for all $A \subset X$ with $A \neq \emptyset$;

(3) X is pseudo-inf-complete if $\inf(A) \neq \emptyset$ for all $A \subset X$ with $\operatorname{lb}(A) \neq \emptyset$;

(4) X is semi-inf-complete if $\inf(A) \neq \emptyset$ for all $A \subset X$ with $A \neq \emptyset$ and $\operatorname{lb}(A) \neq \emptyset$.

Remark 2.2 *The corresponding sup-completeness properties are to be defined analogously.*

Moreover, X may, for instance, be called complete if it is both inf-complete and sup-complete.

Example 2.3 *Note that the set $\mathbb{R}$ of all real numbers, with the usual ordering, is semi-complete, but neither quasi-inf-complete nor pseudo-inf-complete, and neither quasi-sup-complete nor pseudo-sup-complete.*

While, the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of all extended real numbers, with the usual ordering, is already complete.

Example 2.4 *Moreover, note that the set $\overline{\mathbb{R}} \setminus \{-\infty\}$, with the usual ordering, is both pseudo-inf-complete and quasi-sup-complete, but neither quasi-inf-complete nor pseudo-sup-complete.*

While, the set $\overline{\mathbb{R}} \setminus \{+\infty\}$, with the usual ordering, is both quasi-inf-complete and pseudo-sup-complete, but neither pseudo-inf-complete nor quasi-sup-complete.

Remark 2.5 *In addition to Definition 2.1, for instance, we may also naturally say that X is finitely (countably) quasi-inf-complete if $\inf(A) \neq \emptyset$ for all finite (countable) nonvoid subset A of X. Thus, a partially ordered set may be called a meet-semilattice if it is finitely quasi-inf-complete.*

Moreover, some lb, ub, min and max completenesses properties can also be naturally introduced. Namely, for instance, a partially ordered set may be called well-ordered if it is min-complete. And a preordered (partially ordered) set may be called directed upward (totally ordered) if it is finitely ub-complete (max-complete).

In this respect, it is also worth mentioning that a partially ordered set X *may be called inductive (almost inductive) if* $\operatorname{ub}(A) \neq \emptyset$ *for any totally ordered (well-ordered) subset* A *of* X*. And* X *may be called strictly inductive (almost inductive) if* $\sup(A) \neq \emptyset$ *for any totally ordered (well-ordered) subset* A *of* X*. However, in the sequel, we shall only be interested in the completeness properties mentioned in Definition 2.1 and Remark 2.2.*

3 Relationships among infimum completenesses

By Definition 2.1, we evidently have the following two propositions.

Proposition 3.1 *If* X *is inf-complete, then* X *is both quasi-inf-complete and pseudo-inf-complete.*

Proposition 3.2 *If* X *is either quasi-inf-complete or pseudo-inf-complete, then* X *is semi-inf-complete.*

Moreover, by using the corresponding definitions, we can also easily prove the following two theorems.

Theorem 3.3 *The following assertions are equivalent:*

(1) X *is quasi-inf-complete and* $X \neq \emptyset$*;*

(2) X *is semi-inf-complete and* $\operatorname{lb}(X) \neq \emptyset$*.*

Proof. If (1) holds, then in particular we have $\inf(X) \neq \emptyset$. Hence, by using that $\operatorname{lb}(X) = \inf(X)$, we can infer that $\operatorname{lb}(X) \neq \emptyset$. Now, by Proposition 3.2, it is clear that (2) also holds.

On the other hand if (2) holds, then since $\operatorname{lb}(X) \subset X$ we have $X \neq \emptyset$. Moreover, since $\operatorname{lb}(X) \subset \operatorname{lb}(A)$ for all $A \subset X$, we also have $\operatorname{lb}(A) \neq \emptyset$ for all $A \subset X$. Now, by Definition 2.1, it is clear that (1) also holds.

Theorem 3.4 *The following assertions are equivalent:*

(1) X *is pseudo-inf-complete and* $X \neq \emptyset$*;*

(2) X *is semi-inf-complete and* $\operatorname{ub}(X) \neq \emptyset$*.*

Proof. If (1) holds, then since $\operatorname{lb}(\emptyset) = X$ we also have $\inf(\emptyset) \neq \emptyset$. Hence, by using that $\inf(\emptyset) = \operatorname{ub}(X)$, we can infer that $\operatorname{ub}(X) \neq \emptyset$. Now, by Proposition 3.2, it is clear that (2) also holds.

On the other hand, if (2) holds, then since $\operatorname{ub}(X) \subset X$ we have $X \neq \emptyset$. Moreover, since $\inf(\emptyset) = \operatorname{ub}(X)$, we also have $\inf(\emptyset) \neq \emptyset$. Now, by Definition 2.1, it is clear that (1) also holds.

Analogously to Theorems 3.3 and 3.4, one can also easily prove the following extension of the equivalence of (ii) and (iii) in [2, Theorem 2.31, p. 47].

Theorem 3.5 *The following assertions are equivalent:*

(1) X *is inf-complete;*

(2) X *is quasi-inf-complete and* $\operatorname{ub}(X) \neq \emptyset$*;*

(3) X *is pseudo-inf-complete and* $\operatorname{lb}(X) \neq \emptyset$*;*

(4) X *is semi-inf-complete and* $\operatorname{lb}(X) \neq \emptyset$ *and* $\operatorname{ub}(X) \neq \emptyset$.

Remark 3.6 *The above theorems can be reformulated by using that* $\operatorname{lb}(X) = \min(X) = \inf(X)$ *and* $\operatorname{ub}(X) = \max(X) = \sup(X)$.

Moreover, it is also worth noticing that the results of this section can be dualized by writing sup, ub *and* lb *in place of* inf, lb *and* ub*, respectively.*

4 Relationships between infimum and supremum completenesses

The following theorem is a straightforward extension of [1, Theorem 3, p. 112] and the equivalence (i) and (ii) in [2, Theorem 2.31, p. 47]. Due to Theorem 1.3, the proof given here is much shorter and more natural then the usual one.

Theorem 4.1 *The following assertions are equivalent:*

(1) X *is inf-complete;* *(2)* X *is sup-complete.*

Proof. To prove $(1) \Longrightarrow (2)$, note that if (1) holds and $A \subset X$, then by Definition 2.1 we have $\inf\big(\operatorname{ub}(A)\big) \neq \emptyset$. Moreover, by Theorem 1.3, we also have $\sup(A) = \inf\big(\operatorname{ub}(A)\big)$. Therefore, $\sup(A) \neq \emptyset$, and thus (2) also holds.

Hence, it is clear that in particular we also have

Corollary 4.2 *X complete if and only if it is either inf-complete or sup-complete.*

Analogously to Theorem 4.1, we can also easily prove the following improvement of [4, Theorem 17, p. 61] and [2, Lemma 2.30, p. 47].

Theorem 4.3 *The following assertions are equivalent:*

(1) X is quasi-inf-complete; *(2) X is pseudo-sup-complete.*

Proof. If (1) holds, and moreover $A \subset X$ such that $\operatorname{ub}(A) \neq \emptyset$, then by Definition 2.1 we have $\inf\big(\operatorname{ub}(A)\big) \neq \emptyset$. Moreover, by Theorem 1.3, we also have $\sup(A) = \inf\big(\operatorname{ub}(A)\big)$. Therefore, $\sup(A) \neq \emptyset$, and thus (2) also holds.

To prove the converse implication, suppose now that (2) holds, and moreover $A \subset X$ such that $A \neq \emptyset$. Then, by Theorem 1.2, we have $A \subset \operatorname{ub}\big(\operatorname{lb}(A)\big)$. Therefore, $\operatorname{ub}\big(\operatorname{lb}(A)\big) \neq \emptyset$. Hence, by (2), it follows that $\sup\big(\operatorname{lb}(A)\big) \neq \emptyset$. Moreover, by Theorem 1.3, we have $\inf(A) = \sup\big(\operatorname{lb}(A)\big)$. Therefore, $\inf(A) \neq \emptyset$, and thus (1) also holds.

Now, as an obvious dual of the above theorem, we can also state

Theorem 4.4 *The following assertions are equivalent:*

(1) X is quasi-sup-complete; *(2) X is pseudo-inf-complete.*

Hence, it is clear that in particular we also have

Corollary 4.5 *X is quasi-complete if and only if it is pseudo-complete.*

Moreover, by using Theorems 1.2 and 1.3, we can also quite easily prove the following extension of a basic theorem on the conditional completeness of partially ordered sets. (For a related result, see [1, Theorem 8, p. 114].)

Theorem 4.6 *The following assertions are equivalent:*

(1) X is semi-inf-complete; *(2) X is semi-sup-complete.*

Proof. To prove (1) $\Longrightarrow$ (2), suppose that (1) holds, and moreover $A \subset X$ such that $A \neq \emptyset$ and $\operatorname{ub}(A) \neq \emptyset$. Then, by Theorem 1.2, we have $A \subset \operatorname{lb}\big(\operatorname{ub}(A)\big)$. Therefore, $\operatorname{lb}\big(\operatorname{ub}(A)\big) \neq \emptyset$ is also true. Hence, by Definition 2.1, it is clear that $\inf\big(\operatorname{ub}(A)\big) \neq \emptyset$. Moreover, by Theorem 1.3, we also have $\sup(A) = \inf\big(\operatorname{ub}(A)\big)$. Therefore, $\sup(A) \neq \emptyset$, and thus (2) also holds.

Hence, it is clear that in particular we also have

Corollary 4.7 *X is semi-complete if and only if it is either semi-inf-complete or semi-sup-complete.*

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