An. Şt. Univ. Ovidius Constanţa Vol. 16(1), 2008, 67–72

On the inverse inequalities of two new type Hilbert integral inequalities

Zhao Changjian and Mihály Bencze

Abstract

The Hilbert integral inequality is an important inequality in applications. In 1998, two new inequalities similar to Hilbert integral inequality were given by B. G. Pachpatte. The main purpose of the present article is to give two new inverse inequalities similar to these two new inequalities by Jensen inequality and Holder integral inequality.

Hilbert integral inequality plays an important role in mathematical analysis and its applications. In 1998, B. G. Pachpatte gave two new integral inequalities similar to Hilbert integral inequality (see [2], p. 226) in [1]. In this paper we give two new inverse inequalities similar to these new inequalities.

Our main results are given in the following theorems:

Theorem 1. *Let* $h \geq 1, l \geq 1$ *and* $f(\sigma) > 0, g(\tau) > 0$ *for* $\sigma \in (0, x), \tau \in (0, y)$, *where* x, y *are positive real numbers and define* $F(s) = \int_0^s f(\sigma)\, d\sigma$ *and* $G(t) = \int_0^t g(\tau)\, d\tau$ *for* $s \in (0, x), t \in (0, y)$ *and* $\frac{1}{p} + \frac{1}{q} = 1, p < 0$ *or* $0 < p < 1$. *Then*

(1) $\int_0^x \int_0^y \frac{F^h(s) G^l(t)}{C(s,t,p)} ds dt \geq$

Key Words: Hilbert integral inequality; Hölder integral inequality; Jensen inequality.
Mathematics Subject Classification: 26D15.
Received: 31 October, 2007
Accepted: January, 2008

$$\geq hl\left(xy\right)^{1/p}\left(\int\limits_0^x\left(x-s\right)\left(F^{h-1}\left(s\right)\right)^q ds\right)1/q\times\left(\int\limits_0^y\left(y-t\right)\left(G^{l-1}\left(t\right)\right)^q dt\right)^{1/q}$$

where $C\left(s,t,p\right)=\left(\int\limits_0^s f^p\left(\sigma\right)d\sigma\right)^{1/p}\left(\int\limits_0^t g^p\left(\tau\right)d\tau\right)^{1/p}$.

Proof. From the hypotheses, it is easy to observe that

$$F^h\left(s\right)=h\int\limits_0^s F^{h-1}\left(\sigma\right)f\left(\sigma\right)d\sigma, s\in\left(0,x\right)$$

$$G^l\left(t\right)=l\int\limits_0^t G^{l-1}\left(\tau\right)g\left(\tau\right)d\tau, t\in\left(0,y\right)$$

Therefore

(2) $F^h\left(s\right)G^l\left(t\right)=hl\left(\int\limits_0^s F^{h-1}\left(\sigma\right)f\left(\sigma\right)d\sigma\right)\left(\int\limits_0^t G^{l-1}\left(\tau\right)g\left(\tau\right)d\tau\right)$

On the other hand, according to Holder integral inequality (see [2] P. 154) we have

(3) $\int\limits_0^s F^{h-1}\left(\sigma\right)f\left(\sigma\right)d\sigma\geq\left(\int\limits_0^s\left(F^{h-1}\left(\sigma\right)\right)^q d\sigma\right)^{1/q}\cdot\left(\int\limits_0^s f^p\left(\sigma\right)d\sigma\right)^{1/p}$

(4) $\int\limits_0^t G^{l-1}\left(\tau\right)g\left(\tau\right)d\tau\geq\left(\int\limits_0^t\left(G^{l-1}\left(\tau\right)\right)^q d\tau\right)^{1/q}\cdot\left(\int\limits_0^t g^p\left(\tau\right)d\tau\right)^{1/p}$

By (2), (3) and (4) yield that

$$F^h\left(s\right)G^l\left(t\right)\geq hl\left(\int\limits_0^s f^p\left(\sigma\right)d\sigma\right)^{1/p}\left(\int\limits_0^t g^p\left(\tau\right)d\tau\right)^{1/p}\times$$
$$\times\left(\int\limits_0^s\left(F^{h-1}\left(\sigma\right)\right)^q d\sigma\right)^{1/q}\left(\int\limits_0^t\left(G^{l-1}\left(\tau\right)\right)^q d\tau\right)^{1/q}.$$

Thus $\frac{F^h(s)G^l(t)}{C(s,t,p)}dsdt\geq hl\left(\int\limits_0^s\left(F^{h-1}\left(\sigma\right)\right)^q d\sigma\right)^{1/q}\left(\int\limits_0^t\left(G^{l-1}\left(\tau\right)\right)^q d\tau\right)^{1/q}$.

Integrating over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using special case of Holder integral inequality (we take $f\left(x\right)=1$ in to Holder integral inequality), we observe that: $\int\limits_0^x\int\limits_0^y\frac{F^h(s)G^l(t)}{C(s,t,p)}dsdt\geq$

$$\geq hl\left(\int\limits_0^x\left(\int\limits_0^s\left(F^{h-1}\left(\sigma\right)\right)^q d\sigma\right)^{1/q}ds\right)\times\left(\int\limits_0^y\left(\int\limits_0^t\left(G^{l-1}\left(\tau\right)\right)^q d\tau\right)^{1/q}dt\right)\geq$$

$$\geq hlx^{1/p}\left(\int\limits_0^x\left(\int\limits_0^s\left(F^{h-1}\left(\sigma\right)\right)^q d\sigma\right)ds\right)^{1/q}\cdot xy^{1/p}\left(\int\limits_0^y\left(\int\limits_0^t\left(G^{l-1}\left(\tau\right)\right)^q d\tau\right)dt\right)^{1/q}=$$

$$= hl\,(xy)^{1/p}\left(\int\limits_0^x (x-s)\left(F^{h-1}(s)\right)^q ds\right)^{1/q} \times \left(\int\limits_0^y (y-t)\left(G^{l-1}(t)\right)^q dt\right)^{1/q}.$$

The proof is complete.

This is just a inverse inequality similar to following which was given by B. G. Pachpatte in [1].

$$\int\limits_0^x\int\limits_0^y \frac{F^h(s)\,G^l(t)}{s+t}dsdt \le \frac{1}{2}h\,(xy)^{1/2}\left(\int\limits_0^x (x-s)\left(F^{h-1}(s)\right)^2 ds\right)^{1/2} \times$$

$$\times\left(\int\limits_0^y (y-t)\left(G^{l-1}(t)\right)^2 dt\right)^{1/2}.$$

Theorem 2. *Let f, g, F, G be as in Theorem 1. Let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in (0,x), \tau \in (0,y)$ and define $P(s) = \int\limits_0^s p(\sigma)\,d\sigma$, $Q(t) = \int\limits_0^t q(\tau)\,d\tau$ for $s \in (0,x), t \in (0,y)$ where x, y are positive real numbers and p, q are real numbers and $\frac{1}{p}+\frac{1}{q} = 1, p < 0$ or $0 < p < 1$. Let ϕ and ψ be two real-valued nonnegative, concave, and supermultiplicative functions (f is said to be supermultiplicative function if $f(xy) \ge f(x)\,f(y)\,, x, y \in R_+$) defined on $R_+ = [0, +\infty)$. Then*

(5) $\int\limits_0^x\int\limits_0^y \frac{F^h(s)G^l(t)}{D(s,t,p)}dsdt \ge$

$$\ge L(x,y,p)\left(\int\limits_0^x (x-s)\left(\phi\left(\frac{f(s)}{p(s)}\right)\right)^q ds\right)^{1/q} \times \left(\int\limits_0^y (y-t)\left(\psi\left(\frac{g(t)}{q(t)}\right)\right)^q dt\right)^{1/q},$$

where

$$L(x,y,p) = \left(\int\limits_0^x \left(\frac{\phi(P(s))}{P(s)}\right)^p ds\right)^{1/p}\left(\int\limits_0^y \left(\frac{\psi(g(t))}{q(t)}\right)^p dt\right)^{1/p}$$

and

$$D(s,t,p) = \left(\int\limits_0^s p^p(\sigma)\,d\sigma\right)^{1/p}\left(\int\limits_0^t q^p(\tau)\,d\tau\right)^{1/p}.$$

Proof. From the hypotheses and by using Jensen inequality and Holder integral inequality, it is easy to observe that

(6) $\phi\left(F\left(s\right)\right)=\phi\left(\frac{P(s)\int\limits_0^s p(\sigma)\frac{f(\sigma)}{p(\sigma)}d\sigma}{\int\limits_0^s p(\sigma)d\sigma}\right)\geq$

$\geq\phi\left(P\left(s\right)\right)\phi\left(\frac{\int\limits_0^s p(\sigma)\frac{f(\sigma)}{p(\sigma)}d\sigma}{\int\limits_0^s p(\sigma)d\sigma}\right)\geq\frac{\phi(P(s))}{P(s)}\int\limits_0^s p\left(\sigma\right)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)d\sigma\geq$

$\geq\left(\frac{\phi(P(s))}{P(s)}\right)\left(\int\limits_0^s p\left(\sigma\right)d\sigma\right)^{1/p}\left(\int\limits_0^s\left(\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^q d\sigma\right)^{1/q}$

and similarly,

(7) $\psi\left(G\left(t\right)\right)\geq\left(\frac{\psi(Q(t))}{Q(t)}\right)\left(\int\limits_0^t q^p\left(\tau\right)d\tau\right)^{1/p}\left(\int\limits_0^t\left(\psi\left(\frac{g(\tau)}{g(\tau)}\right)\right)^q d\tau\right)^{1/q}.$

By (6) and (7), we get that:

(8) $\frac{\phi(F(s))\psi(G(t))}{D(s,t,p)}\geq$

$\geq\left(\frac{\phi(P(s))}{P(s)}\right)\left(\frac{\psi(Q(t))}{Q(t)}\right)\left(\int\limits_0^s\left(\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^q d\sigma\right)^{1/q}\times\left(\int\limits_0^t\left(\psi\left(\frac{g(\tau)}{g(\tau)}\right)\right)^q d\tau\right)^{1/q}.$

Integrating two sides of (8) over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using Holder integral inequality, we observe that

$\int\limits_0^x\int\limits_0^y\frac{\phi(F(s))\psi(G(t))}{D(s,t,p)}dsdt\geq\left(\int\limits_0^x\frac{\phi(P(s))}{P(s)}\left(\int\limits_0^s\left(\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^q d\sigma\right)^{1/q}ds\right)\times$

$\times\left(\int\limits_0^y\frac{\psi(Q(t))}{Q(t)}\left(\int\limits_0^t\left(\psi\left(\frac{g(\tau)}{g(\tau)}\right)\right)^q d\tau\right)^{1/q}dt\right)\geq$

$\geq\left(\int\limits_0^x\left(\frac{\phi(P(s))}{P(s)}\right)^p ds\right)^{1/p}\left(\int\limits_0^x\left(\int\limits_0^s\left(\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^q d\sigma\right)ds\right)^{1/q}\times$

$\times\left(\int\limits_0^y\left(\frac{\psi(Q(t))}{Q(t)}\right)^p dt\right)\left(\int\limits_0^y\left(\int\limits_0^t\left(\psi\left(\frac{g(\tau)}{g(\tau)}\right)\right)^q d\tau\right)dt\right)^{1/q}=$

$=L\left(x,y,p\right)\left(\int\limits_0^x\left(x-s\right)\left(\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^q ds\right)^{1/q}\times\left(\int\limits_0^y\left(y-t\right)\left(\psi\left(\frac{g(\tau)}{g(\tau)}\right)\right)^q dt\right)^{1/q}.$

This is just another inverse inequality similar to following inequality which was given by B. G. Pachpatte in [1].

(9) $\int\limits_0^x\int\limits_0^y\frac{\phi(F(s))\psi(G(t))}{s+t}dsdt\leq L\left(x,y\right)\left(\int\limits_0^x\left(x-s\right)\left(p\left(s\right)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right)^2 ds\right)^{1/2}\times$

$\times\left(\int\limits_0^y\left(y-t\right)\left(g\left(t\right)\psi\left(\frac{g(\tau)}{g(\tau)}\right)\right)^2 dt\right)^{1/2}$, where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\phi(P(s))}{P(s)}\right)^2 ds\right)^{1/2}\left(\int_0^y\left(\frac{\psi(Q(t))}{Q(t)}\right)^2 dt\right)^{1/2}.$$

Acknowledgment. We wish to express my gratitude to professor Li Wenrong for his valuable help in writing this paper.

References

[1] B. G. Pachpatte, *On some new inequalities similar to Hilbert's inequality* , J. Math. Anal. Appl.,**226**(1)(1998), 166-179.

[2] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

[3] D. S. Mitrinovic, *Analytic inequalities*, Springer-Verlag, 1970.

[4] Gao Minzhe, *On Hilbert inequality and its applications*, J. Math. Anal. Appl., **212**(1997), 316-323.

[5] Hu Ke: On Hilbert inequality and its application [J], Advances Mathematics, 1993, 22(2): 160-163.

[6] Yang Bicheng and L. Debnath, *Generalizations of Hardy integral inequalities*, Internat. J. Math. and Math. Sci. **22**(3)(1999), 535-542.

Department of Mathematics, Binzhou Teachers College, Shandong 256604,
China
e-mail: chjzhao@163.com

Str. Harmanului 6, 505600 Sacele-Négyfalu, Jud. Brasov, Romania
benczemihaly@yahoo.com