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# RANK TWO ULRICH MODULES OVER THE AFFINE CONE OF THE SIMPLE NODE

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**Abstract**

A concrete description of all isomorphism classes of indecomposable rank two graded Ulrich modules over the homogeneous hypersurface ring $k[y_1, y_2, y_3]/\langle y_1^3 + y_1^2 y_3 - y_2^2 y_3\rangle$ is given.

## Introduction

Let $R$ be a homogeneous Cohen–Macaulay $k$–algebra over a field $k$ and let $M$ be a finitely generated graded $R$–module. If $M$ is a maximal Cohen–Macaulay $R$-module (shortly MCM) then $\mu(M) \leq e(M)$ where $\mu(M)$ denotes the minimal number of generators of $M$ and $e(M)$ denotes the multiplicity of $M$. In the case that $M$ is a maximal Cohen–Macaulay module and $\mu(M) = e(M)$, $M$ is called *Ulrich*–module, or maximally generated MCM (see [U]).The corresponding sheaves on Proj$R$ are called Ulrich sheaves. It is known ([ES]) that a line bundle $\mathcal{F}$ on a curve $X$ of genus $g$ embedded in $\mathbb{P}^n$ is an Ulrich sheaf if and only if $\mathcal{F}(-1)$ has degree $g-1$ and no global sections.
In this paper we study the rank two Ulrich sheaves over a singular curve of arithmetic genus 1, the nodal curve. From some points of view, in spite of its singularity, the simple node is very similar to the elliptic curves: for example, the nodal curve has, the same like the elliptic curves, a tame category of vector bundles (see [DG]).
The nodal curve is Proj$R$, where $R = k[y_1, y_2, y_3]/\langle y_1^3 + y_1^2 y_3 - y_2^2 y_3\rangle$, $k$ an algebraically closed field. The existence of a Ulrich module over a homogeneous 2-dimensional CM-ring with an infinite residue class field and over a

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homogeneous hypersurface ring was proven in [BHU] and [BH].

We describe explicitly all indecomposable rank two graded Ulrich $R$–modules, by computing their corresponding matrix factorizations.
The matrix factorizations, introduced by Eisenbud [Ei1], are a powerful tool in the work with MCM–modules over hypersurface rings: in [BEH] and [BHS] the authors have studied connections between matrix factorizations of a homogeneous polynomial $f$ and (a generalized) Clifford algebra of $f$; in [MPP], [LPP], [BEPP] and other, the authors used the matrix factorizations in order to classify different classes of MCM modules.

In the first section we remind some facts about matrix factorizations of homogeneous polynomials and their relation with MCM modules over hypersurface rings; (for more details the reader can consult the book of Y. Yoshino, 'Cohen–Macaulay modules over Cohen–Macaulay rings'); especially, we describe the construction of extensions of MCM modules with known matrix factorizations. We use it in the second section of this paper for the classification of the rank two Ulrich modules. The last section contains the Singular procedures used in the paper.

## 1 Extensions of MCM modules over hypersurface rings

Let $S$ be a polynomial ring over a field $k$ and $f \in S$ an irreducible homogeneous polynomial of degree $d$.
Consider the hypersurface ring $R = S/f$ and $M$ a graded MCM–module over it. As an $S$-module, $M$ has a minimal resolution of the form

$$0 \longrightarrow \bigoplus_{j=1}^{n} S(\beta_j) \xrightarrow{\tilde{A}} \bigoplus_{j=1}^{n} S(\alpha_j) \longrightarrow M \longrightarrow 0,$$

with $\tilde{A}$ the multiplication by a square matrix $A$ with homogeneous entries that are either zero or of strictly positive degree (because of the minimality). Eisenbud proved that there exists another square matrix $A'$ with homogeneous entries (graded matrix) over $S$ such that $(A, A')$ forms a graded matrix factorization of $f$, that is $AA' = A'A = f \cdot Id$. As an $R$–module, $M$ has the following infinite graded minimal 2-periodic $R$-resolution:

$$\ldots \xrightarrow{\cdot A} \bigoplus_{j=1}^{n} R(\alpha_j - d) \xrightarrow{\cdot A'} \bigoplus_{j=1}^{n} R(\beta_j) \xrightarrow{\cdot A} \bigoplus_{j=1}^{n} R(\alpha_j) \longrightarrow M \longrightarrow 0.$$

Conversely, any graded matrix factorization $(A, A')$ of the polynomial $f$ determine (up to shifting) a graded MCM module, $M = \mathrm{Coker}(\tilde{A})$, where $\tilde{A} : \bigoplus_{j=1}^{n} R(\beta_j) \longrightarrow \bigoplus_{j=1}^{n} R(\alpha_j)$ is the multiplication by $A$.

The rank of the module $M$ is precisely the integer $r$ such that $\det A = f^r$. It follows immediately that $e(M) = \deg f \cdot \operatorname{rank} M = \sum_{j=1}^{n}(\alpha_j - \beta_j)$, and, therefore, the minimal number of generators of a MCM $R$–module is smaller equal the multiplicity of the module. Thus, the Ulrich modules are exactly the MCM–modules that have a matrix factorization with linear entries on the diagonal.

Two matrix factorizations $(A, A')$ and $(B, B')$ determine the same MCM module if and only if the matrices $A$ and $B$ are equivalent, that means there exist two graded invertible matrices $U$ and $V$ such that $AU = BV$.
The MCM module given by a reduced matrix factorization $(A, A')$ (reduced means that the entries are either zero or of strictly positive degree) is decomposable if and only if the matrix $A$ is equivalent to a matrix of the form $\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$.

In the following we recall some facts regarding the extensions $\operatorname{Ext}^1_R(N, M)$, with $M, N$ graded MCM modules over a hypersurface ring $R$ (for more details, see [Y]). Let

$$\dots \xrightarrow{\cdot A} \bigoplus_{j=1}^{n} R(\alpha_j - d) \xrightarrow{\cdot A'} \bigoplus_{j=1}^{n} R(\beta_j) \xrightarrow{\cdot A} \bigoplus_{j=1}^{n} R(\alpha_j) \longrightarrow M \longrightarrow 0,$$

and

$$\dots \xrightarrow{\cdot B} \bigoplus_{j=1}^{s} R(\alpha'_j - d) \xrightarrow{\cdot B'} \bigoplus_{j=1}^{s} R(\beta'_j) \xrightarrow{\cdot B} \bigoplus_{j=1}^{s} R(\alpha'_j) \longrightarrow N \longrightarrow 0,$$

be minimal $R$–resolutions of $M$, respectively $N$ and denote with $\Omega^1(M)$ the first syzygy of $M$.
The graded exact sequence

$$(*) \qquad 0 \longrightarrow M \longrightarrow \bigoplus_{j=1}^{n} R(\beta_j + d) \xrightarrow{\cdot A} \Omega^1(M) \otimes_R R(d) \longrightarrow 0,$$

induces the natural surjective mapping

$$\delta : \operatorname{Hom}_R(N, \Omega^1(M) \otimes_R R(d)) \longrightarrow \operatorname{Ext}^1_R(N, M).$$

A morphism $h : N \longrightarrow \Omega^1(M) \otimes_R R(d)$ is given by two graded matrices $D$ and $D'$ such that $A' \cdot D = D' \cdot B$ (the entry $(i, j)$ of the matrix $D$ has the degree $\alpha_i - \beta'_j$), that means, the pair $(D, D')$ makes the following diagram commutative:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{j=1}^{s} S(\beta'_j) & \xrightarrow{\cdot B} & \bigoplus_{j=1}^{s} S(\alpha'_j) & \longrightarrow & N & \longrightarrow & 0 \\
& \downarrow \cdot D & & \downarrow \cdot D' & & \downarrow h & & \\
0 \longrightarrow & \bigoplus_{j=1}^{n} S(\alpha_j) & \xrightarrow{\cdot A'} & \bigoplus_{j=1}^{n} S(\beta_j + d) & \longrightarrow & \Omega^1(M) \otimes_R R(d) & \longrightarrow & 0.
\end{array}$$

By definition, $\delta$ maps the morphism $h$ to

$$\delta(h): \qquad 0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0,$$

with $L$ given by :

$$0 \longrightarrow (\bigoplus_{j=1}^{n} S(\beta_j)) \oplus (\bigoplus_{j=1}^{s} S(\beta'_j)) \xrightarrow{\cdot\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}} (\bigoplus_{j=1}^{n} S(\alpha_j)) \oplus (\bigoplus_{j=1}^{s} S(\alpha'_j)) \longrightarrow L \longrightarrow 0.$$

**Remark.** $\delta(h) = 0$ if and only if there exist two graded matrices $U$ and $V$ such that $D = AU + VB$.

*Proof.* The morphism $h$ is in the kernel of $\delta$ if and only if $h$ factories as $N \xrightarrow{l} \bigoplus_{j=1}^{n} R(\beta_j + d) \xrightarrow{\cdot A} \Omega^1(M) \otimes_R R(d)$. Let $l_1, l_2$ be two graded morphisms such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{j=1}^{s} S(\beta'_j) & \xrightarrow{\cdot B} & \bigoplus_{j=1}^{s} S(\alpha'_j) & \longrightarrow & N & \longrightarrow & 0 \\
 & \downarrow l_2 & & \downarrow l_1 & & \downarrow l & & \\
0 \longrightarrow & \bigoplus_{j=1}^{n} S(\beta_j) & \xrightarrow{\cdot f} & \bigoplus_{j=1}^{n} S(\beta_j + d) & \longrightarrow & \bigoplus_{j=1}^{n} R(\beta_j) & \longrightarrow & 0 \\
 & \downarrow \cdot A & & \downarrow id & & \downarrow \cdot A & & \\
0 \longrightarrow & \bigoplus_{j=1}^{n} S(\alpha_j) & \xrightarrow{\cdot A'} & \bigoplus_{j=1}^{n} S(\beta_j + d) & \xrightarrow{\cdot A} & \Omega^1(M) \otimes_R R(d) & \longrightarrow & 0.
\end{array}$$

Let $W$ and $U$ be the graded matrices defining the morphisms $l_1$, respectively $l_2$. Then, the pairs of matrices $(AU, W)$ and $(D, D')$ define the same morphism, $h$. Therefore, there exists a graded matrix $V$ such that

$$D - AU = VB, \ D' - W = A'U.$$

The first equality is exactly what we want to prove. The inverse direction is evident. □

Therefore, to compute an element of $\mathrm{Ext}^1_R(N, M)$ for two graded MCM–modules $M$ and $N$ with known minimal resolutions given by the matrix factorizations $(A, A')$, respectively $(B, B')$, one has to find a graded matrix $D$ such that there exists another graded matrix $D'$ with $A' \cdot D = D' \cdot B$, or, equivalent, such that $f$ divides the entries of $A'DB'$. Notice that if one replaces the matrix $D$ with $D - AU - VB$, for any matrices $U, V$, one obtains the same extension.

In this paper, the computations of the extensions are made with the help of the Singular procedure `"condext"`, that returns the ideal of conditions on the entries of a matrix $D$ such that $f$ divides the entries of $A'DB'$ (see the last section).

## 2 The classification of rank 2 Ulrich $R$–modules

In this section we describe the isomorphism classes of all graded rank two indecomposable Ulrich $R$–modules over the ring $R = k[y_1, y_2, y_3]/\langle y_1^3 + y_1^2 y_3 - y_2^2 y_3\rangle$, that is the affine cone over the simple node ($k$ is an algebraically closed field).

The graded MCM $R$–modules are exactly the locally torsion-free $R$–modules and they have the following property:

**Lemma 2.1.** *Let $R$ be a homogeneous hypersurface ring $R = k[y_1, y_2, y_3]/f$ with $f$ indecomposable, and $M$ be a locally torsion–free (MCM) $R$–module of rank $r$, $r > 1$. Then, there exist two locally torsion–free $R$–modules $K$, $N$ with rank(K)=1, such that the sequence*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

*is exact. If $M$ is an Ulrich module, then $K$ and $N$ are also Ulrich modules.*

*Proof.* Let $n \in \mathbb{Z}$ and $m_0 \in M_n$, $m_0 \neq 0$. Let $\varphi : R(-n) \longrightarrow M$ be the multiplication with $m_0$. Denote $Q = \operatorname{Coker}\varphi, N = Q/\operatorname{Tors}(Q)$. Then, there exists $K$ of rank 1, such that the sequence $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact. It is easy to see that $K$ and $N$ are indeed locally torsion–free modules.

Consider now that $M$ is an Ulrich–module.
Then $\mu(M) = e(M) = e(K) + e(N) \geq \mu(K) + \mu(N)$.
As we have seen in the previous section, on a hypersurface ring, if $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of MCM modules, $\mu(M) \leq \mu(K) + \mu(N)$. Therefore, if for $M$ holds $\mu(M) = e(M)$, also for $K$ and $N$ holds $\mu(K) = e(K)$ and $\mu(N) = e(N)$. □

Therefore, in order to classify the rank two Ulrich $R$–modules, we need to know the Ulrich $R$–modules of rank one, that are explicitly computed in [Ba3]. We remind here their classification.

Let $f = y_1^3 + y_1^2 y_3 - y_2^2 y_3$ and let $s = (0 : 0 : 1)$ be the unique singular point of the curve $V(f) \subset \mathbb{P}_k^2$. Denote $V(f)_{\text{reg}} = V(f) \backslash \{s\}$.
Then $V(f)_{\text{reg}} = \{(a : b : 1), a^3 + a^2 - b^2 = 0, a \neq 0\} \cup \{(0 : 1 : 0)\}$.

For any $\lambda = (a : b : 1)$ in $V(f)$ denote:

$$\alpha_\lambda = \begin{pmatrix} 0 & y_1 - ay_3 & y_2 - by_3 \\ y_1 & y_2 + by_3 & (a^2 + a)y_3 \\ y_3 & 0 & -y_1 - (a+1)y_3 \end{pmatrix} \text{ and let } \beta_\lambda \text{ be the adjoint of } \alpha_\lambda.$$

Consider also the graded maps given by the multiplication with the matrix $\alpha_\lambda$, $\tilde{\alpha}_\lambda : R(-2)^3 \longrightarrow R(-1)^3$.

**Proposition 2.2.** *1. For each $\lambda = (a : b : 1)$ in $V(f)$, the pair $(\alpha_\lambda, \beta_\lambda)$ is a graded matrix factorization of $f$.*

*2. The rank one, graded, locally free Ulrich $R$–modules are isomorphic, up to shifting, with one of the modules* $\operatorname{Coker} \tilde{\alpha}_\lambda, \lambda \in V(f)_{\text{reg}} \setminus \{(0:1:0)\}$.

*3. Up to shifting, there is only one non–locally free rank one, Ulrich $R$-module, that is* $\operatorname{Coker} \tilde{\alpha_s}$.

For each $m \in \mathbb{N}$, $m \geq 1$, $\lambda = (a : b : 1) \in V(f)_{\text{reg}}$ and $K \in k$, define the following matrices:

$$\delta_\lambda^m = \begin{pmatrix} 0 & y_1 - ay_3 & y_2 - by_3 & 0 & 2by_3^m & (3a^2+2a)y_3^m \\ y_1 & y_2 + by_3 & (a^2+a)y_3 & 0 & -(3a^2+2a)y_3^m & -2b(2a+1)y_3^m \\ y_3 & 0 & -y_1-(a+1)y_3 & 0 & 0 & 2by_3^m \\ 0 & 0 & 0 & 0 & y_1 - ay_3 & y_2 - by_3 \\ 0 & 0 & 0 & y_1 & y_2 + by_3 & (a^2+a)y_3 \\ 0 & 0 & 0 & y_3 & 0 & -y_1-(a+1)y_3 \end{pmatrix},$$

$$\delta_s^m = \begin{pmatrix} 0 & y_1 & y_2 & 0 & 0 & y_3^m \\ y_1 & y_2 & 0 & 0 & -y_3^m & 0 \\ y_3 & 0 & -y_1-y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & y_1 & y_2 & 0 \\ 0 & 0 & 0 & y_3 & 0 & -y_1-y_3 \end{pmatrix}$$

and

$$\delta_K^m = \begin{pmatrix} 0 & y_1 & y_2 & 0 & y_3^m & Ky_3^m \\ y_1 & y_2 & 0 & 0 & -Ky_3^m & -y_3^m \\ y_3 & 0 & -y_1-y_3 & 0 & 0 & y_3^m \\ 0 & 0 & 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & y_1 & y_2 & 0 \\ 0 & 0 & 0 & y_3 & 0 & -y_1-y_3 \end{pmatrix}.$$

For any $\delta \in \{\delta_\lambda^m, \delta_K^m | m \in \mathbb{N}, m \geq 1, \lambda \in V(f), \lambda \neq (0:1:0), K \in k\}$, define the graded map $\tilde{\delta} : R(-2)^3 \oplus R(-m-1)^3 \longrightarrow R(-1)^3 \oplus R(-m)^3$, that is the multiplication with the matrix $\delta$.

**Theorem 2.3.** *Any rank two indecomposable Ulrich module $M$ over the affine cone of the simple node is, up to shifting, isomorphic to one Coker $\tilde{\delta}$, with $\delta \in \{\delta_\lambda^m, \delta_K^m | m \in \mathbb{N}, m \geq 1, \lambda \in V(f), \lambda \neq (0:1:0), K \in k\}$.*

*Proof.* Let $M$ be a graded indecomposable rank two maximally generated MCM $R$-module.
By Lemma 2.1, $M$ fits in a graded exact sequence

$$0 \longrightarrow L_1 \longrightarrow M \longrightarrow L_2 \longrightarrow 0,$$

with $L_1$ and $L_2$ graded, rank one maximally generated MCM $R$–module. Therefore, there exist $\gamma = (c : d : 1)$ and $\lambda = (a : b : 1)$ two points in $V(f)$ and $n \in \mathbb{Z}$ such that, after some shiftings, $M$ fits into a graded exact sequence

$$(**) \qquad 0 \to \operatorname{Coker} \tilde{\alpha}_\gamma \to M \to \operatorname{Coker} \tilde{\alpha}_\lambda \otimes R(n) \to 0.$$

As it was seen in the first section, $M$ has a graded (reduced) matrix factorization $(\delta, \delta')$, with $\delta = \begin{pmatrix} \alpha_\gamma & D \\ 0 & \alpha_\lambda \end{pmatrix}$; the 3–square matrix $D$ has homogeneous entries and it fulfill $\beta_\gamma \cdot D \cdot \beta_\lambda = 0 \mod (f)$.
The corresponding graded map $\tilde{\delta}$ is defined as
$\tilde{\delta} : R(-2)^3 \oplus R(n-2)^3 \longrightarrow R(-1)^3 \oplus R(n-1)^3$, so, the matrix $D$ should have homogeneous entries of degree $m = 1 - n$.
If $n \geq 2$, $D$ is the null-matrix, so the extension splits.
If $n = 1$, $D$ has constant entries, therefore the module $M$ either decomposes or is not maximally generated.
Therefore we should consider only the negative shifting of $\operatorname{Coker} \alpha_\lambda$. □

In the next Lemma we prove that the matrix $D$ can be chosen with a simplified form, without changing the module $\operatorname{Coker} \tilde{\delta}$.

**Lemma 2.4.** *There exists a matrix* $D\,' = \begin{pmatrix} a_1 y_2^m & y_2 B_2 + y_3 A_2 & a_3 y_3^m \\ 0 & a_5 y_3^m & a_6 y_3^m \\ y_2 B_7 + y_3 A_7 & a_8 y_3^m & a_9 y_3^m \end{pmatrix}$ *with homogeneous entries of degree $m$ such that the matrix* $\delta\,' = \begin{pmatrix} \alpha_\gamma & D' \\ 0 & \alpha_\lambda \end{pmatrix}$ *is equivalent with* $\delta$.

**Remark.** In other words, the lemma says, that by some linear transformations, one can eliminate $y_1$ and $y_3$ on the position $[1,1]$ of $D$, $y_1$ and $y_2$ on the positions $[1,3], [2,2], [2,3], [3,2], [3,3]$, only $y_1$ on the positions $[1,2]$ and $[3,1]$, and one can make zero on the position $[2,1]$.

*Proof.* If we prove that there exist two graded matrices $U$ and $V$ such that $D - \alpha_\gamma U - V \alpha_\lambda$ has the form of $D\,'$, we are done the proof.

Let $U = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix}$. The entries of $W = \alpha_\gamma U + V \alpha_\lambda$ are linear in $u_{1,\ldots,9}$ and $v_{1,\ldots,9}$. We denote the coefficient of $y_l$ in $W[i,j]$ with $C_l[i,j]$. $C_l[i,j]$ is an element in the vector space generated by $u_1, ..., u_9, v_1, ..., v_9$ over $k$. An easy but laborious computation shows that the following coefficients are linear independent over $k$:
$\{C_1[i,j], C_3[1,1], C_2[1,3], C_2[2,1], C_3[2,1], C_2[2,2], C_2[2,3], C_2[3,2], C_2[3,3]\}$ $(1 \leq i, j \leq 9)$. (The vector space generated by them is
$\langle v_8, v_7, v_6, v_5 - v_9, v_4, v_3, v_2, v_1 - v_9, u_9 + v_1, u_8, u_7, u_6, u_5 + v_5, u_4, u_3, u_2, u_1 + v_9 \rangle_k$).

Thus, the matrices $U$ and $V$ can be chosen such that the corresponding coefficients in the matrix $D - \alpha_\gamma U - V\alpha_\lambda$ annulate, that is, the matrix has the form of $D\,'$.

□

We can consider now that $D$ has the simplified form from the previous lemma and we impose the condition $\beta_\gamma \cdot D \cdot \beta_\lambda = 0$ modulo $f$, in order to get more information on the entries of $D$. This information is contained in the ideal returned by the procedure `condext` (see the last section).

In the following, $Y$ denotes $y_3^{m-1}$, $d(1) = D[1,1] = a_1 y_2^m$, $d(2) = D[1,2] = y_2 B_2 + y_3 A_2$, $d(7) = D[3,1] = y_2 B_7 + y_3 A_7$, $a(3), a(5), a(6), a(8), a(9)$ are constants, as in Lemma 2.4.

```
ring S=0,(y(1..3),d(1),d(2),a(3),a(5..6),d(7),a(8..9),
          Y,a,b,c,d),(c,dp);
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2,c3+c2-d2;
qring R=std(i);

matrix alphal[3][3]=0,y(1)-a*y(3),     y(2)-b*y(3),
                 y(1),y(2)+b*y(3),     (a2+a)*y(3),
                 y(3),           0,-y(1)-(a+1)*y(3);

matrix alphag[3][3]=0,y(1)-c*y(3),     y(2)-d*y(3),
                 y(1),y(2)+d*y(3),     (c2+c)*y(3),
                 y(3),           0,-y(1)-(c+1)*y(3);

matrix D[3][3]= d(1),       d(2),a(3)*Y*y(3),
                   0,a(5)*Y*y(3),a(6)*Y*y(3),
                d(7),a(8)*Y*y(3),a(9)*Y*y(3);

ideal P=condext(alphag,alphal,D);
P[1];
-y(3)^2*a(9)*Y*a+y(3)^2*a(8)*Y*b-y(3)^2*a(9)*Y*c-y(3)^2*a(6)*Y-
y(2)*y(3)*a(8)*Y-y(3)^2*a(9)*Y-y(3)*d(7)*a^2-y(3)*d(7)*a+
y(3)*d(1)*d+y(2)*d(1)
```

The condition P[1]=0 means that the entry $d(1) = D[1,1]$ is in the ideal $(y_3)$. But actually, $D[1,1]$ is $a_1 y_2^m$, $a_1 \in k$, so $d(1) = D[1,1] = 0$.

```
P=simple(subst(P,d(1),0));
P[6];
y(3)^3*a(9)*Y*c^2-y(3)^3*a(6)*Y*a+y(3)^3*a(5)*Y*b+y(3)^3*a(9)*Y*c-
```

```
y(3)^3*a(3)*Y*d-y(2)*y(3)^2*a(3)*Y-y(2)*y(3)^2*a(5)*Y-y(2)^2*d(7)
```

The condition P[6]=0 shows that $y_3^2 * Y = y_3^{m+1}$ divides $d(7)$, that actually it is a polynomial of degree $m$. So $d(7) = D[3, 1] = 0$.

```
P=simple(subst(P,d(7),0));
P=interred(P);
P[1];
y(3)*a(9)*a-y(3)*a(8)*b+y(3)*a(9)*c+y(3)*a(6)+y(2)*a(8)+y(3)*a(9)
P[2];
y(3)*a(9)*c^2-y(3)*a(6)*a+y(3)*a(5)*b+y(3)*a(9)*c-y(3)*a(3)*d-
y(2)*a(3)-y(2)*a(5)
```

The conditions P[1]=P[2]=0 implies $a(8) = a(3) + a(5) = 0$.

```
P=simple(subst(P,a(8),0,a(5),-a(3)));
P=interred(P);
P[1];
a(9)*a+a(9)*c+a(6)+a(9)
P[6];
y(3)^2*a(3)*Y*a-y(3)^2*a(3)*Y*c+y(3)^2*a(9)*Y*d-y(2)*y(3)*a(9)*Y
-y(3)*d(2)*b+y(2)*d(2)
```

The sixth (last) polynomial of the ideal $P$ shows that $y_3 * Y = y_3^m$ divides the degree $m$ polynomial $d(2)$, so $d(2) = a_2 y_3^m$, $a_2$ constant. The condition P[1]=0 gives $a_6 = -a_9(a + c + 1)$.
We change the ring in which we work just to adjust the variables that we still need:

```
ring S1=0,(y(1..3),d(2),a(2),a(3),a(6),a(9),Y,a,b,c,d),(c,dp);
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2,c3+c2-d2;
qring R1=std(i);
ideal P=imap(R,P);
P=subst(P,a(6),-a(9)*(a+c+1),d(2),a(2)*y(3)*Y);
P=simple(P);
P=interred(P);
P;
P[1]=a(2)*b-a(9)*b+a(2)*d-a(9)*d
P[2]=a(9)*a^2+a(9)*a*c+a(9)*c^2+a(9)*a-a(3)*b+a(9)*c-a(3)*d
P[3]=a(2)*a^2+a(2)*a*c+a(2)*c^2+a(2)*a-a(3)*b+a(2)*c-a(3)*d
P[4]=y(3)*a(3)*a-y(3)*a(9)*b-y(3)*a(3)*c+y(3)*a(2)*d+y(2)*a(2)-
y(2)*a(9)
```

From the last polynomial we get $a_2 = a_9$ and $a_3(a - c) = a_9(b - d)$.
If $a \neq c$, $a_3 = \frac{b-d}{a-c} a_9$.

Therefore, $D = \frac{a_9}{a-c}\begin{pmatrix} 0 & (a-c)y_3^m & (b-d)y_3^m \\ 0 & -(b-d)y_3^m & -(a-c)(a+c+1)y_3^m \\ 0 & 0 & (a-c)y_3^m \end{pmatrix} = \frac{a_9}{a-c}y_3^{m-1}(\alpha_\gamma - \alpha_\lambda)$.
Then the extension (∗∗) splits.

Let $a = c$. Then $a_9(b-d) = 0$ and $a_3(b+d) = a_9(3a^2+2a)$.

The matrix $D$ has the form

$$D = \begin{pmatrix} 0 & a_9 y_3^m & a_3 y_3^m \\ 0 & -a_3 y_3^m & -a_9(2a+1)y_3^m \\ 0 & 0 & a_9 y_3^m \end{pmatrix}.$$

There are five cases to be considered:

1. $a = c, b = d, b \neq 0, a_9 \neq 0, a_3 = \frac{3a^2+2a}{2b}a_9$
2. $a = c = -1, b = d = 0, a_9 = 0, a_3 \neq 0$
3. $a = c, b = -d, b \neq 0, a_9 = 0, a_3 \neq 0$
4. $a = c = b = d = 0, a_9 = 0$
5. $a = c = b = d = 0, a_9 \neq 0$

In the first case, without changing the corresponding module, one can choose $a_9 = 2b$ and the matrix $\delta$ becomes $\delta_\lambda^m$.

In the second case, we can choose $a_3 = 1$, and the matrix $\delta$ becomes $\delta_\lambda^m$, for $\lambda = (-1 : 0 : 1)$.

In the third case, $D = y_3^{m-1}\frac{a_3}{2b}(\alpha_\gamma - \alpha_\lambda)$, so the extension (∗∗) splits.

In the fourth case, without changing the corresponding module, we can choose $a_3 = 1$, and the matrix $\delta$ becomes $\delta_s^m$ (if $a_3 = 0$, the extension given by $\delta$ splits).

In the last case, we can fix $a_9 = 1$ and let $a_3$ to vary. If we denote $a_3 = K$, we obtain the matrix $\delta_K^m$.

The proof of the theorem is finished with the proofs of the following two lemmas.

**Lemma 2.5.** *For all*

$$\delta \in \{\delta_\lambda^m, \delta_K^m | \lambda \in V(f), \lambda \neq (0:1:0), m \in \mathbb{N}, m \geq 1, K \in k\},$$

*the modules Coker $\tilde{\delta}$ are indecomposable.*

*Proof.* Suppose Coker $\tilde{\delta}$ decomposes. Then, there exist two points of $V(f)$, $\mu$ and $\xi$, such that $\delta$ is equivalent to the matrix $T = \begin{pmatrix} \alpha_\mu & 0 \\ 0 & \alpha_\xi \end{pmatrix}$.

Case 1. Consider $m \geq 2$.
For any $\delta \in \{\delta_\lambda^m, \delta_K^m | m \in \mathbb{N}, m \geq 1, \lambda \in V(f), \lambda \neq (0:1:0), K \in k\}$, $Y_3^{m+2} \in \mathrm{Fitt}_3(\delta) \setminus \mathrm{Fitt}_3(T)$. Since two equivalent matrices should have the same fitting-ideals, $\delta$ can not be equivalent to $T$, so Coker $\tilde{\delta}$ is indecomposable.
Case 2. Consider $m = 1$.
The entries of $\delta$ and $T$ are linear forms, so, there exist $U$ and $V$ invertible matrices, with degree zero entries, such that $UT - \delta V = 0$. The Singular-procedure `equiv` (see the last section) checks the existence of such matrices.

```
ring S=0,(y(1..3),a,b,l1,l2,L1,L2,K,u(1..36),v(1..36)),(c,dp);
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2,
        l1^3+l1^2-l2^2,L1^3+L1^2-L2^2;
qring R=std(i);
matrix T[6][6]=
   0,y(1)-l1*y(3),      y(2)-l2*y(3),   0,            0,                 0,
y(1),y(2)+l2*y(3),    (l1^2+l1)*y(3),   0,            0,                 0,
y(3),           0,-y(1)-(l1+1)*y(3),   0,            0,                 0,
   0,           0,                 0,   0,y(1)-L1*y(3),      y(2)-L2*y(3),
   0,           0,                 0,y(1),y(2)+L2*y(3),    (L1^2+L1)*y(3),
   0,           0,                 0,y(3),            0,-y(1)-(L1+1)*y(3);

matrix Sl[6][6]=
   0,y(1)-a*y(3),      y(2)-b*y(3),   0,         2b*y(3),    (3a2+2a)*y(3),
y(1),y(2)+b*y(3),      (a2+a)*y(3),   0,-(3a2+2a)*y(3), -2b*(2a+1)*y(3),
y(3),          0,-y(1)-(a+1)*y(3),   0,              0,          2b*y(3),
   0,          0,                0,   0,   y(1)-a*y(3),      y(2)-b*y(3),
   0,          0,                0,y(1),   y(2)+b*y(3),      (a2+a)*y(3),
   0,          0,                0,y(3),              0,-y(1)-(a+1)*y(3);

matrix SK[6][6]=
   0,y(1),       y(2),    0,   y(3),    K*y(3),
y(1),y(2),          0,    0,-K*y(3),     -y(3),
y(3),   0,-y(1)-y(3),    0,      0,      y(3),
   0,   0,          0,    0,   y(1),      y(2),
   0,   0,          0,y(1),   y(2),         0,
   0,   0,          0,y(3),      0,-y(1)-y(3);

matrix Ss[6][6]=
   0,y(1),       y(2),    0,    0,      y(3),
y(1),y(2),          0,    0,-y(3),         0,
y(3),   0,-y(1)-y(3),    0,    0,         0,
   0,   0,          0,    0, y(1),      y(2),
   0,   0,          0,y(1), y(2),         0,
   0,   0,          0,y(3),    0,-y(1)-y(3);
```

```
equiv(T,Sl);            equiv(T,SK);            equiv(T,Ss);
_[73]=b                 [1]:                    [1]:
_[74]=a                    _[1]=1                  _[1]=1
```

Since either $a \neq 0$ or $b \neq 0$, it follows that, also for $m = 1$, Coker $\tilde{\delta}$ is indecomposable. □

**Lemma 2.6.** *No two modules that up to some shifting are of the form Coker $\tilde{\delta}$, with $\delta \in \{\delta_\lambda^m, \delta_K^m | \lambda \in V(f), \lambda \neq (0:1:0), m \in \mathbb{N}, m \geq 1, K \in k\}$ are isomorphic one with another.*

*Proof.* Case 1. Consider $m \geq 2$.
Since $y_3^3 \in \mathrm{Fitt}_3(\delta_\lambda^m) \setminus \{\mathrm{Fitt}_3(\delta_K^m) \cup \mathrm{Fitt}_3(\delta_s^m)\}$, the matrix $\delta_\lambda^m$ is neither with $\delta_K^m$ nor with $\delta_s^m$ equivalent.
If $K^2 - 1 = 0$, $y_3^{2m+2} \in \mathrm{Fitt}_4(\delta_s^m) \setminus \mathrm{Fitt}_4(\delta_K^m)$, so $\delta_K^m$ and $\delta_s^m$ are not equivalent.
We prove now that $\delta_s^m$ is not equivalent to any $\delta_K^m$, in the case $K^2 \neq 1$. In a very similar one proves that $\delta_1^m$ and $\delta_{-1}^m$ are not equivalent.
Suppose that there exists $d_0 \in \mathbb{Z}$ such that Coker$\delta_s^m$ and Coker$\delta_K^m(d_0)$ are isomorphic.
Then there exist two invertible morphisms

$$\tilde{U} : R(-1)^3 \oplus R(-m)^3 \longrightarrow R(d_0-1)^3 \oplus R(d_0-m)^3$$

and

$$\tilde{V} : R(-2)^3 \oplus R(-m-1)^3 \longrightarrow R(d_0-2)^3 \oplus R(d_0-m-1)^3$$

such that

$$\tilde{U}\tilde{\delta}_s^m = \tilde{\delta}_K^m \tilde{V}.$$

Let $U$ and $V$ be the graded invertible matrices corresponding to these morphisms. Write $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ and $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$, $\delta_s^m = \begin{pmatrix} \alpha_s & D \\ 0 & \alpha_s \end{pmatrix}$ and $\delta_K = \begin{pmatrix} \alpha_s & D_K \\ 0 & \alpha_s \end{pmatrix}$. The entries of $U_1, U_4, V_1, V_4$ have degree $d_0$, the entries of $U_2$ and $V_2$ have degree $m + d_0 - 1$ and the entries of $U_3$ and $V_3$ have degree $d_0 - m + 1$. Therefore, since $U$ is invertible, $d_0 = 0$, $U_1, U_4, V_1, V_4$ have degree zero entries and $U_3 = V_3 = 0$. The relation $U\delta_s^m = \delta_K^m V$ means the following system of equalities:

$$U_1\alpha_s = \alpha_s V_1 \quad (1)$$

$$U_1 D + U_2\alpha_s = \alpha_s V_2 + D_K V_4, \ (2)$$

$$U_4\alpha_s = \alpha_s V_4. \ (3)$$

The procedure `equiv` shows that the equalities (1) and (3) imply $U_1 = V_1 = k_1 Id$ and $U_4 = V_4 = k_4 Id$, where $k_1$ and $k_4$ are constants.

```
ring S=0,(y(1..3),u(1..9),v(1..9),K,Y),(c,dp);
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3);
qring R=std(i);

matrix alphas[3][3]= 0,y(1),      y(2),
                    y(1),y(2),        0,
                    y(3),   0,-y(1)-y(3);
equiv(alphas,alphas);
U=
v(9),0,   0,
0,   v(9),0,
0,   0,   v(9)


V=
v(9),0,   0,
0,   v(9),0,
0,   0,   v(9)
```

Therefore, $k_1D - k_4D_K = \alpha_s V_2 - U_2\alpha_s$. So, the entries [1,2] and [2,2] of $k_1D - k_4D_K$ are in the ideal $(y_1, y_2)$, that is possible only if $k_4 = k_1 = 0$. But then, $U$ is not any more invertible.

Case 2. Consider now $m = 1$.
In this case $\delta_\lambda$, $\delta_s$ and $\delta_K$ have the same fitting ideals. But, since the possible transformation matrices $U$ and $V$ should have only degree zero entries, the equivalence of these matrices can be checked with the Singular-procedure `equiv`:

```
ring S=0,(y(1..3),a,b,c,d,K1,K2,u(1..36),v(1..36)),(c,dp);
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2,c3+c2-d2;
qring R=std(i);
matrix Sl1[6][6]=
   0,y(1)-a*y(3),     y(2)-b*y(3),   0,       2b*y(3),   (3a2+2a)*y(3),
y(1),y(2)+b*y(3),     (a2+a)*y(3),   0,-(3a2+2a)*y(3), -2b*(2a+1)*y(3),
y(3),         0,-y(1)-(a+1)*y(3),   0,             0,         2b*y(3),
   0,         0,               0,   0,   y(1)-a*y(3),     y(2)-b*y(3),
   0,         0,               0,y(1),   y(2)+b*y(3),     (a2+a)*y(3),
   0,         0,               0,y(3),             0,-y(1)-(a+1)*y(3);

matrix Sl2[6][6]=
   0,y(1)-c*y(3),     y(2)-d*y(3),   0,       2d*y(3),   (3c2+2c)*y(3),
y(1),y(2)+d*y(3),     (c2+c)*y(3),   0,-(3c2+2c)*y(3), -2d*(2c+1)*y(3),
y(3),         0,-y(1)-(c+1)*y(3),   0,             0,         2d*y(3),
```

```
   0,          0,                0,   0,   y(1)-c*y(3),      y(2)-d*y(3),
   0,          0,                0,y(1),   y(2)+d*y(3),      (c2+c)*y(3),
   0,          0,                0,y(3),            0,-y(1)-(c+1)*y(3);

matrix Ss[6][6]=
   0,y(1),      y(2),   0,    0,     y(3),
y(1),y(2),         0,   0,-y(3),        0,
y(3),   0,-y(1)-y(3),   0,    0,        0,
   0,   0,         0,   0, y(1),      y(2),
   0,   0,         0,y(1), y(2),         0,
   0,   0,         0,y(3),    0,-y(1)-y(3);

matrix SK1[6][6]=
   0,y(1),      y(2),   0,   y(3),    K1*y(3),
y(1),y(2),         0,   0,-K1*y(3),     -y(3),
y(3),   0,-y(1)-y(3),   0,      0,      y(3),
   0,   0,         0,   0,   y(1),      y(2),
   0,   0,         0,y(1),   y(2),         0,
   0,   0,         0,y(3),      0,-y(1)-y(3);

matrix SK2[6][6]=
   0,y(1),      y(2),   0,   y(3),    K2*y(3),
y(1),y(2),         0,   0,-K2*y(3),     -y(3),
y(3),   0,-y(1)-y(3),   0,      0,      y(3),
   0,   0,         0,   0,   y(1),      y(2),
   0,   0,         0,y(1),   y(2),         0,
   0,   0,         0,y(3),      0,-y(1)-y(3);

equiv(Sl1,Sl2);     equiv(Sl1,Ss);
_[69]=b-d           [1]:
_[70]=a-c              _[1]=1

equiv(Sl1,SK1);    equiv(Ss,SK1);
[1]:               [1]:
   _[1]=1             _[1]=1

equiv(SK2,SK1);
_[71];
K1-K2
```

□

**Remark.** The modules Coker $\tilde{\delta}^m_{-1}$ and Coker $\tilde{\delta}^m_1$ are non-locally free. All other rank two indecomposable Ulrich $R$–modules are locally free (because $\mathrm{Fitt}_4 R_{\langle y_1,y_2\rangle} = R_{\langle y_1,y_2\rangle}$; see for example Prop.1.3, [TJP]).

## 3 Singular-procedures

```
 option(redSB); LIB"matrix.lib"; LIB"homolog.lib"; LIB"linalg.lib";

 proc simple(ideal P) //divides the polynomials by powers of y_2 or y_3
 { int j,i; poly F;
   list L=0;
   for(j=1;j<=size(P);j++)
    { L=factorize(P[j]);
      if(size(L[1])>2)
       { F=1;
         for(i=2;i<=size(L[1]);i++)
          { if(L[1][i]==y(2) or L[1][i]==y(3) or L[1][i]==Y)
             { L[1][i]=1;}
               F=F*L[1][i]^(L[2][i]);}
            P[j]=F;}}
  return(P);}

 proc condext(matrix A,B,D)
 { matrix V; int k,j; ideal P=0; list L=0;
   matrix Aa=adjoint(A); matrix Ba=adjoint(B); matrix G=Aa*D*Ba;
   ideal g=flatten(G);
   for(j=1;j<=size(G);j++)
    { g[j]=reduce(g[j],std(y(1)^3+y(1)^2*y(3)-y(2)^2*y(3)));
      V=coef(g[j],y(1));
      for(k=1;k<=1/2*size(V);k++)
       { P=P+V[2,k];}}
  P=interred(P); P=simple(P);
  return(P);}

proc equiv(matrix X,matrix Y)
{ list z;int n=nrows(X);
  matrix U[n][n]=u(1..n^2); matrix V[n][n]=v(1..n^2);
  matrix C=U*X-Y*V; ideal I=flatten(C);
  ideal I1=transpose(coeffs(I,y(1)))[2];
  ideal I2=transpose(coeffs(I,y(2)))[2];
  ideal I3=transpose(coeffs(I,y(3)))[2];
  ideal J=I1+I2+I3;
  ideal L=std(J);
```

```
if (n==6){z=std(L+(det(U)-1));return(z);}
else{ U=reduce(flatten(U),std(L));z[1]=U;"U=";print(U);
      V=reduce(flatten(V),std(L));z[2]=V;" V=";print(V);}}
```

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