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# ON THE DISTRIBUTION OF A CERTAIN FAMILY OF FIBONACCI TYPE SEQUENCES

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Abstract. Taking the Fibonacci sequence $G_0 = 1, G_1 = b \in \{1, 3, 5\}$ and $G_{n+1} = 3 \cdot G_n + G_{n-1}$ $(n \geq 1)$ with an integer $2 \leq m \in \mathbb{N}$ , we get a purely periodic sequence $\{G_n(\mod m)\}$. Consider any shortest full period and form a frequency block $B_m \in \mathbb{N}^m$ to consist of the frequency values of the residue $d$ when $d$ runs through the complete residue system modulo $m$. The purpose of this paper is to show that such frequency blocks can nearly always be produced by repetition of some multiple of their first few elements a certain number of times. Theorems 3,4 and 5 contains our main results where we show when this repetition does occur, what elements will be repeated, what is the repetition number and how to calculate the value of the multiple.

Let $A, B \neq 0, G_0 = a, G_1 = b$ with $a^2 + b^2 > 0$ be fixed rational integers, let $D = A^2 \Leftrightarrow 4B^2 \neq 0$ and define the Fibonacci type sequence $\{G_n\} = G(A, B, a, b)$ to satisfy the recurrence relation

$$G_{n+1} = A \cdot G_n \Leftrightarrow B \cdot G_{n-1} \quad \text{for } n \geq 1.$$

Let $\{U_n\}$ and $\{V_n\}$ be the Fibonacci and the Lucas sequence deriving from $\{G_n\}$ for $a = 0, b = 1$ and for $a = 2, b = A$, respectively. Then it is easy to check $G_n = \frac{a}{2}V_n + (b \Leftrightarrow \frac{a}{2}A) \cdot U_n$. For $r_{1,2} = \frac{A \pm \sqrt{D}}{2}$ the following equations hold

$$U_n = (r_1^n \Leftrightarrow r_2^n)/(r_1 \Leftrightarrow r_2) \quad \text{and} \quad V_n = r_1^n + r_2^n, \tag{1}$$

which yield also

$$U_{2k+l} = V_{k+l}U_k + B^k U_l \quad \text{and} \quad V_{2k+l} = DU_k U_{k+l},$$

whence

$$V_k^2 = DU_k^2 + 4B^k. \tag{2}$$

We note that in this paper $(x, y)$ and $[x, y]$ will be written instead of $gcd(x, y)$ and $lcm(x, y)$, respectively. Furthermore, $v_{13}(z)$ will denote the greatest power of 13 in the integer $z$, that means $13^{v_{13}(z)} | z$, but $13^{v_{13}(z)+1} \nmid z$.

Now, take $A = 3, B = \Leftrightarrow 1, a = 1, b \in \{1, 3, 5\}$ and $2 \leq m \in \mathbb{N}$. Then $(B, m) = 1$ and for this reason $\{G_n\}$ is purely periodic modulo $m$ (see Theorem 1 in [1]). Let $h(a, b, m) := h(m)$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX denote the shortest period of the sequence $\{G_n(\mod m)\}$. Define $S(m)$ to be the set of residue frequencies within any full period of $\{G_n(\mod m)\}$ and let $A(m,d)$ denote the number of times the residue $d$ appears in a full period of $\{G_n(\mod m)\}$. Hence for a fixed $m$, the range of $A(m,d)$ is the same as the set $S(m)$. We say that $\{G_n\}$ is uniformly distributed modulo $m$ if all residues modulo $m$ occur with the same frequency in any full period. In this case the length of any period will be a multiple of $m$, moreover $|S(m)| = 1$ and $A(m,d)$ is constant. It is known that $G(3, \Leftrightarrow 1, 1, b)$ with $b \in \{1,3,5\}$ is uniformly distributed modulo $13^k$ for all $k \geq 1$ (see [2]). Thus, $|S(13^k)| = 1, A(13^k, d) = 4$ and then $h(13^k) = H(13^k) = 2 \cdot \sigma \cdot 13^k$ (see Corollaries 5 and 6 and Theorem 7 in [1]), where $H(13^k) = h(0,1,13^k)$ and $\sigma = ord_{13}(\Leftrightarrow 1) = 2$ denotes the exact order of $B = \Leftrightarrow 1$ modulo 13, that means $h(13^k) = 4 \cdot 13^k$.

For a fixed $m$ form a number block $B_m \in \mathbb{N}^m$ to consist of the frequency values of the residue $d$ when $d$ runs through the complete residue system modulo $m$. This number block $B_m$ will be called the frequency block modulo $m$, which has properties like $(qB_m)^r = q(B_m)^r$ and $((B_m)^r)^s = (B_m)^{rs}$ with $(B_m)^r := \underbrace{(B_m, ..., B_m)}_{r \text{ times}}$ and $q, r, s \in \mathbb{N}$. Here are some examples for $B_m$ if we take $m = 13c$ with $2 \leq c \in \mathbb{N}$ and $G(3, \Leftrightarrow 1, 1, b)$ with $b \in \{1,3,5\}$.

$$B_2 = (1,2)$$

$$B_{26} = \underbrace{(4,8,...,4,8)}_{13 \text{ times}} = \underbrace{(4B_2, ..., 4B_2)}_{13 \text{ times}} = (4B_2)^{13} = 4(B_2)^{13}$$

$$h(2) = 3, h(26) = 4 \cdot 13 \cdot h(2) = 156$$

$$B_4 = (1,3,1,1)$$

$$B_{52} = \underbrace{(2,6,2,2,...,2,6,2,2)}_{13 \text{ times}} = \underbrace{(2B_4, ..., 2B_4)}_{13 \text{ times}} = (2B_4)^{13} = 2(B_4)^{13}$$

$$h(4) = 6, \quad h(52) = 2 \cdot 13 \cdot h(4) = 156$$

$$B_5 = (0,3,3,3,3)$$

$$B_{65} = \underbrace{(0,3,3,3,3,...,0,3,3,3,3)}_{13 \text{ times}} = \underbrace{(B_5, ..., B_5)}_{13 \text{ times}} = (B_5)^{13}$$

$$h(5) = 12, \quad h(65) = 1 \cdot 13 \cdot h(5) = 156$$

$$B_{53} = (0,2,0,1,1,0,0,1,0,0,0,1,1,1,...)$$

$$B_{689} = (0,2,0,1,1,0,0,1,0,0,0,0,0,2,...) \neq (B_{53})^{13}$$

$$h(53) = 26, \quad h(689) = 52 \neq 1 \cdot 13 \cdot h(53)$$

All examples except the last one show a kind of repetition in the frequency blocks, that means such frequency blocks can be produced by repetition of their first few elements a certain number of times. Moreover, the first few repeting elements of $B_m$ are the elements of $B_c$ or some multiple of them. This fact can be expressed by $A(m,y) = q \cdot A(c,x)$ for

$0 \leq x < c, 0 \leq y < m$, $y \equiv x(\mod c)$ and $q \in \{1, 2, 4\}$. A similar result in connection with the uniform distribution was found in [3] for the Fibonacci sequence.

The purpose of this paper is to investigate such kind of repetition properties in the frequency blocks of $G(3, \Leftrightarrow 1, 1, b)$ with $b \in \{1, 3, 5\}$ modulo $13c$ for $2 \leq c \in \mathbb{N}$. The questions to answer at first are when this repetition does occur and how to calculate the value of the factor $q$. This will be answered in our theorems, but first we prove some necessary lemmas.

**Lemma 1.** Let $m, n \in \mathbb{N}$, $0 < |m \Leftrightarrow n| < h(13) = 52$ and $m \equiv n(\mod s)$ with $1 < s|4$. Then $G_m \not\equiv G_n(\mod 13)$.

*Proof.* The characteristic polynomial of $\{G_n\}$ is $x^2 \Leftrightarrow 3x \Leftrightarrow 1$ with the roots $r_1 = (3+\sqrt{13})/2$ and $r_2 = (3 \Leftrightarrow \sqrt{13})/2$. Then by the Binet equation $G_n = (r_1^n \Leftrightarrow r_2^n)/(r_1 \Leftrightarrow r_2)$ we have

$$G_n = \frac{2^{-n}}{\sqrt{13}} \left( (3 + \sqrt{13})^n \Leftrightarrow (3 \Leftrightarrow \sqrt{13})^n \right) = \frac{2^{1-n}}{\sqrt{13}} \sum_{j \text{ odd}}^{n} \binom{n}{j} \cdot 3^{n-j} \cdot \sqrt{13}^j.$$

Let $s = 4$ and suppose that $m > n$, which can be assumed without loss of generality. From $m \equiv n(\mod 4)$ and $0 < m \Leftrightarrow n < 52$ it follows $m = n + 4t$ with $t \in \{1, 2, ..., 12\}$. Thus

$$\begin{aligned} G_m \Leftrightarrow G_n &= \frac{2^{1-m}}{\sqrt{13}} \sum_{j \text{ odd}}^{m} \binom{m}{j} \cdot 3^{m-j} \cdot \sqrt{13}^j \\ &\Leftrightarrow \frac{2^{1-n}}{\sqrt{13}} \sum_{j \text{ odd}}^{n} \binom{n}{j} \cdot 3^{n-j} \cdot \sqrt{13}^j \\ &= 2^{1-n-4t} \Big( \sum_{j \text{ odd}}^{n+4t} \binom{n+4t}{j} \cdot 3^{n+4t-j} \cdot \sqrt{13}^{j-1} \\ &\Leftrightarrow 2^{4t} \sum_{j \text{ odd}}^{n} \binom{n}{j} \cdot 3^{n-j} \cdot \sqrt{13}^{j-1} \Big), \end{aligned}$$

whence

$$\begin{aligned} 2^{n+4t-1}(G_m \Leftrightarrow G_n) &= \left( \binom{n+4t}{1} \cdot 3^{n+4t-1} \Leftrightarrow 2^{4t} \binom{n}{1} \cdot 3^{n-1} \right) \\ &+ 13 \left( \sum_{j=3\, j \text{ odd}}^{n+4t} \binom{n+4t}{j} \cdot 3^{n+4t-j} \sqrt{13}^{j-3} \Leftrightarrow 2^{4t} \sum_{j=3\, j \text{ odd}}^{n} \binom{n}{j} \cdot 3^{n-j} \sqrt{13}^{j-3} \right) \\ &:= K + 13L, \end{aligned}$$

where $K, L$ are integers. Now, we state that $K$ is not divisible by 13. The reason for this is $K = (n + 4t) \cdot 3^{n+4t-1} \Leftrightarrow 2^{4t} \cdot n \cdot 3^{n-1} = 3^{n-1}((n + 4t) \cdot 3^{4t} \Leftrightarrow 2^{4t} \cdot n)$ and $3K = 3^n((n + 4t) \cdot 81^t \Leftrightarrow 16^t \cdot n) \equiv 4t \cdot 3^{n+t} (\mod 13)$, whence $13 \not| 3K$ since $t \in \{1, 2, ..., 12\}$, and $13 \not| K$ is already true. All these yield $13 \not| (G_m \Leftrightarrow G_n)$, that is $G_m \not\equiv G_n (\mod 13)$. The remaining case $s = 2$ can be carried out in a similar way.

The above statement could have been proved by comparing the residues of $G_m$ and $G_n$ modulo 13 for all possible vaues of $m$ and $n$ with $m > n$ , $m \equiv n \pmod{s}$ , $0 < m \Leftrightarrow n < 52$ and $1 < s|4$. But this comparing would consist of 312 cases for $s = 4$, and 650 cases for $s = 2$. That would be rather boring.

**Lemma 2.** $v_{13}(U_{13^k}) = k$ for all $k \in \mathbb{N}$.

*Proof.* From (1) we generally have

$$2^{l-1} \cdot U_l = \sum_{\mu=0}^{[(l-1)/2]} \binom{l}{2\mu+1} \cdot A^{l-1-2\mu} \cdot D^{\mu},$$

and especially,

$$2^{13^k-1} \cdot U_{13^k} = \sum_{\mu=0}^{[(13^k-1)/2]} \binom{13^k}{2\mu+1} \cdot 3^{13^k-1-2\mu} \cdot 13^{\mu}. \tag{3}$$

The right-hand side of (3) is $3^{13^k-1} \cdot 13^k$ for $\mu = 0$, moreover all terms with $\mu \geq 1$ on the same side of (3) are divisible at least by $13^{k+1}$. Namely, on the basis of

$$\binom{13^k}{2\mu+1} = \frac{13^k}{2\mu+1} \cdot \binom{13^k \Leftrightarrow 1}{2\mu} \tag{4}$$

the preceding statement follows immediatelly for $13 \nmid (2\mu + 1)$, if we consider the factor $13^{\mu}$ on the right-hand side of (3). On the other hand, if $j = v_{13}(2\mu+1)$, then $13^j \leq 2\mu+1$, and this is why 13 occurs in the sum on the right-hand side of (3) at least to the power $k \Leftrightarrow j + \mu \geq k + 5$. Thus the statement is completely proved.

**Lemma 3.** $13^k|U_m \Rightarrow 13^k|m$.

*Proof.* Let $m = 13^j c$ and $n = 13^k$, where $j \in \{0, ..., k \Leftrightarrow 1\}$ and $13 \nmid c$. Then $d = (m, n) = 13^j$. This leads to $(U_m, U_n) = U_{(m,n)} = U_{13^j}$ and to $13^k|U_n$ using Lemma 2 and the well-known identity $(U_m, U_n) = U_d$. The consequence of all this is $13^k|(U_m, U_n) = U_{13^j}$, which is a contradiction since $k > j$.

**Theorem 1.** If for a fixed $\beta \in \{1, ..., h(13^k)\}$ the number $G_\beta$ leaves the remainder $\alpha \in \{0, ..., 13^k \Leftrightarrow 1\}$ modulo $13^k$ , then the numbers $G_{\beta+r\cdot h(13^k)}$ leave the remainders $\alpha + s \cdot 13^k$ modulo $13^{k+1}$ in a certain ordering, where $r, s \in \{0, ..., 12\}$.

*Proof.* For $r \in \{0, ..., 12\}$ we obviously have

$$G_{\beta+r\cdot h(13^k)} = \alpha + 13^k \cdot u_r$$

with some $u_r \in \mathbb{Z}$. Assume

$$0 \leq r' < r \leq 12 \quad \text{and} \quad u_{r'} \equiv u_r \pmod{13}. \tag{5}$$

Then

$$
\begin{aligned}
&13^{k+1}|(G_{\beta+r\cdot h(13^k)} \Leftrightarrow G_{\beta+r'\cdot h(13^k)})\\
&= \frac{a}{2}(V_{\beta+r\cdot h(13^k)} \Leftrightarrow V_{\beta+r'\cdot h(13^k)})\\
&+ (b \Leftrightarrow \frac{a}{2}A)(U_{\beta+r\cdot h(13^k)} \Leftrightarrow U_{\beta+r'\cdot h(13^k)})\\
&= \frac{1}{2}U_{2(r-r')13^k}\left(13aU_{\beta+2(r+r')\cdot 13^k} + (2b \Leftrightarrow Aa)V_{\beta+2(r+r')\cdot 13^k}\right). \qquad (6)
\end{aligned}
$$

If 13 would divide the term in the brackets of (6), then $13|V_{\beta+2(r+r')\cdot 13^k}$ should be also true. But this is impossible becasue of (2). Thus from (6) we have $13^{k+1}|U_{2(r-r')13^k}$ and this is why $13^{k+1}|2(r \Leftrightarrow r')13^k$, that means $13|r \Leftrightarrow r'$, which contradicts (5).

**Theorem 2.** For $2 \le c \in \mathbb{N}$, $(c, 13) = 1$, $v_{13}(h(c)) \le k \Leftrightarrow 1$ and

$$q := \frac{h(13^k c)}{13 \cdot h(13^{k-1}c)}$$

with $k \in \mathbb{N}_0$ we have $q|4$.

*Proof.* From $(c, 13) = 1$ it follows

$$q = \frac{[h(13^k), h(c)]}{13 \cdot [h(13^{k-1}), h(c)]}$$

for all $k \in \mathbb{N}$ (see Theorem 2 in [7]). The case $k = 1$ results from

$$q = \frac{[h(13), h(c)]}{13 \cdot [h(1), h(c)]} = \frac{[52, h(c)]}{13 \cdot h(c)} = \frac{4}{(52, h(c))} = \frac{4}{(4, h(c))},$$

since $13 \nmid h(c)$ because of $v_{13}(h(c)) = 0$ for $k = 1$. Hence $q \cdot (4, h(c)) = 4$, that is $q|4$.

The case $k > 1$ results from

$$q = \frac{[4 \cdot 13^k, h(c)]}{13 \cdot [4 \cdot 13^{k-1}, h(c)]} = \frac{(4 \cdot 13^{k-1}, h(c))]}{(4 \cdot 13^k, h(c))},$$

whence $q = 1$ since $v_{13}(h(c)) \le k \Leftrightarrow 1$, thus $q|4$ is true again.

**Corollary 1.** For $2 \le c \in \mathbb{N}$ , $(c, 13) = 1$ and $k \in \mathbb{N}$, $q = \frac{h(13^k c)}{13h(13^{k-1}c)}$ is an integer iff $v_{13}(h(c)) \le k \Leftrightarrow 1$.

The possible cases are as follows:

$$k = 1 \Rightarrow q = \begin{cases} 4 & \text{if } (4, h(c)) = 1 \Leftrightarrow h(c) \text{ is odd} \\ 2 & \text{if } (4, h(c)) = 2 \Leftrightarrow 2|h(c) \wedge 4 \nmid h(c) \\ 1 & \text{if } (4, h(c)) = 4 \Leftrightarrow 4|h(c) \end{cases}$$

$k > 1 \Rightarrow q = 1$.

**Corollary 2.** For $2 \le c = 13^r \cdot s \in \mathbb{N} 1 \le r, s \in \mathbb{N}$ and $(s, 13) = 1$, $q = \frac{h(13c)}{13h(c)}$ is an integer iff $v_{13}(h(s)) \le r$. The only possible case is $q = 1$.

We note that $q \in \{4/13, 2/13, 1/13\}$ for $k = 1$ and $v_{13}(h(c)) > 0$, moreover $q = 1/13$ for $k > 1$ and $v_{13}(h(c)) > k \Leftrightarrow 1$. This happens for example taking $c \in \{53, 79, 157, ...\}$.

Further on we make use of the known fact that the purely periodic property of $\{G_m(\mod c)\}$ yields the identity of the values $G_{w+jh(c)}$ modulo $c$ for all $w, j \in \mathbb{N}$ and $2 \le c \in \mathbb{N}$.

**Theorem 3.** For $2 \le c \in \mathbb{N}$, $(c, 13) = 1$, $v_{13}(h(c)) = 0$ and $q = \frac{h(13c)}{13h(c)}$ we have $B_{13c} = q(B_c)^{13}$.

*Proof.* According to Corollary 1 for $k = 1$ , we have to consider the three cases when $q \in \{1, 2, 4\}$.

Case 1: $q = 1 \Leftrightarrow (4, h(c)) = 4$, that is $4|h(c)$. Now, $h(13c) = 13h(c)$, and it is to show that for all $w \in \mathbb{N}$ and $j \in \{0, 1, ..., 12\}$, the 13 values of $G_{w+jh(c)}$ are pairwise different modulo 13, and hereby also modulo $13c$. Assume $G_{w+j_1h(c)} \equiv G_{w+j_2h(c)} (\mod 13)$ for $j_1, j_2 \in \{0, 1, ..., 12\}$ and $0 < |j_1 \Leftrightarrow j_2| < 13$, moreover let $k_1$ and $k_2$ denote the values of $G_{w+j_1h(c)}$ respectively reduced modulo 52, that is $0 \le |k_1 \Leftrightarrow k_2| = |j_1 \Leftrightarrow j_2|h(c) < 52$. This gives $G_{k_1} \equiv G_{k_2} (\mod 13)$, since $h(13) = 52$. Now, $0 < |j_1 \Leftrightarrow j_2| < 13$ and $v_{13}(h(c)) = 0$ yield $52 \not| |j_1 \Leftrightarrow j_2| \cdot h(c)$, whence $k_1 \ne k_2$. From $4|h(c)$ follows that the values of $w + j_1h(c)$ and $w + j_2h(c)$ are in the same residue class modulo 4 , and so are $k_1$ and $k_2$ , too. But this contradicts Lemma 1.

Case 2: $q = 2 \Leftrightarrow (4, h(c)) = 2$, that is $2|h(c)$ but $4 \not| h(c)$. Now $h(13c) = 26h(c)$, and it is to show that for all $w \in \mathbb{N}$ and $j \in \{0, 1, ..., 25\}$, among the 26 values of $G_{w+jh(c)}$ there are at most two ones congruent modulo 13, and hereby also modulo $13c$. Assume that there are at least three ones congruent modulo 13, which are $G_{w+j_1h(c)}$, $G_{w+j_2h(c)}$ and $G_{w+j_3h(c)}$ with $j_1, j_2, j_3 \in \{0, 1, ..., 25\}$ and $0 < |j_1 \Leftrightarrow j_2|, |j_1 \Leftrightarrow j_3|, |j_2 \Leftrightarrow j_3| < 26$. Let $k_1, k_2$ and $k_3$ denote the values of $w + j_1h(c)$, $w + j_2h(c)$ and $w + j_3h(c)$ respectively reduced modulo 52. Thus $k_1, k_2$ and $k_3$ are pairwise different, and fall into the same residue class modulo 2. This contradicts Lemma 1 again.

Case 3: $q = 4 \Leftrightarrow (4, h(c)) = 1$, that is $2 \not| h(c)$. Now $h(13c) = 52h(c)$, and it is to show that for all $w \in \mathbb{N}$ and $j \in \{0, 1, ..., 51\}$ among the 52 values of $G_{w+jh(c)}$ with pairwise different indices modulo 52 there are at most four identical ones modulo 13. But this follows from the uniform distribution of the sequence $\{G_n(\mod 13)\}$.

**Theorem 4.** For $2 \le c = 13^r \cdot s \in \mathbb{N}$, $r \ge 0, s \ge 1, (s, 13) = 1, v_{13}(h(s)) \le r$ and $q = \frac{h(13c)}{13h(c)}$ we have $B_{13c} = q(B_c)^{13}$.

*Proof.* The case $r = 0$ leads again to Theorem 3. The case $r \ge 1$ and $s = 1$ is well known uniform distribution. Then $q = 1$ and $B_{13^{r+1}} = 1 \cdot (B_{13^r})^{13}$ is true (see [2]).

Case $r \ge 1$ and $s > 1$:

Now $q = 1$ again and $B_{13^{r+1}s} = 1 \cdot (B_{13^r s})^{13}$ is to prove.

It is to show that for any $w \in \mathbb{N}$ and $j \in \{0, 1, ..., 12\}$ the numbers $G_{w+jh(c)}$ are pairwise different modulo $13c$. Since $(s,13)=1$ and $v_{13}(h(s))\le r$ we have

$$h(c)=H(13^r s)=[h(13^r),h(s)]=h(13^r)\cdot\frac{h(s)}{(h(13^r),h(s))}=h(13^r)\cdot z$$

with some $z\in\mathbb{N}$ and $13\not|z$. Hence for any $w\in\mathbb{N}$ and $j\in\{0,...,12\}$, the numbers $w+jh(c)$ and $w+jh(13^r)$ are always in the same residue class modulo $h(13^r)$, therefore the numbers $G_{w+jh(c)}$ and $G_{w+jh(13^r)}$ are in the same residue class modulo $13^r$, too. But for a fixed $w\in\mathbb{N}$ the numbers $G_{w+jh(13^r)}$ are pairwise different modulo $13^{r+1}$ because of Theorem 1. Thus the numbers $G_{w+jh(c)}$ are again pairwise different modulo $13^{r+1}$, and hereby also modulo $13c$.

**Theorem 5.** For $2\le c\in\mathbb{N}$, $(c,13)=1$, $v_{13}(h(c))\le k\not\Leftrightarrow 1$ and and $q=\frac{h(13^k c)}{13h(13^{k-1}c)}$ with $k\in\mathbb{N}_0$ we have $B_{13^k c}=q(B_{13^{k-1}c})^{13}$.

*Proof.* We proceed by induction on $k$.

For $k=1$ we get Theorem 3. Assume that the statement is true for all $k>1$. Then because of the case $k>1$ in Theorem 2 one has to take $q=1$. Thus

$$\begin{aligned}B_{13^{k+1}c}&=B_{13(13^k c)}=(B_{13^k c})^{13}=(q(B_{13^{k-1}c})^{13})^{13}\\&=q((B_{13^{k-1}c})^{13})^{13}=q(B_{13^k c})^{13}.\end{aligned}$$

**Corollary 3.** For $2\le c\in\mathbb{N}$, $(c,13)=1$, $v_{13}(h(c))\le k\not\Leftrightarrow 1$ and and $q=\frac{h(13^k c)}{13h(13^{k-1}c)}$ with $k\in\mathbb{N}_0$ we have $B_{13^k c}=q(B_c)^{13^k}$.

*Proof.*

$$\begin{aligned}B_{13^k c}&=q(B_{13^{k-1}c})^{13}=q(B_{13(13^{k-2}c)})^{13}\\&=q(B_{13^{k-2}c})^{13^2}=...=q(B_{13c})^{13^{k-1}}=q(B_c)^{13^k}.\end{aligned}$$

**Corollary 4.** For $2\le c\in\mathbb{N}$, $(c,13)=1$, $v_{13}(h(c))\le k\not\Leftrightarrow 1$ and $q=\frac{h(13^k c)}{13h(13^{k-1}c)}$ with $k\in\mathbb{N}_0$ we have $|S(13^k c)|=|S(c)|$, since $q\in\{1,2,4\}$.

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