An. Şt. Univ. Ovidius Constanţa Vol. 15(2), 2007, 29–34

Division Property of the Momentum Map

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Abstract

In this paper we discuss various situations when the momentum map has the division property.

1. Introduction

Momentum maps are at the centre of many geometrical facts that are useful in variety of fields of both pure and applied Mathematics. Also these maps are very useful in Physics and Engineering applications. Here we look at the division property of the momentum maps. In this paper we improve certain results of Y.Karshon and E.Lerman. We generalize a result on division property of momentum map by replacing the compactness of Lie group with proper and effective action. Also improved versions in the cases of torus action and compact connected Lie group action are given here.

2. Division property

Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a Hamiltonian action of a compact connected Lie group G on a symplectic manifold (M, ω). Y.Karshon and E.Lerman in [4] proved that if J is equivariant with respect to the given action of G on M and the coadjoint action on $\mathcal{G}^*$, then the centralizer of the algebra of G-invariant functions in the Poisson algebra on M is the set of

Key Words: Momentum map; Division property.
Mathematics Subject Classification: 58F05.
Received: August, 2007
Accepted: October, 2007
*Research supported by the University Grants Commission of India under the Faculty Improvement Programme

smooth functions that are locally constant on the level sets of the momentum map. As a corollary of the Marle-Guillemin-Sternberg normal form for proper actions, one can prove that any two points in a connected component of a level set of J can be joined by a piece-wise smooth curve that lies in the level set. Using this idea we can reformulate this result by replacing compactness of Lie group using proper and effective action.

Theorem 2.1. *Let (M, ω) be a symplectic manifold and G be a Lie group acting properly and canonically on it. Suppose that this action is Hamiltonian with an associated moment map $J : M \longrightarrow \mathcal{G}^*$. Assume that J be equivariant with respect to the given action of G on M and the coadjoint action on $\mathcal{G}^*$. Then the centralizer of the algebra of G-invariant functions in the Poisson algebra on M is the set of smooth functions that are locally constant on the level sets of the momentum map.*

Definition 2.2. Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a Hamiltonian action of a connected Lie group G on a symplectic manifold (M, ω). Pull backs by J of smooth functions on $\mathcal{G}^*$ are called *collective functions*. They form Poisson subalgebra of the algebra of smooth functions on M. A collective function is clearly constant on the level sets of the momentum map.

Corollary 2.3. *Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a proper Hamiltonian action of a connected Lie group G on a symplectic manifold (M, ω). The algebra of collective functions and the algebra of invariant functions are mutual centralizers in the Poisson algebra $C^\infty(M)$ if and only if every smooth function on M that is locally constant on the level sets of the momentum map is collective.*

Definition 2.4. A smooth map $J : M \longrightarrow N$ between two smooth manifolds has the *division property* if any smooth function on M that is locally constant on the level sets of J is the pull back via J of a smooth function on N.

The Corollary 2.3 can be restated as follows:

Corollary 2.5. *Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a proper Hamiltonian action of a connected Lie group G on a symplectic manifold (M, ω). The algebra of collective functions and the algebra of invariant functions are mutual centralizers in the Poisson algebra $C^\infty(M)$ if and only if the momentum map$J$ has division property.*

Definition 2.6. Let $J : M \longrightarrow N$ be a smooth map between two smooth manifolds. A smooth function f on M is a *formal pullback* with respect to J if for every point y in the image $J(M)$ there exists a function, φ on N such that $f - J^*\varphi$ is flat at all the points of $J^{-1}(y)$.

Since the pull back of functions induces a well defined pull back of Taylor's series, being a formal pull back with respect to a smooth function $J : M \longrightarrow N$ if and only if for every $y \in N$ there exists a power series φ on N, centered at y, such that for all x in the level set $J^{-1}(y)$, the power series of f at x is the pull back of the power series φ. Every formal pull back with respect to J is constant on the level sets of J. For,if $f - J^*\varphi$ is flat, $f(x) = \varphi(y)$ for all $x \in J^{-1}(y)$.

Definition 2.7. Let $\psi : A \longrightarrow B$ is *semi-proper* if for every compact set $L \subset B$ there is a compact set $K \subset A$ such that $\psi(K) = L \cap \psi(A)$.

Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a Hamiltonian action of a compact Lie group G on a connected symplectic manifold (M, ω). If this map J is proper, Y.Karshon and E.Lerman in [4] proved that every formal pull back with respect to J is a collective function. We can generalize this theorem by replacing the compactness condition on the Lie group by proper and effective action.

Theorem 2.8. *Let G be a connected abelian Lie group acting properly and effectively on a connected symplectic manifold (M, ω). Let $J : M \longrightarrow \mathcal{G}^*$ be a proper momentum map associated to this action. Then J has the division property if and only if every smooth function on M that is locally constant on the level sets of J is a formal pull back with respect to J.*

Proof. Let x be a point in M, and let G_α be the stabilizer of its image , $\alpha = J(m) \in \mathcal{G}^*$ under the coadjoint action. Since the action is effective and G is a connected abelian Lie group, the G-orbits are isotropic [1]. So α is fixed under the coadjoint action of G, for every $\alpha \in \mathcal{G}^*$. Since α is fixed, the translation $J - \alpha$ of the moment map by $-\alpha$ is still a momentum map. So, without loss of generality , we can assume that $\alpha = 0$.

For the proper action, by Theorem 7.5.5 of [6], we have a neighborhood of an isotropic orbit $G.x$,

$$Y = G \times_{G_x} (\mathcal{G}_x^o \oplus V),$$

where G_x is the stabilizer of x , $\mathcal{G}_x$ is its Lie algebra , $\mathcal{G}_x^0$ is the annihilator of $\mathcal{G}_x$ in $\mathcal{G}^*$, and V is the symplectic slice at x. The action of G on Y is Hamiltonian with a moment map $J_Y : Y \longrightarrow \mathcal{G}^*$ given by

$$J_Y([g, \eta, \upsilon]) = Ad^*(g)(\eta + i(J_V(\upsilon))),$$

where Ad^* is the coadjoint action, and $J_V : V \longrightarrow \mathcal{G}_x^*$ is the quadratic momentum map for the slice representation of G_x and i is a G_x-equivariant embedding of $\mathcal{G}_x^*$ in $\mathcal{G}^*$. Moreover there exists a neighborhood U_x the orbit $G.x$ in M and an equivariant embedding $i : U_x \longrightarrow Y$, of U_x onto a neighborhood of the zero section of the bundle $Y \longrightarrow G/G_x$, such that $J = J_Y \circ i$.

As a consequence of the normal form the image under the momentum map of a small neighborhood of an orbit $G.x$ does not change as x varies along a connected component of the level set $J^{-1}(J(x))$. Also this image is the intersection of the cone $J_Y(Y)$ with a neighborhood of the origin in $\mathcal{G}^*$. Note that the hypothesis that the momentum map is proper can be replaced by the hypothesis that it is semi-proper as a map into some open subset of $\mathcal{G}^*$ and that its level sets are connected. So we can choose a neighborhood W_x of the origin in $\mathcal{G}^*$ and shrink the neighborhood U_x of $G.x$ so that

$J(U_x) = J(M) \cap W_x = J_Y(Y) \cap W_x$. (1)

The map J_Y is analytic with respect to the natural real analytic structures of the model Y and of the vector space $\mathcal{G}^*$. If we endow U_x with the real analytic structure induced by its embedding , i, into Y, then

the restriction $J\mid_{U_x}: U_x \longrightarrow W_x$ is a real analytic map. (2)

Consider the action of $\Re_+$ on Y given by $\lambda.[g, \eta, \upsilon] = [g, \lambda\eta, \sqrt{\lambda}\upsilon]$. The map $J_Y : Y \longrightarrow \mathcal{G}^*$ is homogeneous of degree one with respect to the action of $\Re_+$. After possibly shrinking U_x and W_x further, we can assume that the open set $i(U_x) \subseteq Y$ is preserved under multiplication by any $\lambda < 1$;for such λ we define $\lambda : U_x \longrightarrow U_x$ by $i(\lambda.m) = \lambda.i(m)$. Let K be a compact subset of the open set W_x. Then there exist a positive number $\lambda < 1$ such that K is contained in λW_x. By homogeneity $K \cap J(U_x) \subset J(\lambda.U_x)$. Then $L := closure(\lambda.U_x) \cap J^{-1}K$ is a compact subset of U_x whose image is $K \cap J(U_x)$. Thus

the restriction $J\mid_{U_x}: U_x \longrightarrow W_x$ is semi- proper. (3)

Since the map J_V is algebraic, its image $J_V(V)$ is a semi algebraic subset of $\mathcal{G}_x^*$. Furthermore, since $Ad^*(G) \subseteq GL(\mathcal{G}^*)$ is algebraic, the set $J_Y(Y) = Ad^*(G)(\mathcal{G}_x)^0 \times J_V(V))$ a semi algebraic subset of $\mathcal{G}^*$. Restricting to the open subset W_x, we see that

$J(U_x) = J_Y(Y) \cap W_x$ is a semi-analytic subset of W_x. (4)

Thus there exist a neighborhood U_x of the orbit $G.x$ in M and a neighborhood W_x of the point $J(x)$ in $\mathcal{G}^*$ with the following properties.

1. $J(U_x) = J(M) \cap W_x$.
2. The restriction $J\mid_{U_x}: U_x \longrightarrow W_x$ is semi-proper.
3. There exist real analytic structures on U_x and on W_x, compatible with their smooth structures, such that the restriction $J\mid_{U_x}: U_x \longrightarrow W_x$ is a real analytic map and the image $J(U_x)$ is a semi analytic subset of W_x.

Moreover the neighborhoods U_x and W_x can be chosen to be arbitrarily small, that is, can be chosen to be contained in any given neighborhoods U' of $G.x$ and W' of $J(x)$.

Let N be an open subset of $\mathcal{G}^*$ containing the moment image $J(M)$ with the property that the momentum map$J : M \longrightarrow N$ is semi-proper. Also the

image of any semi-proper map is closed. Therefore
$J(M)$ is closed subset of N. (5)

Then we can prove that the set of pull backs by the map J coincides with the set of formal pull backs with respect to J.

Clearly every pull back is a formal pull back. Conversely, let $f \in C^{\infty}(M)$ be a formal pull back with respect to J. Let x be a point in M, and let U_x and W_x be as obtained above. Since f is a formal pull back with respect to J, its restriction $f\mid_{U_x}$ is a formal pull back with respect to the map $J\mid_{U_x}: U_x \longrightarrow W_x$.

Bierstone and Milman in [2] proved that, if M and N are real analytic manifolds. Let $J : M \longrightarrow N$ be a real analytic map that is semi-proper and whose image $J(M)$ is semi-analytic. Then a function f is a formal pull back with respect to J if and only if it is the pull back via J of a smooth function on N. So we can apply this theorem to the map $J\mid_{U_x}$ because of conditions (2) and (4). Hence there exists a smooth function φ_x on W_x such that $f = \varphi_x \circ J$ on U_x. This equality holds on all $J^{-1}(J(U_x))$ because f, being formal pull back with respect to J, is constant on the level sets of J.

Condition (1) implies that $J^{-1}(J(U_x)) = J^{-1}(W_x)$ so, $f = \varphi_x \circ J$ on all of $J^{-1}(W_x)$. The open sets W_x together with the complement of the image $J(M)$ form an open cover of N. Using a partition of unity subordinate to this cover we piece together the functions φ_x to form a function φ on N such that $f = \varphi \circ J$.

Then taking $N = \mathcal{G}^*$, we have the theorem. •

Remark 2.9. Let $J_T : M \longrightarrow \mathcal{T}^*$ be a momentum map associated to a Hamiltonian action of a torus T on a connected symplectic manifold (M, ω). If this map J_T is proper, E.Lerman in [L] proved that it has the division property. We recall the result in [3] that for a Hamiltonian torus action if the associated momentum map J_T is closed, then the level sets of J_T are connected. So we can prove that torus action has division property if J_T is closed and semi-proper.

Theorem 2.10. *Let M be a paracompact connected symplectic manifold on which a torus $\mathcal{T}$ acts in a Hamiltonian fashion. If the associated momentum map $J_{\mathcal{T}}$ is closed and semi-proper as a map into some open subset of $\mathcal{T}^*$, then J has the division property.*

Remark 2.11. The elements of $\mathcal{G}^*$ whose stabilizers under the coadjoint action of G are tori is denoted by $\mathcal{G}^*_{reg}$.

Let $J_G : M \longrightarrow \mathcal{G}^*$ be a proper momentum map associated to a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω). Suppose the image $J(M)$ is contained the $\mathcal{G}^*_{reg}$. Then Lerman in [5] proved that J has the division property. We give a generalization of this theorem using Theorem 3.19 of [3], which states that M be a paracompact connected symplectic Hamiltonian G-manifold with G a compact connected Lie group with the associated momentum map J_G is closed then the level sets of J_G are connected.

Theorem 2.12. *Let M be a paracompact connected symplectic Hamiltonian G-manifold with G a compact connected Lie group. If the associated momentum map J is closed and semi-proper as a map into some open subset of $\mathcal{G}^*$, then J has the division property, if the image $J(M)$ is contained the $\mathcal{G}^*_{reg}$.*

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