# Incidence algebras and algebraic fundamental group

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*AMS classification* : 16E40 ; 16G20 ; 06A11 ; 55Q05

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One of the main tools for the study of the category of finite dimensional modules over a basic algebra, over an algebraically closed field $k$ is its presentation as quiver and relations. This theory is mainly due to P. Gabriel (see for example [GRo]). More precisely, it has been proved that for all finite dimensional and basic algebras over an algebraically closed field $k$, there exists a unique quiver $Q$ and an admissible ideal $I$ of the algebra $kQ$, the path quiver algebra of $Q$, such that $A$ is isomorphic to $kQ/I$. Such a couple $(Q, I)$ is called a presentation of $A$ by quiver and relations. For each presentation $(Q, I)$, we can construct an algebraic fundamental group $\Pi_1(Q, I)$.

I will present here three results. First, the fundamental group of an incidence algebra has a geometric representation (see [Rey1] or [Bus]). Indeed, the algebraic fundamental group is isomorphic to a topological fundamental group of a simplicial complex. Second, to give a geometric vision of all algebraic fundamental groups, we construct for each presentation $(Q, I)$ an incidence algebra $A$ and show that there is an exact sequence of groups of the following form :

$$1 \longrightarrow H \longrightarrow \Pi_1(Q, I) \longrightarrow \Pi_1(A) \longrightarrow 1$$

in which $H$ is a sub-group of $\Pi_1(Q, I)$ which we can describe by generators and relations. We will give a list of sufficient conditions to have this subgroup equal to 1 : for example, it happens in the case of Schurian algebras.

Third, we describe an algorithm to calculate all fundamental groups, which allows to quickly present the fundamental groups by generators and relations. To calculate the fundamental group of a presentation $(Q, I)$, if $Q$ has more than one vertex, we can prove that it is isomorphic to the fundamental group of a couple $(Q', I')$ in which $Q'$ contains a vertex less than $Q$. Then by applying this process again and again, we obtain that the fundamental group $\Pi_1(Q, I)$ is isomorphic to the fundamental group of a presentation of an algebra in which the quiver only contains one vertex, using this we give a presentation of $\Pi_1(Q, I)$ by generators and relations.

We assume in this paper that the definition of the algebraic fundamental group is known. Otherwise, it can be found in [Ap] or [Red], for example.

This paper gives some results of my phd and I want to thank all the persons who have helped me in its realization, particularly C.Cibils, my thesis director, M. Saorín and M.J. Redondo.

## 1. Simplicial complexes and incidence algebras

Given a *poset* (i.e. a partially ordered set), there is an associated *ordered quiver*, that is to say a finite oriented graph without loops and such that if there exists an arrow from $a$ to $b$, there does not exist any

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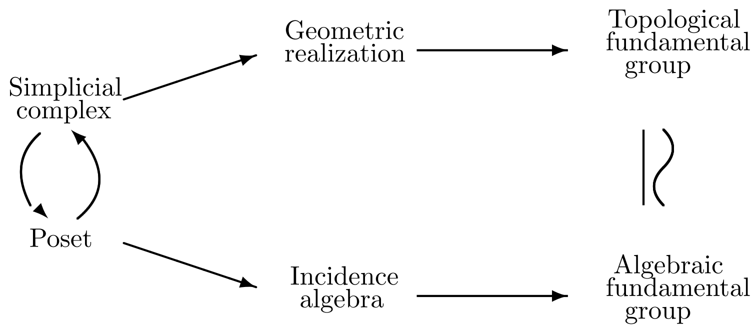
other path from $a$ to $b$. To each element of the poset corresponds a vertex of the graph. Moreover, let $S_1$ and $S_2$ be vertices in the graph; there exists an arrow from $S_1$ to $S_2$ if and only if the element associated to $S_1$ in the poset is smaller than the element associated to $S_2$ and if there does not exist any element of the poset strictly between these two elements. The graph obtained is an ordered quiver. Conversely, by this operation all ordered graphs arise from a poset. There is a bijection between the set of ordered graphs and the set of posets. Moreover, a poset is said to be connected if its ordered quiver is connected. All the posets considered will be connected, finite and non empty.

Let now $Q$ be a quiver, and $k$ be a field. We denote $kQ$ the $k$-vector space with basis the paths of $Q$ (the paths of length 0 being the vertices), with the multiplication given by the composition of two paths if possible and 0 otherwise. Two paths of $Q$ are *parallel* if they have the same beginning and the same end. The k-space generated by the set of differences of two parallel paths is a two-sided ideal of $kQ$, denoted $I_Q$ and called *parallel ideal*. The quotient algebra $kQ/I_Q$ is the *incidence algebra* of $Q$.

Let $C$ be a simplicial complex. The set of non empty simplexes of $C$ ordered by inclusion is a poset which we will denote $Pos(C)$. To each poset $P$ we associate a simplicial complex $Sim(P)$, in which the n-simplexes are the subsets of $P$ which are totally ordered. These procedures are of course not inverse one of each other, their composition is in fact the barycentric decomposition. The application $Sim$ is surjective but not injective, for instance the simplicial complexes which are associated to $a < b < c$, $a < b < d$ and to $a < c < b$, $a < d < b$ are the same.

Then the fundamental group $\Pi_1(|C|)$ defined on the geometric realization of a finite simplicial complex $C$ is isomorphic to the fundamental group $\Pi_1(Pos(C))$ of the incidence algebra of the poset deduced from the complex. A proof of this result can be found in [Rey] and [Bus].

The following diagram summarizes the situation :




## 2. Algorithm of calculus of $\pi_1(Q, I)$

Before giving the algorithm, we must give some notations. Let $Q$ be a quiver and $x_0$ be a vertex of $Q$. We also consider $(c_i)_{i\in A}$ and $(c'_i)_{i\in A}$ two families of paths of $Q$ indexed by a set $A$. Then, we denote by $< (c_i \sim c'_i)_{i\in A} >$, the smallest equivalence relation on $P(Q)$, the set of walks of $Q$, which is compatible with the concatenation and which verifies :

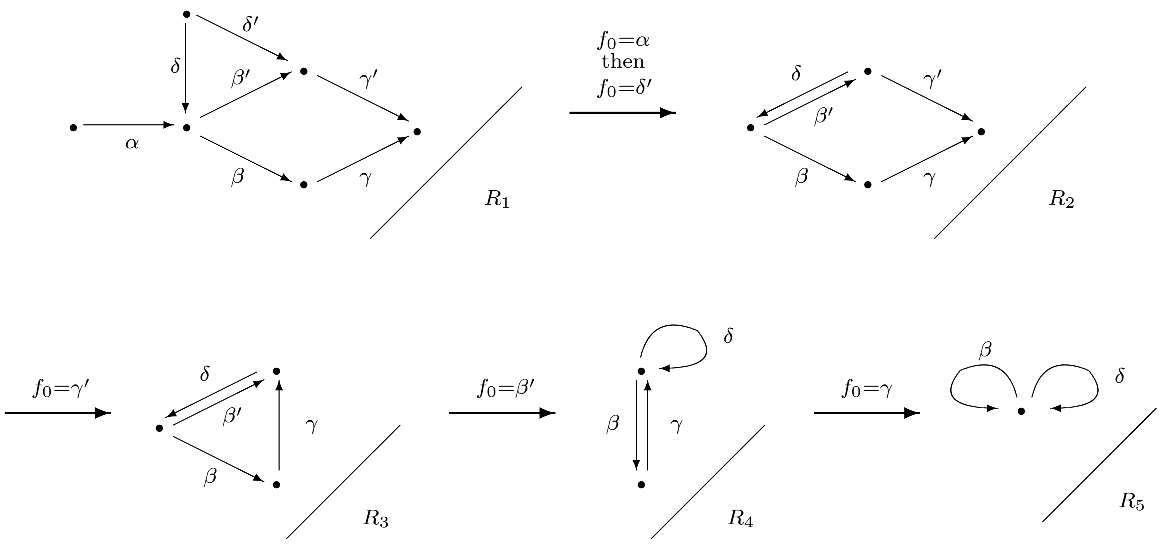
1. if $f$ is an arrow of $Q$ from $a$ to $b$ then $f.f^{-1} \sim b$ and $f^{-1}.f \sim a$,
2. for all $i$ of $A$, we have : $c_i \sim c'_i$.

Moreover, we consider an arrow $f_0$ of $Q$ from $a_0$ to $b_0$ with $a_0 \neq b_0$, and we denote by $Q'$ the quiver obtained from $Q$ by merging the vertices $a_0$ and $b_0$. More precisely, the vertices of $Q'$ are the vertices of $Q$ in which we identify the vertices $a_0$ and $b_0$ and denote the new vertex obtained by this identification by $c_0$. The arrows of $Q'$ are the arrows of $Q$ minus $f_0$. All the other arrows have the same source and the same terminus than in $Q$. Let's notice that a parallel arrow of $f_0$ becomes a loop in $Q'$.

We also denote by $p$ the morphism from $P(Q)$ to $P(Q')$ defined by $p(s) = s$ for any vertex $s$ not equal to $a_0$ and $b_0$, $p(a_0) = p(b_0) = c_0$, $p(f) = f$ for any arrow $f$ not equal to $f_0$ and $p(f_0) = c_0$. Finally, let's define $R'$, the equivalence relation on $kQ'$ : $< (p(c_i) \sim p(c'_i))_{i \in A} >$. Then it has been proved that there exists an isomorphism between $P_{x_0}(Q)/R$ and $P_{p(x_0)}(Q')/R'$ (See [Rey2]).

**The algorithm.** First, we notice that the equivalence relation defined in the construction of the algebraic fundamental group $\Pi_1(Q, I)$ is indeed the equivalence relation $< (\omega_i \sim \omega'_i)_{i \in A} >$ in which for all $i$ of $A$, the paths $\omega_i$ and $\omega'_i$ are paths in the support of a minimal relation. By applying the theorem a number of times equal to the number of vertices of $Q$ minus one, and after the simplification of equivalent loops, we obtain that the algebraic fundamental group $\Pi_1(Q, I)$ is isomorphic to the algebraic fundamental group of a quiver containing only one vertex. This gives a presentation of the group $\Pi_1(Q, I)$ by generators and relations.

**Example II.1.** Let's consider the following quiver and the ideal $I =< \gamma\beta\alpha - \gamma'\beta'\alpha, \gamma\beta\delta - \gamma'\delta', \gamma'\beta'\delta - \gamma'\delta' >$. We are going to calculate the fundamental group of this presentation by using the algorithm.




With

$$\begin{cases} R_1 =< & \gamma\beta\alpha \sim \gamma'\beta'\alpha, & \gamma\beta\delta \sim \gamma'\delta', & \gamma'\beta'\delta \sim \gamma'\delta' >, \\ R_2 =< & \gamma\beta \sim \gamma'\beta', & \gamma\beta\delta \sim \gamma', & \gamma'\beta'\delta \sim \gamma' >, \\ R_3 =< & \gamma\beta \sim \beta', & \gamma\beta\delta \sim t(\gamma), & \beta'\delta \sim t(\gamma) >, \\ R_4 =< & \gamma\beta \sim s(\beta), & \gamma\beta\delta \sim t(\gamma), & \delta \sim t(\gamma) >, \\ R_5 =< & \beta \sim s(\beta), & \beta\delta \sim s(\beta), & \delta \sim s(\beta) >. \end{cases}$$

Then, after simplification of these two arrows, we have that the fundamental group of $(Q, I)$ is trivial.

## 3. Incidence algebra associated to a presentation

In this paragraph, to each presentation by quiver and relations $(Q, I)$ of a $k$-algebra $A$ we are going to construct an incidence algebra $A'$. The goal of this paragraph is then to compare the algebraic fundamental group of $(Q, I)$ to those of $A'$. The algebra $A'$ is interesting because it can give - in the case in which the fundamental groups of $A$ and $A'$ are isomorphic - a geometric vision of the algebraic fundamental group.

We first consider the set $C(Q)/\sim$ in which $C(Q)$ is the set of paths of $Q$ and $\sim$ is the equivalence relation used to construct the fundamental group. The set $\Sigma$ will be the set $C(Q)/\sim$ minus the classes containing paths in $I$. Then, by construction, the paths in a class of $\Sigma$ are never equivalent to any other path in $I$.

Since the beginnings and the ends of equivalent paths are the same, the notions of origin, end, parallelism can be extended to the set $\Sigma$. Moreover, since the equivalence relation $\sim$ is compatible with the concatenation, the set $\Sigma$ inherits a law of composition partially defined. Moreover, we say that $\overline{a}$ divide $\overline{b}$ and we denote it by $\overline{a}/\overline{b}$ if there exists $\overline{w}$ and $\overline{w'}$ in $\Sigma$ such that $\overline{b} = \overline{w}.\overline{a}.\overline{w'}$. Since the ideal is admissible the relation divide is an order relation on $\Sigma$ (see [Rey2])

These two objects, the presentation and the poset, have their own algebraic fundamental group and they are linked by the following theorem :

**Theorem III.1.** Let $(Q, I)$ be a presentation of an algebra and $\Sigma$, the associated incidence poset. Then the sequence

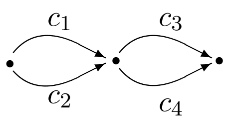
$$0 \longrightarrow H \longrightarrow \Pi_1(Q, I) \longrightarrow \Pi_1(Q_\Sigma) \longrightarrow 0$$

is exact, in which $H$ is defined as follows. Let $\sim$ be the equivalence relation defined in the construction of the algebraic fundamental group $\Pi_1(Q, I)$. We also consider the equivalence relation $R$ in $P(Q)/_\sim$, the set of isoclasses by the equivalence relation $\sim$ of walks of $Q$, the smallest equivalence relation compatible with the multiplication and verifying the following property : if two parallel paths $c_1$ and $c_2$ of $Q$ divide two $\sim$-equivalent paths not in $I$, then the classes $\overline{c_1}$ and $\overline{c_2}$ are $R$-equivalent. Then $H$ is the normal sub-group of $\Pi_1(Q, I)$ associated to the relation $R$, that is to say the smallest normal sub-group $H$ of $\Pi_1(Q, I)$ containing the walks $p^{-1}\overline{c_1}^{-1}.\overline{c_2}p$, with $p$ a walk which begins in $X_0$, the vertex chosen to calculate $\Pi_1(Q, I)$, and with $c_1$ and $c_2$ parallel paths dividing respectively two paths $\sim-$equivalent non equal to zero in $kQ/I$. The sub-group $H$ can be written as follows :

$$H = < \overline{p}^{-1}.\overline{c_1}^{-1}.\overline{c_2}.\overline{p} \ \ / \ \ \overline{p} \text{ a walk which begins in } X_0,\ \overline{c} \in \Sigma, \text{ and } \overline{c_1}/\overline{c},\ \overline{c_2}/\overline{c},\ \overline{c_1}//\overline{c_2} \ > .$$

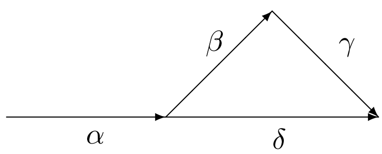
In [Rey2], it has been given some sufficient conditions to have an isomorphism between the two fundamental groups described in the previous paragraphs, that is to say to have a sub-group $H$ restricted to the neutral element. We recall here the main conditions :

1. The ideal $I$ is restricted to 0.
2. The couple $(Q, I)$ is the presentation of an incidence algebra, more generally a presentation of a Schurian algebra.
3. The quiver $Q$ does not contain cycles and sub-quivers in the form of eight, that is to say sub-quivers in the form :

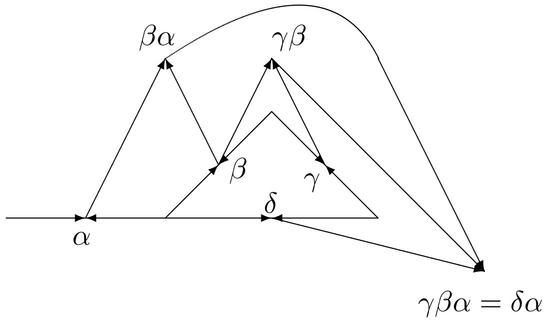


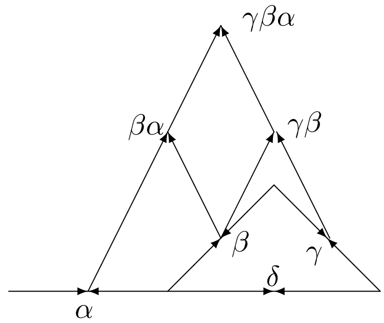

in which arrows represent in fact paths.

**Example III.2.** This example shows that the incidence algebra associated to a presentation $(Q, I)$ is not an invariant of the algebra. Indeed, let's consider the quiver $Q$ :




and the two ideals $I_1 = < \gamma\beta\alpha - \delta\alpha >$ and $I_2 = < \delta\alpha >$ of $kQ$. The two algebra $kQ/I_1$ and $kQ/I_2$ are isomorphic. Let's construct now the incidence quivers $\Sigma_{(Q,I_1)}$ and $\Sigma_{(Q,I_2)}$ :






For the presentation $(Q, I_1)$, the fundamental group is trivial, then by using the previous theorem the fundamental group of the associated incidence quiver is restricted to 1. For the second presentation $(Q, I_2)$, the fundamental group of the associated incidence quiver is isomorphic to $\mathbb{Z}$. To see that, we are going to calculate the sub-group $H$ of $\Pi_1(Q, I_2)$. Only the paths $\delta$ and $\gamma\beta$ are parallel and not equivalent. The subgroup $H$ is then generated by the relation $\beta\alpha - \delta\gamma\alpha$. We choose the vertex $\alpha$ as base point to calculate the fundamental group. But this relation is trivial in $\Pi_1(Q, I_2)$ and then $H$ is restricted to 1. Moreover, by using the previous theorem again, the fundamental group of $\Sigma_{(Q,I_2)}$ is isomorphic to $\Pi_1(Q, I_2)$ which is isomorphic to $\mathbb{Z}$.

Then, there exists two presentations of an algebra whose incidence quivers give different fundamental groups ; the fundamental groups of the associated incidence algebras also depend on the presentation of the algebra.

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[AMA - Algebra Montpellier Announcements - 01-2003] [September 2003]