Invariance properties of automorphism groups of algebras

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1. Introduction and terminology

In this paper K will be a commutative ring, which will become an algebraically closed field in section 2. All algebras will be associative K-algebras with identity and it will be assumed that if M is a bimodule over two K-algebras, then the action of K on M is the same on the left and on the right. For a given algebra A, the category of (left) A-modules will be denoted by $A-Mod$, while $A-mod$ will stand for its full subcategory of finitely generated (f.g.) A-modules. All functors between module categories are K-linear, i.e., they induce K-linear maps between the corresponding Hom K-vector spaces. Likewise, all automorphisms of algebras are K-automorphisms. For a given K-algebra A, we shall denote by $Aut(A)$ the group of K-linear automorphisms of A and by $Out(A)$ the quotient of $Aut(A)$ by the normal subgroup $Inn(A)$ of inner automorphisms. We shall denote by $J(A)$ the Jacobson radical of A and the normal subgroup of $Aut(A)$ given by the inner automorphisms induced by elements of the form $1-x$, with $x \in J(A)$, will be denoted by $Inn^*(A)$. In the particular case when K is an algebraically closed field and A is finite dimensional over K, all these automorphism groups are algebraic groups.

The Picard group of A over K will be denoted by $Pic(A)$. It is classically defined as the one having as elements the isoclasses of invertible $A-A-$bimodules, with the tensor product as operation. Due to Morita's theorem, it can be redefined as the one having as elements the natural isoclasses of K-linear Morita equivalences $A-Mod \xrightarrow{\cong} A-Mod$, with the composition of functors as operation. In both interpretations we shall abuse of notation and identify a bimodule or a Morita equivalence with its isoclass. When $\varphi \in Aut(A)$, we will denote by ${}_1A_\varphi$ the invertible $A-A-$bimodule coinciding with ${}_AA$ as a left module, but with the right A-module structure given by $x \cdot a = x\varphi(a)$, for all $x, a \in A$. Also, for a left A-module M, we shall denote by ${}^\varphi M$ the left A-module having the same undelying K-module structure as M, but with external multiplication by elements of A defined by $a \cdot x = \varphi(a)x$, for all $a \in A$, $x \in M$. For all the remaining concepts not explicitly defined in the paper, the reader is referred to references [1], [13] and [9], on what concerns rings, algebras and algebraic groups, respectively.

The goal of this paper is twofold. On one side, we present a brief history of the (recent) known results about invariance properties of groups of automorphisms of finite dimensional algebras over an algebraically closed field. That is the content of section 2. The second goal, which is the content of the last section, is to present a list of new results on the Morita invariance of some automorphism groups of algebras over an arbitrary commutative ring. The results of the first section are given without proofs, referring the reader to the original references. The proofs of the results in the last section are just sketched, referring the reader to a forthcoming paper for complete proofs.

2. A brief history of known results

All through this section, unless especifically stated otherwise, K will be an algebraically closed field and A will be a finite dimensional algebra.

By taking the simple example $A = K$, $B = \mathcal{M}_{n\times n}(K)$ $(n > 1)$, one clearly sees that the group of automorphisms $Aut(A)$ is not a Morita invariant of the algebra A. Intuitively speaking, the group $Aut(A)$ is of a very arithmetic nature and has not much categorical information. The following result, which is essentially included in [3], seems to say that things could be rather different with the group $Out(A)$.

Theorem 1 *Let K be a commutative ring and A a K-algebra. The assignment $\varphi \longrightarrow_1 A_\varphi$ defines a group homomorphism $\Omega : Aut(A) \longrightarrow Pic(A)$ with kernel $Inn(A)$ and image $\{P \in Pic(A) : {}_AP \cong_A A\}$ or, equivalently, $\{F :_A Mod \longrightarrow_A Mod : F$ is a Morita equivalence and $F({}_AA) \cong_A A\}$*

In spite of the above theorem, the following examples show that $Out(A)$ is not Morita invariant either.

Examples 1 *1. If $A = K \times K$ and $B = \mathcal{M}_m(K) \times \mathcal{M}_n(K)$, with $m \neq n$, then the commutative condition of A gives that $Out(A) = Aut(A) \cong \mathbf{C}_2$ is cyclic of order 2 while $Aut(B) \cong Aut(\mathcal{M}_m(K)) \times Aut(\mathcal{M}_n(K))$, which coincides with $Inn(\mathcal{M}_m(K)) \times Inn(\mathcal{M}_n(K)) = Inn(B)$ due to Skoler-Noether's theorem (cf. [7][Theorem 4.3.1]). Hence $Out(B)$ is trivial.*

2. *Suppose now that* $A = [\begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}]$,

$$B = [\begin{pmatrix} K & K & K & K \\ 0 & K & 0 & 0 \\ 0 & 0 & K & K \\ 0 & 0 & K & K \end{pmatrix}]$$

The algebra A is basic and if e_1, e_2, e_3 are the canonical primitive idempotents of A, then B is isomorphic to $End_A(P_1 \oplus P_2 \oplus P_3^{(2)})$, where $P_i = Ae_i$ for $i = 1, 2, 3$. Hence, A and B are Morita equivalent and, after the suitable identifications using Theorem 1, $Pic(B) \cong Pic(A) = Out(A)$. The algebra A is isomorphic to the path algebra of the quiver $2 \longleftarrow 1 \longrightarrow 3$, which has an outer automorphism obtained by permuting the vertices 2 and 3 and the corresponding arrows arriving at them (see, e.g., [5][Corollary 4.9(b)]). The element F of the Picard group represented by that automorphism permutes the isoclasses of simples left A-modules via the trasposition (23), i.e. $F(S_1) \cong S_1$, $F(S_2) \cong S_3$ and $F(S_3) \cong S_2$. We claim that no outer automorphism of B can induce such a permutation of the simple modules when viewed as an element of $Pic(B)$. Indeed, if $\varphi \in Aut(B)$ it induces an automorphism of the semisimple algebra $B/J(B) \cong K \times K \times M_2(K)$ which cannot induce the transposition (23) when permuting the blocks of $B/J(B)$. That means that the element $\bar{\varphi} \in Out(B) \subseteq Pic(B)$ cannot induce the trasposition (23) when permuting the simple modules. Hence, $F \notin Out(B)$ and $Out(B) \subsetneq Pic(B)$. Then a Morita equivalence $H :_A Mod \longrightarrow_B Mod$ induces an isomorphism $Pic(A) \cong Pic(B)$ and not an isomorphism $Out(A) \cong Out(B)$.

These examples tend to indicate that, in case A and B are Morita equivalent algebras, the groups $Out(A)$ and $Out(B)$ are isomorphic 'up to a discrete part'. The first result stating rigorously what that means is the following one. It is attributed to Brauer and a proof of (a generalization of) it can be found in [14][Theorem 2.1]

Theorem 2 *The identity component $O(A)$ of the algebraic group $Out(A)$ is Morita invariant.*

After this result, it was natural to ask whether the group $O(A)$ was a tighter invariant, namely, if it was invariant under generalizations of Morita equivalence. The following result, appeared in [6] (cf. Theorem 2.5), gave support to that idea. We state it with the same terminology of that paper.

Theorem 3 *Let A and B be two finite dimensional algebras which are tilting-cotilting equivalent. There is an algebraic group G together with two morphisms of algebraic groups $G \longrightarrow Out(A)$ and $G \rightarrow Out(B)$, such that their restrictions to the identity component G^0 induce isomorphisms $G^0 \cong O(A)$ and $G^0 \cong O(B)$*

Tilting-cotilting equivalences are a particular instance of derived equivalences. Recall that the derived category of A (or, more properly, of its category of modules), here denoted $D(A - Mod)$, is the category whose objects are the chain complexes of A-modules, and where the morphisms are obtained by formally inverting the quasi-isomorphisms of chain complexes. Two algebras A and B are said to be **derived equivalent** when there is a triangulated equivalence $D(A - Mod) \xrightarrow{\cong} D(B - Mod)$. The following step in the natural process was to question about the invariance of $O(A)$ under derived equivalence. The following result is an affirmative answer to the question, which was first given in [8] and slightly later, but independently, in [16].

Theorem 4 *If A and B are finite dimensional derived equivalent algebras, then there is an isomorphism of algebraic groups $O(A) \cong O(B)$*

Remark 1 *For a beautiful conceptual interpretation of the above invariance property of $O(A)$, we refer the reader to [10].*

Recall that the projectively (resp. injectively) stable category of A is the category $A - \underline{mod}$ (resp.$A - \overline{mod}$), where the objects are the finitely generated A-modules and the morphisms are obtained from those in $A - mod$ by killing the A-homomorphisms which factor through projectives (resp. injectives). In case A is selfinjective, $A - \underline{mod} = A - \overline{mod}$ is a triangulated factor of the bounded derived category $D^b(A - mod)$ (cf. [15][Theorem 2.1]). A stable equivalence between two selfinjective algebras A and B is just an equivalence of triangulated categories $A - \underline{mod} \xrightarrow{\cong} B - \underline{mod}$. It is then natural to ask whether $O(A)$ is invariant under stable equivalences. To the best of our knowledge, the general answer is not known yet, but the following is a partial positive answer given by Rouquier (cf. [16][Théorème 4.3]). We need to introduce some terminology. A particular instance of stable equivalences are those induced by exact functors $M \otimes_B - : B - mod \longrightarrow A - mod$ and $N \otimes_A - : A - mod \longrightarrow B - mod$, where ${}_AM_B$ and ${}_BN_A$ are bimodules which are projective on both sides and have the property that, as bimodules, $M \otimes_B N \cong A \oplus P$ and $N \otimes_A M \cong B \oplus Q$, with ${}_AP_A$ and ${}_BQ_B$ projective bimodules. Two selfinjective algebras A and B are called **stably equivalent of Morita type** when there is a stable equivalence between them induced by such exact functors.

Theorem 5 *Let A and B be two selfinjective algebras which are stably equivalent of Morita type. Then $O(A)$ and $O(B)$ are isomorphic algebraic groups.*

3. New results

In this section we come back to the general situation in which K is an arbitrary commutative ring. Although the algebraic geometric approach to the groups of automorphisms does not go on anymore, some of the isomorphisms for finite dimensional algebras over a field can be extended, as isomorphisms of abstract groups, to much more general situations. For a given K-algebra, we shall always view $Out(A)$ as a subgroup of $Pic(A)$ via Theorem 1. We define the subgroup of $Pic(A)$ given by $N_A = \{F \in Pic(A) : F(P) \cong P$, for all f.g. projectives ${}_AP\}$

Recall that an algebra A is called **Von Neumann regular** when its finitely generated left (resp. right) ideals are direct summands of ${}_AA$ (respectively A_A). It is called **semiregular** (resp. **semiperfect**) when $A/J(A)$ is Von Neumann regular (resp. semisimple) and idempotents lift modulo $J(A)$. If, in addition, $A/J(A)$ is a finite direct product of division algebras, then A is called **basic**. It is well-known that every semiperfect algebra is Morita equivalent to a basic semiperfect algebra, which is uniquely determined up to isomorphism. When A is semiperfect, we shall say that an automorphism $\varphi \in Aut(A)$ preserves the block decomposition of $A/J(A)$ when so does the induced automorphism $\bar{\varphi}$ of $A/J(A)$

Proposition 1 *The subgroup N_A is always contained in $Out(A)$ and is Morita invariant. In case A is semiregular, $N_A = H/Inn(A)$, where H is the subgroup of $Aut(A)$ consisting of those automorphisms φ which satisfy one (or both) of the following equivalent conditions:*

1. *$Ae \cong A\varphi(e)$ (as left A-modules), for every idempotent $e \in A$*
2. *For every idempotent $e \in A$, there exist $a \in eA\varphi(e)$ and $b \in \varphi(e)Ae$ such that $ab = e$ and $ba = \varphi(e)$*

Moreover, in case A is semiperfect, one has $H = \{\varphi \in Aut(A) : \varphi$ preserves the block decomposition of $A/J(A)\}$

Sketch of proof: Since the class of f.g. projective modules is invariant under Morita equivalences, the group N_A is Morita invariant. Moreover, by Theorem 1, it is contained in $Out(A)$. Then, from the description of f.g. projective modules over semiregular rings given in [12][Corollary 1.13] the equality $N_A = H/Inn(A)$ follows, with H as in the statement of the proposicion. For the last statement, one should notice that semiperfect rings are characterized as the rings for which every simple module has a projective cover. Then, in that case, N_A gets identified with the subgroup of $Out(A)$ given by those $\bar{\varphi}$ such that ${}^{\varphi}S \cong S$, for every simple module S. That is equivalent to say that φ preserves the block decomposition.

We next denote by $Aut(A)_1$ the (normal) subgroup of $Aut(A)$ consisting of those $\varphi \in Aut(A)$ which induce the identity on $A/J(A)$, and $Out(A)_1$ will be the image of $Aut(A)_1$ by the canonical projection $Aut(A) \longrightarrow Out(A)$. The next result deeply generalizes [14][Theorem 2.1], except in its algebraic geometric part.

Theorem 6 *Let A be any K-algebra. The group $Out(A)_1$ is Morita invariant.*

Sketch of proof: Every Morita equivalence $F : A - Mod \xrightarrow{\cong} B - Mod$ induces another one $\bar{F} : A/J - Mod \xrightarrow{\cong} B/J - Mod$, where J is the Jacobson radical. From that one derives that the group homomorphism $p_A : N_A \longrightarrow N_{A/J}$ is Morita invariant, i.e., the isomorphism : $\tilde{F} : N_A \cong N_B$ induced by F and the isomorphism $\hat{F} : N_{A/J} \cong N_{B/J}$ induced by $\bar{F}$ satisfy that $\hat{F} \circ p_A = p_B \circ \tilde{F}$. Then one proves that the kernel of p_A is precisely $Out(A)_1$ and the result follows.

In [4][Corollary 19], it was proved that, in case A is a finite dimensional split algebra over a field, then the group $Aut(A)_1/Inn^*(A)$ is Morita invariant. The next result extends that to much more general situations. For any K-algebra R, we denote by $U(R)$ the group of multiplicatively invertible elements of R and by $Z(R)$ the center of R.

Theorem 7 *Let A be a K-algebra satisfying one of the following conditions:*

1. *$J(A)$ is nilpotent and $A/J(A)$ is a separable K-algebra which is projective as a K-module*
2. *The canonical group homomorphism $U(Z(A)) \longrightarrow U(Z(A/J(A)))$ is surjective*

Then the group $Aut(A)_1/Inn^*(A)$ is Morita invariant

Sketch of proof: . As in the proof Theorem 6, by taking a pair of associated Morita equivalences $F : A - Mod \xrightarrow{\cong} B - Mod$ and $\bar{F} : A/J - Mod \xrightarrow{\cong} B/J - Mod$, one sees that the canonical algebra homomorphism $p_A : Z(A) \longrightarrow Z(A/J)$ is Morita invariant and taking the groups of multiplicatively invertible elements, we see that the hypothesis 2 is Morita invariant. Then one derives from that hypothesis that $Aut(A)_1 \cap Inn(A) = Inn^*(A)$, so that $Aut(A)_1/Inn^*(A) \cong Out(A)_1$ is Morita invariant. That proves the theorem under hypothesis 2.

On the other hand, under the hypothesis 1, he Principal Wedderburn-Malcev Theorem yields a decomposition $A = B \oplus J(A)$, where B is a semisimple subalgebra of A. The point here is that $Aut(A)_1/Inn^*(A)$ is canonically isomorphic to $H_A/H_A \cap Inn^*(A)$, where $H_A = \{\varphi \in Aut(A) : \varphi(B) = B$ and φ induces the identity on $B\}$. By taking a suitable idempotent $e \in B$, one has that eAe is the basic algebra of A. Now, by restricting automorphisms, we get a group homomorphism $H_A/H_A \cap Inn^*(A) \longrightarrow H_{eAe}/H_{eAe} \cap Inn^*(eAe)$, which one proves is an isomorphism using the existence of an exact sequence of groups $1 \to Coker(p_A) \longrightarrow H_A/H_A \cap Inn^*(A) \longrightarrow Out(A)_1 \to 1$, where the extreme nontrivial terms are Morita invariant.

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