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Quenching time of solutions for some nonlinear parabolic equations

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Abstract

In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_t(x,t) = \varepsilon Lu(x,t) + f(u(x,t)) \quad in \quad \Omega \times (0,T), \\ u(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad in \quad \Omega \end{cases}$$

where ε is a positive parameter, L is an elliptic operator, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $f(s)$ is positive, nondecreasing, convex function for $s \in (-\infty, b)$, $\lim_{s\to b} f(s) = +\infty$ with $b = const > 0$ and $\int_0^b \frac{ds}{f(s)} < +\infty$. We show that if ε is small enough, the solution u of the above problem quenches in a finite time and its quenching time tends to the one of the solution of the following differential equation

$$\begin{cases} \alpha'(t) = f(\alpha(t)), \quad t > 0, \\ \alpha(0) = M, \end{cases}$$

when ε goes to zero, where $M = \sup_{x\in\Omega} u_0(x)$.
Finally, we give some numerical results to illustrate our analysis.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem for a nonlinear parabolic equation of the form:

$$u_t(x,t) = \varepsilon Lu(x,t) + f(u(x,t)) \quad in \quad \Omega \times (0,T), \tag{1}$$

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$$u(x,t)=0 \quad on \quad \partial\Omega\times(0,T), \tag{2}$$

$$u(x,0)=u_0(x)\geq 0 \quad in \quad \Omega, \tag{3}$$

where $f(s)$ is a positive, nondecreasing, convex function for $s\in(-\infty,b)$, $\lim_{s\to b}f(s)=+\infty$ and $\int_0^b \frac{ds}{f(s)}<+\infty$ with $b=const>0$, ε is a positive parameter,

$$Lu=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}),$$

where $a_{ij}:\overline{\Omega}\to\mathbb{R}$, $a_{ij}\in C^1(\overline{\Omega})$, $a_{ij}=a_{ji}$, $1\leq i,\ j\leq N$ and there exists a constant $C>0$ such that

$$\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\geq C\|\ \xi\ \|^2 \quad \forall\ x\in\overline{\Omega} \quad \forall\ \xi=(\xi_1,...,\xi_N)\in\mathbb{R}^N,$$

where $\|\ .\ \|$ stands for the Euclidean norm of $\mathbb{R}^N$.
The initial data $u_0\in C^1(\overline{\Omega})$, $u_0(x)$ is nonnegative in Ω, $\sup_{x\in\Omega}u_0(x)=M<b$, $u_0(x)=0$ on $\partial\Omega$.
Here $(0,T)$ is the maximal time interval on which the solution u exists. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u reaches the value b in a finite time, namely

$$\lim_{t\to T}\|u(.,t)\|_\infty=b$$

where $\|u(.,t)\|_\infty=\sup_{x\in\Omega}|u(x,t)|$. In this last case, we say that the solution u quenches in a finite time and the time T is called the quenching time of the solution u. Using standard methods based on the maximum principle, it is not hard to prove the local existence and uniqueness of the solution (see for instance [3]). On the other hand, since the initial data u_0 is nonnegative in Ω, from the maximum principle, we see that the solution u is also nonnegative in $\Omega\times(0,T)$. Solutions of nonlinear parabolic equations which quench in a finite time have been the subject of investigation of many authors (see [3], [4], [5], [9] and the references cited therein). In [6], Friedman and Lacey have considered the problem (1)–(3) in the case where the operator L is replaced by the Laplacian and the term of the source by a function $f(s)$ which is positive, increasing, convex for nonnegative values of s and $\int_0^\infty\frac{ds}{f(s)}<+\infty$. Under some additional conditions on the initial data, they have shown that the solution u of (1)–(3) blows up in a finite time and its blow-up time goes to the one of the solution of the following differential equation

$$\alpha^{'}(t)=f(\alpha(t)), \quad t>0, \quad \alpha(0)=M, \tag{4}$$

when ε goes to zero, where $M = \sup_{x\in\Omega} u_0(x)$ (we say that a solution u blows up in a finite time if it reaches the value infinity in a finite time).
The proof developed in [6] is based on the construction of upper and lower solutions and it is difficult to extend the above result using the method described in [6]. In this paper, we obtain the same result using both a modification of Kaplan's method (see [7]) and a method based on the construction of upper solutions. This method is simple and may be extended to other classes of parabolic equations. Our paper is written in the following manner. In the next section, we show that when ε is sufficiently small, the solution u of (1)–(3) quenches in a finite time and its quenching time goes to the one of the solution of the differential equation defined in (4) when ε decays to zero. We also extend this result to other classes of nonlinear parabolic equations in the third section. Finally, in the last section we give some numerical results to illustrate our analysis.

2 Quenching solutions

In this section, we show that the solution u of the problem (1)–(3) quenches in a finite time for ε sufficiently small. In addition, we prove that its quenching time goes to the one of the solution of the differential equation defined in (4) as ε tends to zero.
Before starting, let us recall a well known result.
Consider the eigenvalue problem

$$-L\varphi(x) = \lambda\varphi(x) \quad in \quad \Omega, \tag{5}$$

$$\varphi(x) = 0 \quad on \quad \partial\Omega, \tag{6}$$

$$\varphi(x) > 0 \quad in \quad \Omega. \tag{7}$$

We know that the above problem has a solution (φ, λ) such that $\lambda > 0$. Without loss of generality, we may suppose that $\int_\Omega \varphi(x)dx = 1$.
Our first result is the following.

Theorem 1 *Assume that $u_0(x) = 0$. Suppose that $\varepsilon < \frac{1}{A}$ where $A = \lambda\int_0^b \frac{ds}{f(s)}$. Then the solution u of (1)–(3) quenches in a finite time and its quenching time T satisfies the following relation*

$$0 \leq T - T_0 \leq \varepsilon T_0 A + o(\varepsilon),$$

where $T_0 = \int_0^b \frac{ds}{f(s)}$ is the quenching time of the solution $\alpha(t)$ of the differential equation defined in (4).

Proof. Since $(0,T)$ is the maximal time interval on which $\|u(.,t)\|_\infty < b$, our aim is to show that T is finite and satisfies the above relation. Since the initial data u_0 is nonnegative in Ω, the maximum principle implies that the solution u is also nonnegative in $\Omega \times (0,T)$. Introduce the function $v(t)$ defined as follows

$$v(t) = \int_\Omega u(x,t)\varphi(x)dx.$$

Take the derivative of v in t and use (1) to obtain

$$v^{'}(t) = \varepsilon \int_\Omega \varphi(x)Lu(x,t)dx + \int_\Omega f(u(x,t))\varphi(x)dx.$$

Applying Green's formula, we arrive at

$$v^{'}(t) = \varepsilon \int_\Omega u(x,t)L\varphi(x)dx + \int_\Omega f(u(x,t))\varphi(x)dx.$$

Since $\int_\Omega \varphi(x)dx = 1$ and $f(s)$ is a convex function for nonnegative values of s, using Jensen's inequality and taking into account (5), we deduce that

$$v^{'}(t) \geq -\lambda\varepsilon v(t) + f(v(t)),$$

which implies that

$$v^{'}(t) \geq f(v(t))(1 - \frac{\lambda\varepsilon v(t)}{f(v(t))}).$$

We observe that

$$\int_0^b \frac{dt}{f(t)} \geq \sup_{0\leq s\leq b} \int_0^s \frac{dt}{f(t)} \geq \sup_{0\leq s\leq b} \frac{s}{f(s)}$$

because $f(s)$ is a nondecreasing function for $0 \leq s \leq b$. We deduce that $v^{'}(t) \geq (1-A\varepsilon)f(v(t))$, which implies that $\frac{dv}{f(v)} \geq (1-A\varepsilon)dt$. Integrating the above inequality over $(0,T)$, we discover that

$$T \leq \frac{1}{1-A\varepsilon}\int_0^b \frac{ds}{f(s)}. \tag{8}$$

This implies that the solution u quenches in a finite time and we have an upper bound of the quenching time.
Now introduce the function $z(x,t)$ defined as follows

$$z(x,t) = \alpha(t) \quad in \quad \overline{\Omega} \times (0,T_0).$$

A routine computation yields

$$z_t(x,t) - Lz(x,t) - f(z(x,t)) = 0 \quad in \quad \Omega \times (0, T_*),$$
$$z(x,t) \geq 0 \quad on \quad \partial\Omega \times (0, T_*),$$
$$z(x,0) \geq u(x,0) \quad in \quad \Omega,$$

where $T_* = \min\{T, T_0\}$. The maximum principle implies that

$$0 \leq u(x,t) \leq z(x,t) = \alpha(t) \quad in \quad \Omega \times (0, T_*).$$

Consequently, we find that

$$T \geq T_0 = \int_0^b \frac{ds}{f(s)}. \tag{9}$$

In fact, suppose that $T_0 > T$, which implies that $\alpha(T) \geq \|u(.,T)\|_\infty = b$. But this contradicts the fact that$(0, T_0)$ is the maximal time interval of existence of the solution $\alpha(t)$. Apply Taylor's expansion to obtain $\frac{1}{1-A\varepsilon} = 1 + A\varepsilon + o(\varepsilon)$. Use (8), (9) and the above relation to complete the rest of the proof.

Remark 1 *Let us consider the case where $u_0 = 0$. The above theorem shows that for ε small enough, the solution u of (1)–(3) quenches in a finite time. We want to know what happens when ε is large enough. Introduce the function $w(x)$ defined as follows*

$$\begin{cases} Lw(x) + 1 = 0 \quad in \quad \Omega, \\ w(x) = 0 \quad on \quad \partial\Omega \end{cases}$$

We know that $w(x)$ exists and is positive in Ω. Let α be a positive constant such that $\|w\|_\infty < \frac{b}{\alpha}$. It is not hard to see that if $\varepsilon \geq \frac{1}{\alpha} f(\alpha\|w\|_\infty)$ then the solution u of (1)–(3) exists globally and is bounded from above by $\alpha\|w\|_\infty$. Indeed, let $z(x,t) = \alpha w(x)$ in $\Omega \times (0,T)$. A straightforward computation reveals that

$$z_t(x,t) - \varepsilon Lz(x,t) = \alpha\varepsilon \quad in \quad \Omega \times (0,T).$$

Since $\alpha\varepsilon \geq f(\alpha\|w\|_\infty) \geq f(z(x,t))$, we deduce that

$$\begin{cases} z_t(x,t) - \varepsilon Lz(x,t) = f(z(x,t) \quad in \quad \Omega \times (0,T), \\ z(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ z(x,0) = \alpha w(x) \quad in \quad \Omega. \end{cases}$$

It follows from the maximum principle that

$$0 \leq u(x,t) \leq z(x,t) = \alpha w(x) \quad in \quad \Omega \times (0,T).$$

Consequently, we get $\|u(.,T)\|_\infty \leq \alpha\|w\|_\infty < b$, which leads us to the desired result.

Now, let us consider the case where the initial data is not null.
Let $a \in \Omega$ be such that $u(a) = M$ and consider the eigenvalue problem below

$$-L\psi(x) = \lambda_\delta \psi(x) \quad in \quad B(a,\delta),$$

$$\psi(x) = 0 \quad on \quad \partial B(a,\delta),$$

$$\psi(x) > 0 \quad in \quad B(a,\delta),$$

where $\delta > 0$, such that, $B(a,\delta) = \{x \in \mathbb{R}^N; \parallel x - a \parallel < \delta\} \subset \Omega$. It is well known that the above eigenvalue problem has a solution (ψ, λ_δ) such that $0 < \lambda_\delta \leq \frac{D}{\delta^2}$ where D is a positive constant which depends only on the upper bound of the coefficients of the operator L and the dimension N.
We can normalize ψ so that $\int_{B(a,\delta)} \psi(x)dx = 1$.
Now, we are in a position to state our result in the case where the initial data is not null.

Theorem 2 *Assume that* $\sup_{x\in\Omega} u_0(x) = M > 0$ *and let* K *be an upper bound for the first derivatives of* u_0. *Suppose that* $\varepsilon < \min\{A^{-3}, (K\, dist(a, \partial\Omega))^3\}$, *where* $A = DK^2 \int_0^b \frac{ds}{f(s)}$. *Then the solution* u *of (1)–(3) quenches in a finite time* T *which obeys the following relation*

$$0 \leq T - T_0 \leq \left(AT_0 + 1/f(\frac{M}{2}) \right) \varepsilon^{1/3} + o(\varepsilon^{1/3}),$$

where $T_0 = \int_M^b \frac{ds}{f(s)}$ *is the quenching time of the solution* $\alpha(t)$ *of the differential equation defined in (4).*

Proof. Since $(0, T)$ is the maximal time interval on which $\|u(.,t)\|_\infty < b$, our goal is to prove that T is finite and obeys the above relation. The fact that the initial data u_0 is nonnegative in Ω implies that the solution u is also nonnegative in $\Omega \times (0,T)$ owing to the maximum principle. Since $u_0 \in C^1(\overline{\Omega})$, from the mean value theorem, we get

$$u_0(x) \geq u_0(a) - \varepsilon^{1/3} \quad for \quad x \in B(a,\delta) \subset \Omega$$

where $\delta = \frac{\varepsilon^{1/3}}{K}$.
Let w be the solution of the following initial-boundary value problem

$$w_t(x,t) = \varepsilon Lw(x,t) + f(w(x,t)) \quad in \quad B(a,\delta) \times (0, T_*),$$

$$w(x,t) = 0 \quad on \quad \partial B(a,\delta) \times (0, T_*),$$

$$w(x,0)=u_0(x) \quad in \quad B(a,\delta),$$

where $(0,T_*)$ is the maximal time interval of existence of the solution w. Introduce the function

$$v(t)=\int_{B(a,\delta)} w(x,t)\psi(x)dx.$$

As in the proof of Theorem 2.1, we find that

$$v^{'}(t)\geq -\varepsilon\lambda_\delta v(t)+f(v(t)),$$

which implies that

$$v^{'}(t)\geq f(v(t))(1-\frac{\varepsilon\lambda_\delta v(t)}{f(v(t))})\geq f(v(t))(1-\varepsilon^{1/3}DK^2\frac{v(t)}{f(v(t))})$$

because $\lambda_\delta\leq\frac{D}{\delta^2}=\frac{DK^2}{\varepsilon^{2/3}}$. As in the proof of Theorem 2.1, we discover that

$$v^{'}(t)\geq f(v(t))(1-\varepsilon^{1/3}A).$$

This inequality may be rewritten as follows

$$\frac{dv}{f(v)}\geq(1-\varepsilon^{1/3}A)dt.$$

Integrate the above inequality over $(0,T_*)$ to obtain

$$(1-\varepsilon^{1/3}A)T_*\leq\int_{v(0)}^{b}\frac{ds}{f(s)}\leq\int_{M-\varepsilon^{1/3}}^{b}\frac{ds}{f(s)}$$

because $v(0)\geq M-\varepsilon^{1/3}$. We deduce that

$$T_*\leq\frac{1}{1-\varepsilon^{1/3}A}\int_{M-\varepsilon^{1/3}}^{b}\frac{ds}{f(s)}.$$

Consequently, w quenches in a finite time because the quantity on the right hand side of the above estimate is finite. Since u is nonnegative in $\Omega\times(0,T)$, we get

$$u_t(x,t)=\varepsilon Lu(x,t)+f(u(x,t)) \quad in \quad B(a,\delta)\times(0,T^*),$$

$$u(x,t)\geq 0 \quad on \quad \partial B(a,\delta)\times(0,T^*),$$

$$u(x,0) = u_0(x) \quad in \quad B(a,\delta),$$

where $T^* = \min\{T, T_*\}$. It follows from the maximum principle that

$$u(x,t) \geq w(x,t) \quad \text{in} \quad B(a,\delta) \times (0,T^*).$$

We deduce that

$$T \leq T_* \leq \frac{1}{1-\varepsilon^{1/3}A} \int_{M-\varepsilon^{1/3}}^{b} \frac{ds}{f(s)}. \tag{10}$$

Indeed, suppose that $T > T_*$. This implies that $\|u(.,T_*)\|_\infty \geq \|w(.,T_*)\|_\infty = b$ which contradicts the fact that $(0,T)$ is the maximal time interval of existence of the solution u. On the other hand, as in the proof of Theorem 2.1, it is not hard to see that

$$z_t(x,t) - Lz(x,t) - f(z(x,t)) = 0 \quad in \quad \Omega \times (0, T_*^*),$$

$$z(x,t) \geq 0 \quad \text{on} \quad \partial\Omega \times (0, T_*^*),$$

$$z(x,0) \geq u(x,0) \quad \text{in} \quad \Omega,$$

where $z(x,t) = \alpha(t)$ in $\overline{\Omega} \times (0,T_0)$ and $T_*^* = \min\{T_0, T\}$. The maximum principle implies that $0 \leq u(x,t) \leq z(x,t) = \alpha(t) \quad in \quad \Omega \times (0,T_*^*)$. Therefore we have

$$T \geq T_0 = \int_M^b \frac{ds}{f(s)}. \tag{11}$$

Indeed, suppose that $T < T_0$ which implies that $b = \|u(.,T)\|_\infty \leq \alpha(T) < b$. But this is a contradiction. Obviously

$$\int_{M-\varepsilon^{1/3}}^{b} \frac{ds}{f(s)} = \int_M^b \frac{ds}{f(s)} + \int_{M-\varepsilon^{1/3}}^{M} \frac{ds}{f(s)}.$$

Due to the fact that $f(s)$ is a nondecreasing function for $s \in (0,b)$, we find that

$$\int_{M-\varepsilon^{1/3}}^{M} \frac{ds}{f(s)} \leq \frac{\varepsilon^{1/3}}{f(M-\varepsilon^{1/3})} \leq \frac{\varepsilon^{1/3}}{f(\frac{M}{2})},$$

which implies that

$$\int_{M-\varepsilon^{1/3}}^{b} \frac{ds}{f(s)} \leq \int_M^b \frac{ds}{f(s)} + \frac{\varepsilon^{1/3}}{f(\frac{M}{2})}. \tag{12}$$

Use Taylor's expansion to obtain

$$\frac{1}{1-\varepsilon^{1/3}A} = 1 + \varepsilon^{1/3}A + o(\varepsilon^{1/3}).$$

It follows from (10), (11), (12) and the above relation that

$$0 \le T - T_0 \le \left(AT_0 + 1/f(\frac{M}{2}) \right) \varepsilon^{1/3} + o(\varepsilon^{1/3}),$$

and the proof is complete.

3 Other quenching solutions

Consider the following initial-boundary value problem

$$(\varphi(u))_t = \varepsilon Lu + f(u) \quad in \quad \Omega \times (0,T), \tag{13}$$

$$u(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \tag{14}$$

$$u(x,0) = u_0(x) \quad in \quad \Omega, \tag{15}$$

where $\varphi(s)$ is a nonnegative and increasing function for the positive values of s. In addition $\int_0^b \frac{\varphi'(s)}{f(s)} < +\infty$. Using the methods described in the proofs of the above theorems, we have the following results.

Theorem 3 *Assume that* $u_0(x) = 0$. *Suppose that* $\varepsilon < \frac{1}{B}$ *where* $B = \lambda \int_0^b \frac{\varphi'(s)}{f(s)} ds$. *Then the solution u of (13)–(15) quenches in a finite time and its quenching time T satisfies the following relation*

$$0 \le T - T_0 \le \varepsilon T_0 B + o(\varepsilon),$$

where $T_0 = \int_0^b \frac{\varphi'(s)}{f(s)} ds$ *is the quenching time of the solution* $\alpha(t)$ *of the differential equation defined as follows*

$$\begin{cases} \varphi'(\alpha(t))\alpha'(t) = f(\alpha(t)), & t > 0, \\ \alpha(0) = 0. \end{cases}$$

Theorem 4 *Assume that* $\sup_{x\in\Omega} u_0(x) = M > 0$ *and let K be an upper bound for the first derivatives of* u_0. *Suppose that* $\varepsilon < \min\{B^{-3}, (Kdist(a, \partial\Omega))^3\}$, *where* $B = DK^2 \int_0^b \frac{\varphi'(s)ds}{f(s)}$. *Then the solution u of (13)–(15) quenches in a finite time and its quenching time T satisfies the following relation*

$$0 \leq T - T_0 \leq \left(BT_0 + \varphi'(\frac{M}{2})/f(\frac{M}{2}) \right) \varepsilon^{1/3} + o(\varepsilon^{1/3}),$$

where $T_0 = \int_M^b \frac{\varphi'(s)}{f(s)} ds$ *is the quenching time of the solution* $\alpha(t)$ *of the differential equation defined as follows*

$$\begin{cases} \varphi'(\alpha(t))\alpha'(t) = f(\alpha(t)), \quad t > 0, \\ \alpha(0) = M. \end{cases}$$

4 Numerical results

In this section, we consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = \varepsilon \Delta u + (1-u)^{-p} \quad in \quad B \times (0,T),$$

$$u(x,t) = 0 \quad on \quad S \times (0,T),$$

$$u(x,0) = u_0(x) \quad in \quad B,$$

where $B = \{x \in \mathbb{R}^N;\ \| x \| < 1\}$, $S = \{x \in \mathbb{R}^N;\ \| x \| = 1\}$. The above problem may be rewritten in the following form

$$u_t = \varepsilon(u_{rr} + \frac{N-1}{r} u_r) + (1-u)^{-p}, \quad r \in (0,1), \quad t \in (0,T), \tag{16}$$

$$u(1,t) = 0, \quad t \in (0,T), \tag{17}$$

$$u(r,0) = \varphi(r), \quad r \in (0,1). \tag{18}$$

Here, we take $\varphi(r) = a\sin(\pi r)$ with $a \in [0,1)$.
Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution u of (16)–(18) by the solution $U_h^{(n)} = (U_0^{(n)}, ..., U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \varepsilon N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (1 - U_0^{(n)})^{-p},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \varepsilon\Big(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}\Big) + (1 - U_i^{(n)})^{-p}, \quad 1 \le i \le I-1,$$

$$U_I^{(n)} = 0,$$

$$U_i^{(0)} = a\sin(\pi ih), \quad 0 \le i \le I.$$

We also approximate the solution u of (16)–(18) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \varepsilon N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (1 - U_0^{(n)})^{-p},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \varepsilon\Big(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h}\Big) + (1 - U_i^{(n)})^{-p}, 1 \le i \le I-1,$$

$$U_I^{(n+1)} = 0,$$

$$U_i^{(0)} = a\sin(\pi ih), \quad 0 \le i \le I.$$

We take $\Delta t_n = \min\{\frac{h^2}{2N\varepsilon}, h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}\}$ for the explicit scheme and $\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}$ for the implicit scheme where

$$\|U_h^{(n)}\|_\infty = \max_{0 \le i \le I} |U_i^{(n)}|.$$

We remark that $\lim_{r\to 0}\frac{u_r(r,t)}{r} = u_{rr}(0,t)$. Hence, if $t = 0$, we have

$$u_t(0,t) = \varepsilon N u_{rr}(0,t) + (1 - u(0,t))^{-p}.$$

This remark has been used in the construction of our schemes when $i = 0$.
Let us notice that in the explicit scheme, the restriction on the time step ensures the nonnegativity of the discrete solution. For the implicit scheme, existence and nonnegativity are also guaranteed by standard methods (see for instance [2]).
We need the following definition.

Definition 1 *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if* $\lim_{n\to+\infty} \|U_h^{(n)}\|_\infty = 1$ *and the series* $\sum_{n=0}^{+\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{+\infty} \Delta t_n$ *is called the numerical quenching time of the solution* $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $|T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $a = 0$, $N = 2$, $p = 1$.
First case: $\varepsilon = \frac{1}{10}$**.**

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.501259	4078	-	-
32	0.500475	15625	-	-
64	0.500274	59688	-	1.97
128	0.500222	227442	7	1.96
256	0.500208	864473	56	1.89

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.501302	4078	-	-
32	0.500484	15626	1	-
64	0.500277	59689	3	1.99
128	0.500223	227444	20	1.95
256	0.500208	864473	142	1.95

Second case: $\varepsilon = \frac{1}{50}$**.**

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler

method

I	T^n	n	CPU time	s
16	0.500978	4074	-	-
32	0.500244	15608	-	-
64	0.500061	59614	3	2.01
128	0.500015	227120	20	2.00
256	0.500004	863074	141	2.07

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.500978	4074	-	-
32	0.500244	15608	-	-
64	0.500061	59614	3	2.01
128	0.500015	227120	20	2.00
256	0.500004	863074	141	2.07

Numerical experiments for $a = \frac{1}{2}$, $N = 2$, $p = 1$.
First case: $\varepsilon = \frac{1}{10}$**.**

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.161101	4007	-	-
32	0.161389	15446	1	-
64	0.161480	59332	1	1.67
128	0.161509	227203	9	1.65
256	0.161518	866278	60	1.69

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.161196	4007	-	-
32	0.161455	15446	1	-
64	0.161482	59333	2	3.27
128	0.161510	227202	21	0.05
256	0.161518	866279	148	1.81

Second case: $\varepsilon = \frac{1}{100}$.

Table 7: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.128272	3843	-	-
32	0.128155	14700	-	-
64	0.128131	56045	1	2.29
128	0.128126	213087	8	2.27
256	0.128125	807893	57	2.32

Table 8: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.128282	3843	-	-
32	0.128157	14700	-	-
64	0.128132	56045	3	2.33
128	0.128127	213087	19	2.33
256	0.128126	807894	139	2.33

Third case: $\varepsilon = \frac{1}{500}$.

Table 9: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.125842	3829	-	-
32	0.125672	14630	-	-
64	0.125631	55715	1	2.06
128	0.125622	211577	8	2.20
256	0.125619	801115	55	1.59

Table 10: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.125844	3829	-	-
32	0.125673	14630	-	-
64	0.125632	55715	3	2.07
128	0.125622	211577	19	2.04
256	0.125619	80115	136	1.74

Remark 2 *If we consider the problem (16)–(18) in the case where the initial data is null and* $p = 1$*, it is not hard to see that the quenching time of the solution of the differential equation defined in (4) equals* 0.5*. We observe from Tables 1-4 that when* ε *diminishes, the numerical quenching time decays to* 0.5*. This result has been proved in Theorem 2.1. When the initial data* $\varphi(r) = \frac{1}{2}\sin(\pi r)$ *and* $p = 1$*, we find that the quenching time of the solution of the differential equation defined in (4) equals* 0.125*. We discover from Tables 5-10 that when* ε *diminishes, the numerical quenching time decays to* 0.125 *which is a result proved in Theorem 2.2.*

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