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REGULARITY FOR CERTAIN CLASSES OF MONOMIAL IDEALS*

Mircea Cimpoeaş

Abstract

We introduce a new class of monomial ideals, called strong Borel type ideals, and we compute the Mumford-Castelnouvo regularity for principal strong Borel type ideals. Also, we describe the **d**-fixed ideals generated by powers of variables and we compute their regularity.

Introduction.

Let K be an infinite field, and let $S = K[x_1, ..., x_n], n \geq 2$ be the polynomial ring over K. Bayer and Stillman [2] note that a Borel fixed ideal I satisfies the following property $(I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty)$ for all $j = 1, \ldots, n$. Herzog, Popescu and Vladoiu state that a monomial ideal is of Borel type if it fulfill the previous condition. We mention that this concept appears also in [3, Definition 1.3] as the so called weakly stable ideal. In fact, Herzog, Popescu and Vladoiu notice that a monomial ideal I is of Borel type, if and only if for any monomial $u \in I$ and for any $1 \leq j < i \leq n$, there exists an integer $t > 0$ such that $x_j^t u / x_i^{\nu_i(u)} \in I$, where $\nu_i(u) > 0$ is the exponent of x_i in u. (See [7, Proposition 1.2].) This property suggest us to define the so called ideals of strong Borel type (Definition 1.1), or simply, (SBT)-ideals. In the first section, we give the explicit form of a principal (SBT)-ideal (Lemma 1.4) and we compute its regularity (Theorem 1.6).

Let $\mathbf{d} : 1 = d_0 | d_1 | \cdots | d_s$ be a strictly increasing sequence of positive integers. We say that $\mathbf{d}$ is a $\mathbf{d}$-sequence. In [4] it was proved that for any

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$a \in \mathbb{N}$ there exists a unique sequence of positive integers $a_0, a_1, \dots, a_s$ such that: $a = \sum_{t=0}^{s} a_t d_t$ and $0 \leq a_t < \frac{d_{t+1}}{d_t}$, for any $0 \leq t < s$. The decomposition $a = \sum_{t=0}^{s} a_t d_t$ is called the **d**-decomposition of a. In particular, if $d_t = p^t$ we get the p-adic decomposition of a. Let $a, b \in \mathbb{N}$ and consider the decompositions $a = \sum_{t=0}^{s} a_t d_t$ and $b = \sum_{t=0}^{s} b_t d_t$. We say that $a \leq_{\mathbf{d}} b$ if $a_t \leq b_t$ for any $0 \leq t \leq s$. We say that a monomial ideal $I \subset S$ is **d**-fixed, if for any monomial $u \in I$ and for any indices $1 \leq j < i \leq n$, if $t \leq_{\mathbf{d}} \nu_i(u)$ then $u \cdot x_j^t / x_i^t \in I$ (see [4, Definition 1.4]).

In [4], it was proved a formula for the regularity of a principal **d**-fixed ideal, i.e the smallest **d**-fixed ideal which contains a given monomial $u \in S$. This formula generalizes the Pardue's formula for the regularity of a principal p-Borel ideal, proved in [1] and [8], and later in [7]. In the section 2, we describe the **d**-fixed ideals generated by powers of variables (Proposition 2.2) and we give a formula for their regularity (Corollary 2.8).

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1 Monomial ideals of strong Borel type.

Let K be an infinite field, and let $S = K[x_1, ..., x_n], n \geq 2$, be the polynomial ring over K.

Definition 1.1. *We say that a monomial ideal $I \subset S$ is of strong Borel type (SBT) if for any monomial $u \in I$ and for any $1 \leq j < i \leq n$, there exists an integer $0 \leq t \leq \nu_i(u)$ such that $x_j^t u / x_i^{\nu_i(u)} \in I$, where $\nu_i(u) > 0$ is the exponent of x_i in u.*

Remark 1.2. *Obviously, an ideal of strong Borel type is also an ideal of Borel type, but the converse is not true. Take for instance $I = (x_1^3, x_2^2) \subset K[x_1, x_2]$.*

The sum of two ideals of (SBT) is still an ideal of (SBT). The same is true for an intersection or a product of two ideals of (SBT).

Definition 1.3. *Let $\mathcal{A} \subset S$ be a set of monomials. We say that I is the (SBT)-ideal generated by $\mathcal{A}$, if I is the smallest, with respect to inclusion, ideal of (SBT) containing $\mathcal{A}$. We write $I = SBT(\mathcal{A})$.*

In particular, if $\mathcal{A} = \{u\}$, where $u \in S$ is a monomial, we say that I is the principal (SBT)-ideal generated by u, and we write $I = SBT(u)$.

Lemma 1.4. *Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be some integers, $\alpha_1, \dots, \alpha_r$ be some positive integers and $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r} \in S$. Then, the principal*

(SBT)-ideal generated by u, is:

$$I = SBT(u) = \prod_{q=1}^{r} (\mathbf{m}_q^{[\alpha_q]}),$$

where

$$\mathbf{m}_q = \{x_1, \ldots, x_{i_q}\} \textit{ and } \mathbf{m}_q^{[\alpha_q]} = \{x_1^{\alpha_q}, \ldots, x_{i_q}^{\alpha_q}\}.$$

Proof. Denote $I' = \prod_{q=1}^{r}(\mathbf{m}_q^{[\alpha_q]})$. If v is a minimal monomial generator of I', then $v = x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_r}^{\alpha_r}$, for some $1 \leq j_q \leq i_q$, where $1 \leq q \leq r$.
Since

$$v = \frac{x_{j_r}^{\alpha_r}}{x_{i_r}^{\alpha_r}} \cdots \frac{x_{j_2}^{\alpha_2}}{x_{i_2}^{\alpha_2}} \cdot \frac{x_{j_1}^{\alpha_1}}{x_{i_1}^{\alpha_1}} u,$$

and I is of (SBT) it follows that $v \in I$ and thus $I' \subseteq I$. For the converse, simply notice that I' is itself an (SBT)-ideal. □

Remark 1.5. *For any monomial ideal $I \subset S$, we denote $m(I) = \max\{m(u) : u \in G(I)\}$, where $G(I)$ is the set of the minimal generators of I and $m(u) = \max\{i : x_i | u\}$. Also, if M is a graded S-module of finite length, we denote $s(M) = \max\{t : M_t \neq 0\}$.*

Let $I \subset S$ be a Borel type ideal. In [7], it is defined a chains of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ as follows. We let $I_0 = I$. Suppose I_ℓ is already defined. If $I_\ell = S$ then the chain ends. Otherwise, we let $n_\ell = m(I_\ell)$ and set $I_{\ell+1} = (I_\ell : x_{n_\ell}^{\infty})$. Notice that $r \leq n$, since $n_\ell > n_{\ell+1}$ for all $0 \leq \ell < r$. The chain $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ is called the sequential chain *of I. [7, Corollary 2.5] states that*

$$(1) \quad I_{\ell+1}/I_\ell \cong (J_\ell^{sat}/J_\ell)[x_{n_\ell+1}, \ldots, x_n],$$

for all $0 \leq \ell < r$, where $J_\ell \subset S_\ell = K[x_1, \ldots, x_{n_\ell}]$ is the ideal generated by $G(I_\ell)$. Also, [7, Corollary 2.5] gives a formula for the regularity of I, more precisely,

$$(2) \quad reg(I) = \max\{s(J_0^{sat}/J_0), s(J_1^{sat}/J_1), \cdots, s(J_{r-1}^{sat}/J_{r-1})\} + 1.$$

Our next goal is to give a formula for the regularity of a principal (SBT)-ideal. In order to do it, we shall use the previous remark.

Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be some integers, $\alpha_1, \ldots, \alpha_r$ be some positive integers and $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r} \in S$. For each $1 \leq q \leq r$, $1 \leq f \leq q$ with $\alpha_f \leq \alpha_q$ and $1 \leq j \leq i_q$, we define the numbers:

$$\chi_{qj}^{(f)} := \begin{cases} \alpha_j + \alpha_q - 1, & if\ j < q \text{ and } \alpha_j \geq \alpha_f \\ \alpha_f - 1, & otherwise \end{cases},$$

$$\chi_q^{(f)} := \sum_{j=1}^{i_q} \chi_{qj}^{(f)} \text{ and } \chi_q = \max_f \chi_q^{(f)}.$$

Theorem 1.6. *With the above notations, we have* $reg(SBT(u)) = \max_{q=1}^{r} \chi_q + 1$.

Proof. Firstly, we describe the sequential chain of I.
Since $I_r := I = \prod_{q=1}^{r}(\mathbf{m}_q^{[\alpha_q]})$, it follows that $I_{r-1} := (I_r : x_{i_r}^{\infty}) = \prod_{q=1}^{r-1}(\mathbf{m}_q^{[\alpha_q]})$. Analogously, we get $I_q := (I_{q+1} : x_{i_{q+1}}^{\infty}) = \prod_{e=1}^{q}(\mathbf{m}_e^{[\alpha_e]})$, for all $0 \leq q < r$. Therefore, the sequential chain of I is

$$I = I_r \subset I_{r-1} \subset \cdots \subset I_1 \subset I_0 = S.$$

Let J_q be the ideal of $S_q = K[x_1, \ldots, x_{i_q}]$ generated by $G(I_q)$, for $1 \leq q \leq r$. Denoting $s_q = s(J_q^{sat}/J_q)$, (2) from Remark 1.5 implies $reg(I) = \max\{s_q : 1 \leq q \leq r\}$, so, in order to compute the regularity of I, we must determine the numbers s_q. We claim that $s_q = \chi_q$.

First of all, note that $J_q = I_q \cap S_q$ and $J_q^{sat} = I_{q-1} \cap S_q$. Let $1 \leq f \leq q$ with $\alpha_f \leq \alpha_q$ and $w = x_1^{\chi_{q1}^{(f)}} \cdots x_{i_q}^{\chi_{q,i_q}^{(f)}}$. Since $\chi_{qe}^{(f)} \geq \alpha_e$ for any $1 \leq e \leq q-1$ we get $x_1^{\chi_{q1}^{(f)}} \cdots x_{q-1}^{\chi_{q,q-1}^{(f)}} \in J_q^{sat} = \prod_{e=1}^{q-1}(\mathbf{m}_e^{[\alpha_e]})S_q$, therefore $w \in J_q^{sat}$. On the other hand, one can easily see that $w \notin J_q$, so w is a nonzero element in J_q^{sat}/J_q with $\deg(w) = \chi_q$, thus $s_q \geq \chi_q$.

In order to prove the converse inequality, we consider a monomial $u \in J_q^{sat}$ with $\deg(u) \geq \chi_q + 1$ and we show that $u \in J_q$. Assume, by contradiction, that $u \notin J_q$. Since $u \in J_q^{sat}$, it follows that $u = x_{j_1}^{\alpha_1} \cdots x_{j_{q-1}}^{\alpha_{q-1}} \cdot x_1^{\beta_1} \cdots x_{i_q}^{\beta_{i_q}}$, where $1 \leq j_e \leq i_e$ for $1 \leq e \leq q-1$ and $\beta_1 + \cdots + \beta_{i_q} \geq \chi_q - \sum_{e=1}^{q-1}\alpha_e$. Let $A = \{1, \ldots, i_q\} \setminus \{j_1, \ldots, j_{q-1}\}$. Since $u \notin J_q$ and $x_{j_1}^{\alpha_1} \cdots x_{j_{q-1}}^{\alpha_{q-1}} \in J_q^{sat}$ it follows $\beta_j \leq \alpha_q - 1$ for all $j \in A$.

Write $\{1, \ldots, q-1\} = \cup_{i=1}^{m} E_i$, where $E_i = \{e_{i1}, \ldots, e_{ik_i}\}$, such that $j_{e_{ik}} = j_{e_i}$ for all $1 \leq k \leq k_i$ and $E_i \cap E_{i'} = \emptyset$ whenever $i \neq i'$. With these notations,

$$u = x_{j_{e_1}}^{\alpha_{e_{11}} + \cdots + \alpha_{e_{1k_1}} + \beta_{j_{e_1}}} \cdots x_{j_{e_m}}^{\alpha_{e_{m1}} + \cdots + \alpha_{e_{mk_m}} + \beta_{j_{e_m}}} \cdot \prod_{j \in A} x_j^{\beta_j}.$$

Let $1 \leq f \leq q$ be such that $\alpha_f \leq \alpha_q$, $\beta_j < \alpha_f$ for all $j \in A$ and α_f be the largest integer among all the $\alpha_{f'}$, with f' satisfying the above conditions. Suppose that there exist some $1 \leq i \leq m$ and $1 \leq k \leq k_i$ such that $\alpha_{e_{ik}} < \alpha_q$. It follows that $\beta_{j_{e_i}} \leq \alpha_f - \alpha_{e_{ik}} - 1$, otherwise $u \in J_q$. One can immediately conclude that $\sum_{e=1}^{q-1}\alpha_e + \sum_{j=1}^{i_q}\beta_j \leq \chi_q^{(f)}$. □

Example 1.7. Let $u = x_2^6 x_3^7 \in S = K[x_1, x_2, x_3]$. From Lemma 1.4 it follows that $I = SBT(u) = (x_1^6, x_2^6)(x_1^7, x_2^7, x_3^7)$. With the notations of 1.5 and 1.6, we have $J_1 = (x_1^6, x_2^6) \subset K[x_1, x_2]$ and $J_2 = I$. Also, $J_1^{sat} = K[x_1, x_2]$ and $J_2^{sat} = (x_1^6, x_2^6) \subset S$. Obviously, $\chi_1 = \chi_1^{(1)} = 2 \cdot 5 = 10$, i.e. $s(J_1^{sat}/J_1) = s(K[x_1, x_2]/(x_1^6, x_2^6)) = 10$. We have $\chi_2^{(1)} = (6 + 7 - 1) + 2 \cdot 5 = 23$ and $\chi_2^{(2)} = 3 \cdot 6 = 18$, therefore $\chi_2 = 23$ and thus $reg(I) = \max\{10, 23\} + 1 = 24$.

In the end of this section, we mention the following result, which generalizes a result of Eisenbud-Reeves-Totaro (see [6, Proposition 12]).

Proposition 1.8. *[5, Corollary 8] If I is a Borel type ideal, then*

$$reg(I) = \min\{e : \ e \geq \deg(I), \ I_{\geq e} \text{ is stable}\},$$

where $\deg(I)$ is the maximal degree of a minimal monomial generator of I.

In particular, this holds for (SBT)-ideals, and thus we get the following corollary.

Corollary 1.9. *With the notations of Theorem 1.5, if $I = SBT(u)$ and $e \geq \max_{q=1}^{r} \chi_q + 1$, then $I_{\geq e}$ is stable.*

Remark 1.10. *Note also that the regularity of an (SBT)-ideal, $I \subset S$, is upper bounded by $n(\deg(I) - 1) + 1$, (see [9, Theorem 2.2]). In fact, $\deg(I)$ is the maximum degree of a minimal generator of I as an (SBT)-ideal!*

2 d-fixed ideals generated by powers of variables.

Let us fix some notations. Let $u_1, \ldots, u_m \in S$ be some monomials. We say that I is the **d**-fixed ideal generated by $u_1, \ldots, u_m$, if I is the smallest **d**-fixed ideal , w.r.t inclusion, which contains $u_1, \ldots, u_m$, and we write

$$I = < u_1, \ldots, u_m >_{\mathbf{d}} .$$

In particular, if $m = 1$, we say that I is the principal **d**-fixed ideal generated by $u = u_1$ and we write $I = < u >_{\mathbf{d}}$.

In the case when I is a principal **d**-fixed ideal , [4, Theorem 3.1] gives a formula for the Castelnuovo-Mumford regularity of I. Using similar techniques as in [4], we shall compute the regularity for **d**-fixed ideals generated by powers of variables. We recall some results proved in [4] which are useful. Let α be a positive integer and let $I = < x_n^{\alpha} >_{\mathbf{d}} \subset S = K[x_1, \ldots, x_n]$. Suppose $\alpha = \sum_{t=0}^{s} \alpha_t d_t$ with $\alpha_s \neq 0$. Then:

- $I = \prod_{t=0}^{s} (\mathbf{m}^{[d_t]})^{\alpha_t}$, where $\mathbf{m} = \{x_1, \dots, x_n\}$ and $\mathbf{m}^{[d]} = \{x_1^d, \dots, x_n^d\}$ [4, 1.6].
- $Soc(S/I) = (J+I)/I$, with

$$J = \sum_{t=0}^{s} (x_1 \cdots x_n)^{d_t - 1} (\mathbf{m}^{[d_t]})^{\alpha_t - 1} \prod_{j>t} (\mathbf{m}^{[d_j]})^{\alpha_j} [4, 2.1].$$

- $reg(I) = \max\{e : ((J+I)/I)_e \neq 0\} = \alpha_s d_s + (n-1)(d_s - 1)$ (see [4, 3.1]).
- If $e \geq reg(I)$ then $I_{\geq e}$ is stable (see [4, 3.6] or apply Proposition 1.8, since any d-fixed ideal is of Borel type, see [4, 1.11]).

Lemma 2.1. *If $1 \leq j \leq j' \leq n$ and $\alpha \geq \beta$ are positive integers, then $< x_j^{\alpha} > \subset < x_{j'}^{\beta} >$.*

Proof. Indeed, using [4, 1.7] it is enough to notice that $< x_j^{\alpha} > \subset < x_{j'}^{\alpha} >$, since $x_j^{\alpha} \in < x_{j'}^{\alpha} >$. □

Our next goal is to give the set of the minimal generators of a $\mathbf{d}$-fixed ideal generated by some powers of variables. Using the previous lemma, we had reduced to the next case:

Proposition 2.2. *Let $n \geq 2$ and let $1 \leq i_1 < i_2 < \cdots < i_r = n$ be some integers. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ be some positive integers. Then*

$$I = < x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}, \dots, x_{i_r}^{\alpha_r} >_{\mathbf{d}} = \sum_{q=1}^{r} I^{(q)},$$

with

$$I^{(q)} = \sum_{\substack{\gamma_1, \dots, \gamma_q \leq_{\mathbf{d}} \alpha_q, \\ \gamma_1 + \cdots + \gamma_i < \alpha_i,\ for\ i<q \\ \gamma_1 + \cdots + \gamma_i <_d \alpha_q,\ for\ i<q \\ \gamma_1 + \cdots + \gamma_q = \alpha_q}} \prod_{e=1}^{q} \prod_{t=0}^{s} (\mathbf{n}_e^{[d_t]})^{\gamma_{et}},$$

where $\mathbf{n}_e = \{x_{i_{e-1}+1}, \dots, x_{i_e}\}$, $\mathbf{n}_e^{[d_t]} = \{x_{i_{e-1}+1}^{d_t}, \dots, x_{i_e}^{d_t}\}$, $i_0 = 0$ and $\gamma_e = \sum_{t=0}^{s} \gamma_{et} d_t$.

Proof. Let $\mathbf{m}_q = \{x_1, \dots, x_{i_q}\}$ for $1 \leq q \leq r$. Obviously, $\mathbf{n}_q = \mathbf{m}_q \setminus \mathbf{m}_{q-1}$ for $q > 1$ and $\mathbf{m}_1 = \mathbf{n}_1$. Using the simple fact that I is the sum of principal

d-fixed ideals generated by the **d**-generators of I together with [4, Proposition 1.6], we get:

$$I = \sum_{q=1}^{r} \prod_{t=0}^{s} (\mathbf{m}_q^{[d_t]})^{\alpha_{qt}},\ where\ \alpha_q = \sum_{t=0}^{s} \alpha_{qt} d_t.$$

Denote $S_q = K[x_1, \dots, x_{i_q}]$ for $1 \leq q \leq r$. In order to obtain the required formula, we use induction on $r \geq 1$, the case $r = 1$ being obvious. Let $r > 1$ and assume that the assertion is true for $r-1$, i.e

$$I' = < x_{i_1}^{\alpha_1}, \dots, x_{i_{r-1}}^{\alpha_{r-1}} >_{\mathbf{d}} =$$

$$= \sum_{q=1}^{r-1} \sum_{\substack{\gamma_1, \dots, \gamma_q \leq_{\mathbf{d}} \alpha_q, \\ \gamma_1 + \dots + \gamma_i < \alpha_i,\ for\ i<q \\ \gamma_1 + \dots + \gamma_i <_d \alpha_q,\ for\ i<q \\ \gamma_1 + \dots + \gamma_q = \alpha_q}} \prod_{e=1}^{q} \prod_{t=0}^{s} (\mathbf{n}_e^{[d_t]})^{\gamma_{et}} \subset S_{r-1}.$$

Obviously, $I = I'S + < x_n^{\alpha_r} >_{\mathbf{d}} = I'S + \prod_{t=0}^{s} (\mathbf{m}_r^{[d_t]})^{\alpha_{rt}}$. Also, $I'S$ and I' have the same set of minimal generators and none of the minimal generators of $I'S$ is in $I^{(r)}$. But, a minimal generator of $< x_n^{\alpha_r} >_{\mathbf{d}}$ is of the form $w = \prod_{t=0}^{s} \prod_{j=1}^{n} x_j^{\lambda_{tj} d_t}$ with $0 \leq \lambda_{tj}$ and $\sum_{j=1}^{n} \lambda_{tj} = \alpha_{rt}$. Suppose $w \notin I'S$. In order to complete the proof, we shall show that $w \in I^{(r)}$. Let $v_q = \prod_{t=0}^{s} \prod_{j=i_{q-1}+1}^{i_q} x_j^{\lambda_{tj} d_t}$ and let $w_q = \prod_{e=1}^{q} v_e$. Obviously, $w = v_1 \cdots v_r = w_r$. Since $w \notin I'$ it follows that $w_q \notin I^{(q)}$ for any $1 \leq q \leq r-1$. But $w_q \notin I^{(q)}$ implies $(*)$ $\sum_{t=0}^{s} \sum_{j=1}^{i_q} \lambda_{tj} d_t < \alpha_q$, otherwise $w_q \in < x_{i_q}^{\alpha_q} S_q >_{\mathbf{d}} S_{r-1} \subset I'$ and thus $w \in I'$, a contradiction. We choose $\gamma_e = \sum_{t=0}^{s} \sum_{j=i_{e-1}+1}^{i_e} \lambda_{tj} d_t$ for $1 \leq e \leq r$. For $1 \leq q < r$, the inequality $(*)$ implies $\gamma_1 + \dots + \gamma_q < \alpha_q$. On the other hand, it is obvious that $\gamma_1 + \dots + \gamma_e \leq_d \alpha_r$ for any $1 \leq e \leq r$ and $\gamma_1 + \dots + \gamma_r = \alpha_r$. Thus $w \in I^{(r)}$ as required. □

Example 2.3. *Let* $\mathbf{d} : 1|2|4|12$ *and let* $I = < x_2^7, x_3^{10}, x_5^{17} >_{\mathbf{d}} \subset K[x_1, \dots, x_5]$. *We have* $7 = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 4$, $10 = 1 \cdot 2 + 2 \cdot 4$, $17 = 1 \cdot 1 + 1 \cdot 4 + 1 \cdot 12$. *We have*

$$I^{(1)} = < x_2^7 >_{\mathbf{d}} = (x_1, x_2)(x_1^2, x_2^2)(x_1^4, x_2^4).$$

In order to compute $I^{(2)}$, *we need to find all the pairs* (γ_1, γ_2) *such that* $\gamma_1 < 7$, $\gamma_1 <_{\mathbf{d}} 10$ *and* $\gamma_2 = 10 - \gamma_1$. *We have* 4 *pairs, namely* $(0, 10)$, $(2, 8)$, $(4, 6)$ *and* $(6, 4)$, *thus*

$$I^{(2)} = (x_1^2, x_2^2)(x_1^4, x_2^4)x_3^4 + (x_1^4, x_2^4)x_3^6 + (x_1^2, x_2^2)x_3^8 + (x_3^{10}).$$

In order to compute $I^{(3)}$, *we need to find all* $(\gamma_1,\gamma_2,\gamma_3)$ *such that* $\gamma_1<7$, $\gamma_1+\gamma_2<10$, $\gamma_1<_{\mathbf{d}}17$, $\gamma_1+\gamma_2<_{\mathbf{d}}17$ *and* $\gamma_3=17-\gamma_1+\gamma_2$. *If* $\gamma_1=0$ *then, the pair* (γ_2,γ_3) *is one of the following pairs:* $(0,17)$,$(1,16)$,$(4,13)$ *or* $(5,12)$. *If* $\gamma_1=1$ *then, the pair* (γ_2,γ_3) *is one of the following pairs:* $(0,16)$ *and* $(4,12)$. *If* $\gamma_1=4$ *then, the pair* (γ_2,γ_3) *is one of the pairs:* $(0,13)$ *and* $(1,12)$. *If* $\gamma_1=5$ *then, the pair* (γ_2,γ_3) *is* $(0,12)$. *Thus*

$$I^{(3)}=(x_1,x_2)(x_1^4,x_2^4)(x_4^{12},x_5^{12})+(x_1^4,x_2^4)x_3(x_4^{12},x_5^{12})+$$
$$+(x_1^4,x_2^4)(x_4,x_5)(x_4^{12},x_5^{12})+(x_1,x_2)x_3^4(x_4^{12},x_5^{12})+$$
$$+(x_1,x_2)(x_4^4,x_5^4)(x_4^{12},x_5^{12})+x_3(x_4^4,x_5^4)(x_4^{12},x_5^{12})+$$
$$+x_3^4(x_4,x_5)(x_4^{12},x_5^{12})+x_3^5(x_4^{12},x_5^{12})+(x_4,x_5)(x_4^4,x_5^4)(x_4^{12},x_5^{12}).$$

By Proposition 2.2, *we get* $I=I^{(1)}+I^{(2)}+I^{(3)}$.

Remark 2.4. *For any* $1\leq q\leq r$ *and any nonnegative integers* $\gamma_1,\ldots,\gamma_q\leq_{\mathbf{d}}\alpha_q$ *such that* $\gamma_1+\cdots+\gamma_i<\alpha_i$, $\gamma_1+\cdots+\gamma_i<_{\mathbf{d}}\alpha_q$ *for* $1\leq i<q$ *and* $\gamma_1+\cdots+\gamma_q=\alpha_q$ *we denote* $I^{(q)}_{\gamma_1,\ldots,\gamma_q}=\prod_{e=1}^q\prod_{t=0}^s(\mathbf{n}_e^{[d_t]})^{\gamma_{et}}$. *Proposition 2.2 implies:*

$$I=\sum_{q=1}^r\sum_{\gamma_1,\ldots,\gamma_q}I^{(q)}_{\gamma_1,\ldots,\gamma_q}.$$

Let $\mathbf{m}=(x_1,\ldots,x_n)\subset S$ *be the irrelevant ideal of* S. *We have:*

$$(I:_S\mathbf{m})=\bigcap_{j=1}^n(I:x_j)=\bigcap_{j=1}^n((\sum_{q=1}^r\sum_{\gamma_1,\ldots,\gamma_q}I^{(q)}_{\gamma_1,\ldots,\gamma_q}):x_j)=$$
$$=\bigcap_{j=1}^n(\sum_{q=1}^r\sum_{\gamma_1,\ldots,\gamma_q}(I^{(q)}_{\gamma_1,\ldots,\gamma_q}:x_j)).$$

On the other hand, if $x_j\in\mathbf{n}_p$ *for some* $1\leq p\leq q$, *then*

$$J^{(q),j}_{\gamma_1,\ldots,\gamma_q}:=(I^{(q)}_{\gamma_1,\ldots,\gamma_q}:x_j)=$$
$$=\prod_{e\neq p}^q\prod_{t=0}^s(\mathbf{n}_e^{[d_t]})^{\gamma_{et}}\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]}(\mathbf{n}_{\mathbf{p}}{}^{[d_t]})^{\gamma_{pt}-1}(\sum_{\gamma_{pt}>0}\prod_{j\neq t}(\mathbf{n}_e^{[d_t]})^{\gamma_{jt}}),$$

where $\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]}=(x_{i_{p-1}+1}^{d_t},\ldots,x_j^{d_t-1},\ldots,x_{i_p}^{d_t})$ *and* $\mathbf{n}_{\mathbf{p},\hat{\mathbf{j}}}{}^{[d_t]}(\mathbf{n}_{\mathbf{p}}{}^{[d_t]})^{\gamma_{pt}-1}:=S$ *if* $\gamma_{pt}=0$. *Thus*

$$(I:_S\mathbf{m})=\sum_{q^1=1}^r\sum_{\gamma_1^1,\ldots,\gamma_{q^1}^1}\cdots\sum_{q^n=1}^r\sum_{\gamma_1^n,\ldots,\gamma_{q^n}^n}\bigcap_{j=1}^n J^{(q^j),j}_{\gamma_1^j,\ldots,\gamma_{q^j}^j},$$

where, for a given $q = q^j$, *we take the second* j^{th} *sum for* $\gamma_1^j, \dots, \gamma_q^j \leq_{\mathbf{d}} \alpha_q$ *such that* $\gamma_1^j + \dots + \gamma_i^j < \alpha_i$, $\gamma_1^j + \dots + \gamma_i^j <_{\mathbf{d}} \alpha_q$ *for* $1 \leq i < q^j$ *and* $\gamma_1^j + \dots + \gamma_q^j = \alpha_q$.

Proposition 2.5. *Let* $n \geq 2$ *and let* $1 \leq i_1 < i_2 < \dots < i_r = n$ *be some integers. Let* $\alpha_1 < \alpha_2 < \dots < \alpha_r$ *be some positive integers. We consider the ideal* $I = \sum_{q=1}^{r} I_q$, *where* $I_q =< x_{i_q}^{\alpha_q} >_{\mathbf{d}}$. *Then, we have:* $reg(I) \leq reg(I_r)$ (We will see later in which conditions we have equality).

Proof. From [4, Corollary 3.6] it follows that $(I_q)_{\geq e}$ is stable, if $e \geq reg(I_q)$ so $(I_q)_{\geq e}$ is stable for $e = \max\{reg(I_1), \dots, reg(I_r)\}$. Since $I_{\geq e} = \sum_{q=1}^{r} (I_q)_{\geq e}$ and since a sum of stable ideals is still a stable ideal, it follows that $I_{\geq e}$ is stable. Therefore, from [6, Proposition 12], we get $reg(I) \leq e$. On the other hand, if we denote $s_q = \max\{t | \alpha_{qt} > 0\}$ for any $1 \leq q \leq r$, from [4, Theorem 3.1] we get $reg(I_q) = \alpha_{qs_q} d_{s_q} + (i_q - 1)(d_{s_q} - 1)$, thus $\max\{reg(I_1), \dots, reg(I_r)\} = reg(I_r)$. In conclusion, $reg(I) \leq reg(I_r)$. □

Proposition 2.6. *With the above notations, for any* $1 \leq q \leq r$ *we have:*

$$(I_q : \mathbf{m}_q) + (I_1 + \dots + I_q) \subset ((I_1 + \dots + I_q) : \mathbf{m}_q) \subset$$

$$\subset ((I_1 + \dots + I_q) : \mathbf{n}_q) = (I_q : \mathbf{n}_q) + (I_1 + \dots + I_q).$$

Proof. Fix $1 \leq q \leq r$. The first two inclusions are obvious. In order to prove the last equality, it is enough to show that

$$((I_1 + \dots + I_q) : x_j) \subset (I_q : x_j) + (I_1 + \dots + I_q),$$

for any $x_j \in \mathbf{n}_q$. Indeed, suppose $u \in ((I_1 + \dots + I_q) : x_j)$, therefore $x_j \cdot u \in I_1 + \dots + I_q$. If $x_j \cdot u \notin I_q$ it follows that $x_j \cdot u \in I_e$ for some $e < q$. Thus $u \in I_e$, since x_j does not divide any minimal generator of I_e. □

Let $n \geq 2$ and let $1 \leq i_1 < i_2 < \dots < i_r = n$ be some integers. Let $\alpha_1 < \alpha_2 < \dots < \alpha_r$ be some positive integers. We write $\alpha_q = \sum_{t \geq 0} \alpha_{qt} d_t$. Let $s_q = \max\{t | \alpha_{qt} > 0\}$ for any $1 \leq q \leq r$. Notice that $s_1 \leq s_2 \leq \dots \leq s_r$. Indeed, assume, by contradiction, that there exist $q < q'$ such that $s_q > s_{q'}$. Then, from the **d** - decomposition of $\alpha_{q'}$ and α_q, we have

$$\alpha_{q'} = \sum_{t=0}^{s_{q'}} \alpha_{q't} d_t \leq \sum_{t=0}^{s_{q'}} (\frac{d_{t+1}}{d_t} - 1) d_t = d_{s_{q'}+1} - d_0 \leq d_{s_{q'}+1} \leq d_{s_q} \leq \alpha_q,$$

absurd.
Let $1 \leq q_1 < q_2 < \dots < q_k = r$ be such that:

$$s_1 = \dots = s_{q_1} < s_{q_1+1} = \dots = s_{q_2} < \dots < s_{q_{k-1}+1} = \dots = s_{q_k}.$$

For $1 \leq j \leq k$, we define some positive integers χ_j as follows. If $i_{q_j} - i_{q_j-1} \geq 2$, we put $\chi_j = (d_{s_{q_j}} - 1)(i_{q_j} - i_{q_{j-1}}) + d_{s_{q_j}}(\alpha_{q_j s_{q_j}} - 1)$. Otherwise, suppose that $q = q_j$ and there exists a positive integer $1 \leq l \leq r - q + 1$ such that $s_{q-1} < s_q < \cdots < s_{q+l-1}$ and $i_{q+l-1} = i_{q-1} + l$. Denote $i = i_q$. We define recursively the numbers χ_{i+m-1}, for $1 \leq m \leq l$, starting with $m = l$. Suppose that we have already defined $\chi_{i+m}, \ldots, \chi_{i+l-1}$. If $\alpha_{q+m-2,s_{q+m-2}} > \alpha_{q+m-1,s_{q+m-1}}$, we put $\chi_{q+m-1} := \sum_{t=s_{q+m-2}+1}^{s_{q+m-1}} \alpha_{q+m-1,t} d_t - 1$ and we switch from m to $m-1$. Otherwise, if $\alpha_{q+m-2,s_{q+m-2}} \leq \alpha_{q+m-1,s_{q+m-1}}$ we put

$$\chi_{q+m-1} := (\alpha_{q+m-1,s_{q+m-2}} - \alpha_{q+m-2,s_{q+m-2}} + 1) \cdot d_{s_{q+m-2}} + \\ + \sum_{t=s_{q+m-2}+1}^{s_{q+m-1}} \alpha_{q+m-1,t} d_t - 1$$

and, if $m \geq 2$, we put also $\chi_{q+m-2} := \alpha_{q+m-2,s_{q+m-2}} \cdot d_{s_{q+m-2}} - 1$. We switch from m to $m-2$. We continue this procedure until $m \leq 0$.

With these notations, for the ideal $I =< x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}, \ldots, x_{i_r}^{\alpha_r} >_{\mathbf{d}}$, we have the following theorem:

Theorem 2.7. $\max\{e : (Soc(S/I))_e \neq 0\} = \sum_{j=1}^{k} \chi_j$.

Proof. For each integer $1 \leq j \leq k$, we consider the following ideal:

$$J_j = \begin{cases} (x_{i_{q_j}}^{\chi_j}), & if\ i_{q_j} - i_{q_j-1} = 1, \\ (x_{i_{q_{j-1}}+1} \cdots x_{i_{q_j}})^{d_{s_{q_j}}-1} \cdot \sum_{e=q_{j-1}+1}^{q_j} (\mathbf{n}_e^{[d_{s_{q_j}}]})^{\alpha_{e s_e}-1}, & otherwise. \end{cases}$$

Let $J = J_1 \cdot J_2 \cdots J_k$. We claim the following:

(1) $J \subset (I : \mathbf{m})$,

(2) $G(J) \cap G(I) = \emptyset$,

(3) $\max\{e|\ (Soc(S/I))_e \neq 0\} = \max\{e|\ ((J+I)/I)_e \neq 0\}$.

Suppose that we proved (1), (2) and (3). (1) and (2) implies

$$\max\{e|\ ((J+I)/I)_e \neq 0\} = \deg(J) := \max\{\deg(u)|\ u \in G(J)\}.$$

On the other hand, it is obvious that $\deg(J) = \sum_{j=1}^{k} \chi_j$ and thus, by (3), we complete the proof of the theorem.

In order to prove (1), we pick $x_i \in \mathbf{n}_q$ a variable, where $q \in \{1, \ldots, r\}$. Let j be the unique integer with the property that $q \in \{q_{j-1}+1, \ldots, q_j\}$. We want to show that $x_i \cdot J \subset I$. We consider two cases. First, we assume $i_{q_j} - i_{q_{j-1}} \geq 2$. We claim that $x_i J_j \subset I_{q_{j-1}+1} + \cdots + I_{q_j}$. Indeed, for any $e \in \{q_{j-1}+1, \ldots, q_j\}$,

$x_i(x_{i_{q_{j-1}}+1}\cdots x_{i_{q_j}})^{d_{s_{q_j}}-1}(\mathbf{n}_e^{[d_{s_{q_j}}]})^{\alpha_{es_e}-1} \subset I_e$, thus $x_iJ_j \subset I_{q_{j-1}+1}+\cdots+I_{q_j}$, as required. (See the proof of [4, Lema 2.1] for details.)

Suppose now $i_{q_j}-i_{q_j-1}=1$. Let $j' \leq j$, such that if we denote $q=q_{j'}$, there exists a positive integer $j-j'+1 \leq l$ with $s_{q-1}<s_q<\cdots<s_{q+l-1}$, $i_{q+l-1}=i_{q-1}+l$ and $i_{q_{j'+l}}>i_{q+l-1}+1$ when $q+l-1<r$. We prove in fact that $x_i \cdot J_{j'}\cdots J_j \subset I_j$. Note that $i=i_{q+m-1}$, where $m=j-j'+1$. Assume $m \geq 2$. If $\alpha_{q+m-2,s_{q+m-2}}>\alpha_{q+m-1,s_{q+m-2}}$, then

$$x_i \cdot J_{q+m-2}J_{q+m-1} = (x_{i-1}^{\cdots+\alpha_{q+m-2,d_{s_{q+m-2}}}-1}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t}) \subset I_j,$$

because $\alpha_{q+m-2,d_{s_{q+m-2}}}-1 \geq \alpha_{q+m-1,d_{s_{q+m-2}}}+d_{s_{q+m-2}}-1$ and therefore

$$x_i \cdot J_{q+m-2}J_{q+m-1} \subset (x_{i-1}^{d_{s_{q+m-2}}-1}\cdot x_{i-1}^{\alpha_{q+m-1,d_{s_{q+m-2}}}}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t}).$$

Now, the above assertion is obvious. If $m=1$, the same trick works, with the only difference that the first "=" is replaced by "$\subseteq$".

If $m \geq 2$ and $\alpha_{q+m-2,s_{q+m-2}} \leq \alpha_{q+m-1,s_{q+m-2}}$, then $x_i \cdot J_{q+m-2}J_{q+m-1}$ is the ideal generated by the product of the monomial $x_{i-1}^{\alpha_{q+m-2,d_{s_{q+m-2}}}d_{s_{q+m-2}}-1}$ with

$$x_i^{(\alpha_{q+m-1,s_{q+m-2}}-\alpha_{q+m-2,s_{q+m-2}}+1)d_{s_{q+m-2}}+\sum\limits_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t}.$$

By regrouping, we see that $x_i \cdot J_{q+m-2}J_{q+m-1} =$

$$=(x_{i-1}^{d_{s_{q+m-2}}-1}\cdot(x_{i-1}^{(\alpha_{q+m-2,d_{s_{q+m-2}}}-1)d_{s_{q+m-2}}}\cdot$$

$x_i^{(\alpha_{q+m-1,s_{q+m-2}}-\alpha_{q+m-2,s_{q+m-2}}+1)d_{s_{q+m-2}}})\cdot x_i^{\sum\limits_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t}) \subset I_j$, as required. If $m=1$ the same trick works, with the only difference that the first "=" is replaced by "$\subseteq$".

In order to prove (2) it is enough to show for any $1 \leq j \leq k$ that $G(J_1\cdots J_j)\cap G(I_e)=\emptyset$ for any $e \in \{q_{j-1}+1,\ldots,q_j\}$, because each of the minimal generators of $J_1\cdots J_j$ does not contain variables x_i with $i>i_{q_j}$. We use induction on $1 \leq j \leq k$. If $j=1$, then $G(J_1)\cap G(I_1)=\emptyset$ from [4, Lemma 2.1]. Suppose the assertion is true for $j-1$. We must consider two cases.

First, suppose $i_{q_j}-i_{q_{j-1}} \geq 2$. It follows $J_j=(x_{i_{q_{j-1}}+1}\cdots x_{i_{q_j}})^{d_{s_{q_j}}-1}\cdot \sum_{e=q_{j-1}+1}^{q_j}(\mathbf{n}_e^{[d_{s_{q_j}}]})^{\alpha_{es_e}-1}$. Since $s_{q_{j-1}}<s_{q_j}$, it follows that $J_1\cdots J_{j-1}\cdot J_j \subset$

$(x_1,\dots,x_{i_{q_{j-1}}})^{d_{s_{q_j}}-1}J_j$, and it is easy to note that none of the minimal generator of the ideal from left is included in some I_e with $q_{j-1}+1\le e\le q_j$.

Suppose now $i_{q_j}-i_{q_{j-1}}=1$. Let $j'\le j$, such that if we denote $q=q_{j'}$, there exists an positive integer $j-j'+1\le l$ with $s_{q-1}<s_q<\dots<s_{q+l-1}$, $i_{q+l-1}=i_{q-1}+l$ and $i_{q_{j'+l}}>i_{q+l-1}+1$ when $q+l-1<r$. We prove in fact that $x_i\cdot J_{j'}\cdots J_j\subset I_j$. Note that $i=i_{q+m-1}$, where $m=j-j'+1$. Assume $m\ge 2$. If $\alpha_{q+m-2,s_{q+m-2}}>\alpha_{q+m-1,s_{q+m-2}}$, then

$$J_1\cdots J_j=(J_1\cdots J_{j-2})\cdot(x_{i-1}^{\dots+\alpha_{q+m-2,d_{s_{q+m-2}}}-1}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t-1})\subset$$

$$(x_1,\dots,x_{i_{q_{j-2}}})^{d_{s_{q_{j-1}}}-1}(x_{i-1}^{\dots+\alpha_{q+m-2,d_{s_{q+m-2}}}-1}\cdot x_i^{\sum_{t=s_{q+m-2}+1}^{s_{q+m-1}}\alpha_{q+m-1,t}d_t-1}),$$

and it is easy to see that none of the minimal generators of the last ideals is in I_j. The subcase $\alpha_{q+m-2,s_{q+m-2}}\le\alpha_{q+m-1,s_{q+m-2}}$ is similar. Also, the case $m=1$.

In order to prove (3) it is enough to show the "$\le$" inequality, since obviously $(J+I)/I\subset Soc(S/I)$. Let $u=x_1^{\beta_1}\cdots x_n^{\beta_n}\in(I:\mathbf{m})$ be a monomial such that $u\notin I$. We claim that $\deg(u)\le\sum\limits_{j=0}^{k}\chi_j$. More precisely, we claim the following:

(a) $\sum_{i=i_{q_{j-1}}+1}^{i_{q_j}}\beta_i\le\chi_j$, for all $1\le j\le r$ such that $i_{q_j}-i_{q_{j-1}}\ge 2$.

(b) For each j with the property that there exists an positive integer $1\le l\le r-q+1$ (where $q=q_j$) such that $s_{q-1}<s_q<\dots<s_{q+l-1}$, $i_{q_j}-i_{q_{j-1}}\ge 2$ and $i_{q+l-1}=i_{q-1}+l$, we have $\sum_{i=i_{q_{j-1}+1}}^{i_{q_{j-1+l}}}\beta_i\le\sum_{m=1}^{l}\chi_{j+m-1}$.

Obviously, (a) and (b) implies (3).

In order to prove (a), assume that $\sum_{i=i_{q_{j-1}}+1}^{i_{q_j}}\beta_i>\chi_j$, therefore

$$\sum_{i=i_{q_{j-1}}+1}^{i_{q_j}}\beta_i\ge(d_{s_{q_j}}-1)(i_{q_j}-i_{q_{j-1}}-1)+\alpha_{q_js_{q_j}}d_{s_{q_j}}.$$

It follows that we can write $u_j=x_i^{d_{s_{q_j}}-1}\cdot w$, with

$$w\in(x_{i_{q_{j-1}}+1}^{d_{s_{q_j}}},\dots,x_{i_{q_j}}^{d_{s_{q_j}}})^{\alpha_{q_js_{q_j}}},$$

for some $i\in\{x_{i_{q_{j-1}}+1},\dots,x_{i_{q_j}}\}$, and thus $u_j\in I_{q_j}$, a contradiction.

Consider now the case (b) and assume that

$$\sum_{i=i_{q_{j-1}+1}}^{i_{q_{j-1+l}}}\beta_i>\sum_{m=1}^{l}\chi_{j+m-1}.$$

Using similar arguments as in the case (a), we get $u_j \in I_{q_j}$, a contradiction. □

Corollary 2.8. *With the previous notations,* $reg(I) = \sum_{j=1}^{k} \chi_k + 1$.

Proof. Since I is an artinian ideal, $reg(I) = \max\{e : Soc(S/I)_e \neq 0\} + 1$ so the required result follows immediately from the previous theorem. □

Remark 2.9. *We have already seen that* $reg(I) \leq reg(I_r)$*. Now, we are able to say when we have equality, and this is only in the case when* $k = 1$*, i.e.* $s_1 = s_2 = \cdots = s_r$*. Indeed, if* $k = 1$*, by [4, 3.1],* $reg(I_r) = (d_{s_r} - 1)(n-1) + d_{s_r}(\alpha_{rs_r} - 1) + 1 = \chi_1 + 1$*. Conversely, if* $k > 1$ *then* $\chi_1 + \cdots + \chi_k < reg(I_r)$*, because* $\chi_j < (d_{s_r} - 1)(i_{q_j} - i_{q_j - 1}) + d_{s_r}(\alpha_{rs_r} - 1)$ *for any* $j < k$.

Example 2.10.
1. *Let* $\mathbf{d} : 1|2|6|12$ *and* $I = < x_2^7, x_3^{10}, x_5^{17} >_{\mathbf{d}} \subset K[x_1, \ldots, x_5]$*. We have* $k = 2$*,* $\chi_1 = 15$ *and* $\chi_2 = 22$*. Therefore,* $reg(I) = 27$*. An element of maximal degree in* $Soc(S/I)$ *is* $x_1^5 x_2^5 x_3^5 x_4^{11} x_5^{11}$.

2. *Let* $\mathbf{d} : 1|4|12$ *and* $I = < x_1^2, x_2^7, x_3^{16} >_{\mathbf{d}} \subset K[x_1, x_2, x_3]$*. We have* $k = 3$*. Since* $2 = 2 \cdot 1$*,* $7 = 3 \cdot 1 + 1 \cdot 4$ *and* $16 = 1 \cdot 4 + 1 \cdot 12$*, we get* $\chi_1 = 1$*,* $\chi_2 = 3$ *and* $\chi_3 = 19$*. Therefore,* $reg(I) = 23$*. An element of maximal degree in* $Soc(S/I)$ *is* $x_1 x_2^3 x_3^{19}$.

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Institute of Mathematics of the Romanian Academy
Bucharest, Romania
E-mail: mircea.cimpoeas@imar.ro